

# **HYPERG**

Maple Package

User's Reference Manual

**Version 1.0**

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# Abstract

**HYPERG** is a *Maple* package for the manipulation of generalized hypergeometric series denoted by the usual symbol  $F(a_1, a_2, \dots; b_1, b_2, \dots; x)$ . It also provides tools for automatic proofs of identities, but they are just tools: every step in a series of manipulations has to be controlled by the user. The goal of this package is to make the algorithms in this area better known, and to help you enter the wonderful world of hypergeometric series. Many features of this package are due to C. Krattenthaler and his *Mathematica* package HYP [15]. Many thanks to C. Krattenthaler, P. Paule, M. Petkovšek, A. Tefera and D. Zeilberger for their helpful comments.

*“There must be many universities to-day, where 95 per cent, if not 100 per cent, of the functions studied by physics, engineering, and even mathematics students, are covered by the single symbol  $F(a, b; c; x)$ .”*

— W. W. Sawyer [27]



# Part I

## Installation



This part is devoted to the installation of HYPERG. This package is developed at the University of Marne-la-Vallée, it is available on the *World Wide Web* at the URL <http://www-igm.univ-mlv.fr/~gauthier/>.

For the moment, this program only works under *Maple V* Release 2 or later.

## 0.1 Installing HYPERG on a UNIX/LINUX system

You must have the following files:

README	
HYPERG	The code file
HYPERG.tst	The test file
HYPERG.ps.gz	This manual (gzipped postscript-file)

To compile the package, you must move the HYPERG file in your maple packages directory (for example: /usr/people/joe/maple\_pack) then type the UNIX command:

```
maple -s -q < HYPERG
```

Then, you have to tell *Maple* where to find the compiled file HYPERG.m. You must add the two following lines to the .mapleinit file in your HOME directory, or create such a file if you don't yet have one:

☛ For *Maple V* Release 1:

```
HYPERGLib:='/usr/people/joe/maple_pack':
_liblist:=[HYPERGLib]:
```

☛ For *Maple V* Release 2, 3 or 4:

```
HYPERGLib:='/usr/people/joe/maple_pack':
libname:=libname, HYPERGLib:
```

## 0.2 Using the package

You can use HYPERG by running *Maple V* and then loading the package by the *Maple* command with(HYPERG);. To get help, you can type ?HYPERG, and to know the loaded version of HYPERG, you can type 'HYPERG/Version';.

Set the global variable 'HYPERG/Verbose' to a value greater than 0 to display information about the execution of some procedures (the greater the value, the more informations displayed). Its default value is 0 (for no information). You can for instance set it to 1 to trace your mathematical work.

## 0.3 Contact

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HYPERG's main page: <http://www-igm.univ-mlv.fr/~gauthier/HYPERG.html>