Bordeaux Graph Workshop
21/11/2012 - Bordeaux

(Nearly)-tight bounds on the linearity and contiguity of cographs

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Outline

- Contiguity and linearity
- Cographs and cotrees
- A min-max theorem on the rank of a tree
- Upper bounds with caterpillar decompositions
- Lower bounds with claws
- Perspectives

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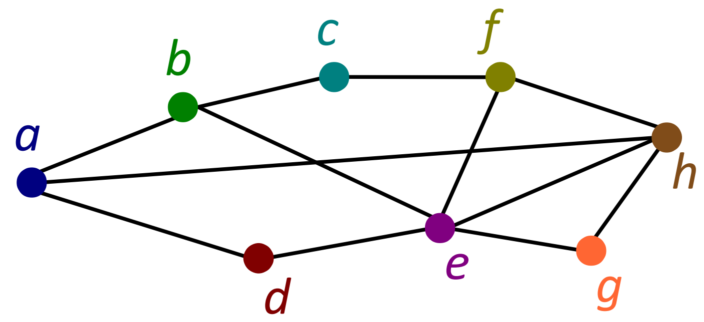
Contiguity

a graph G



a **closed-*k*-interval model** of G

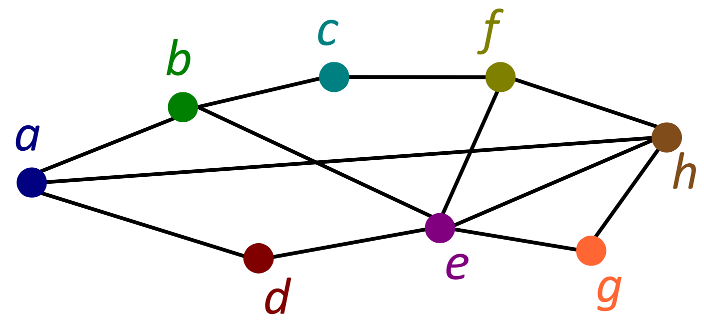
a d b c e f h g

→ order σ where the closed neighborhood of each vertex is the union of at most k intervals of σ

closed contiguity of G, cc(G): smallest k / G has a closed-k-interval model

Contiguity

a graph G



a **closed-k-interval model** of G

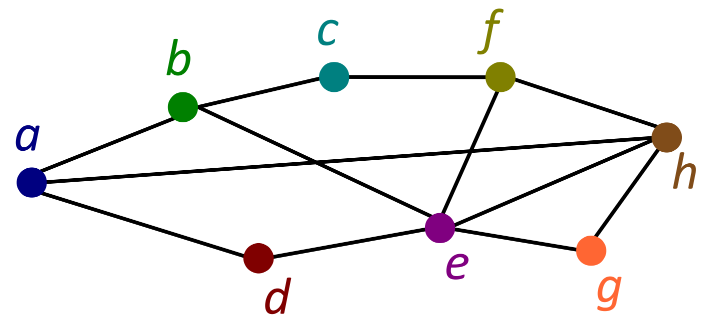
→ order σ where the closed neighborhood of each vertex is the union of at most k intervals of σ

$N[a]$: $\underline{a\ d\ b}\ c\ e\ f\ \underline{h}\ g$

closed contiguity of G, cc(G): smallest k / G has a closed-k-interval model

Contiguity

a graph G



a **closed-k-interval model** of G

$N[a]$: $a\ d\ b\ c\ e\ f\ h\ g$ $\Longrightarrow k \geq 2$

→ order σ where the closed neighborhood of each vertex is the union of at most k intervals of σ

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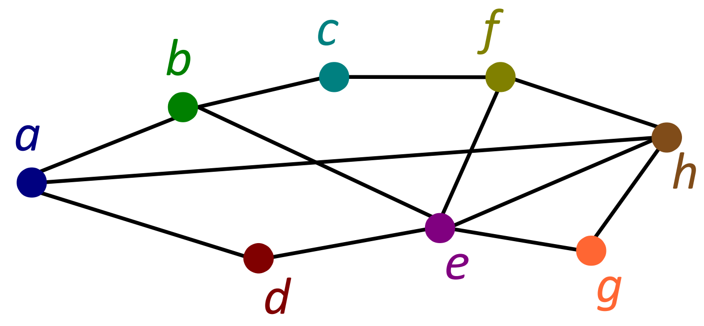
Contiguity

a graph G



a **closed-k-interval model** of G

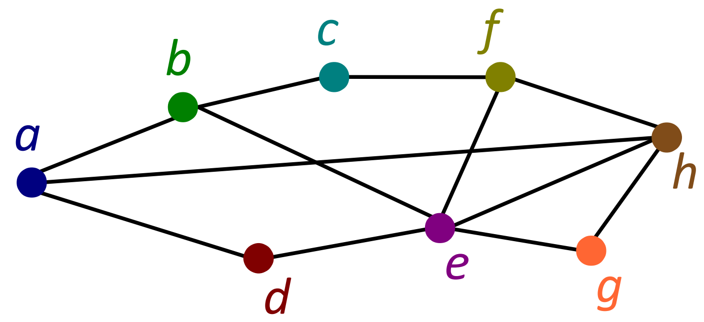
→ order σ where the closed neighborhood of each vertex is the union of at most k intervals of σ

$N[b]$: a d b c e f h g $\implies k \geq 2$

closed contiguity of G, cc(G): smallest k / G has a closed-k-interval model

Contiguity

a graph G



a **closed-*k*-interval model** of G

→ order σ where the closed neighborhood of each vertex is the union of at most k intervals of σ

$N[c]$: $a\ d\ \underline{b\ c}\ e\ \underline{f}\ h\ g$ $\Longrightarrow k \geq 2$

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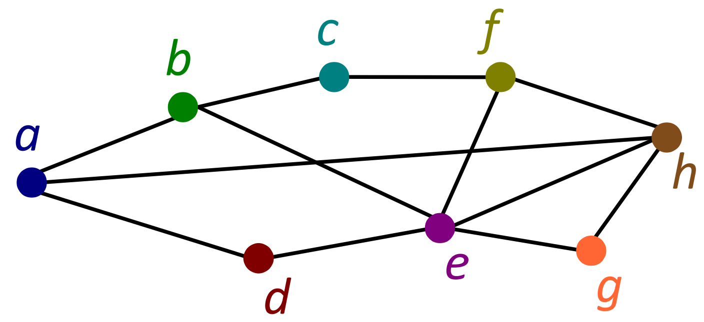
Contiguity

a graph G



a **closed-k-interval model** of G

→ order σ where the closed neighborhood of each vertex is the union of at most k intervals of σ

…

$a\ d\ b\ c\ e\ f\ h\ g$

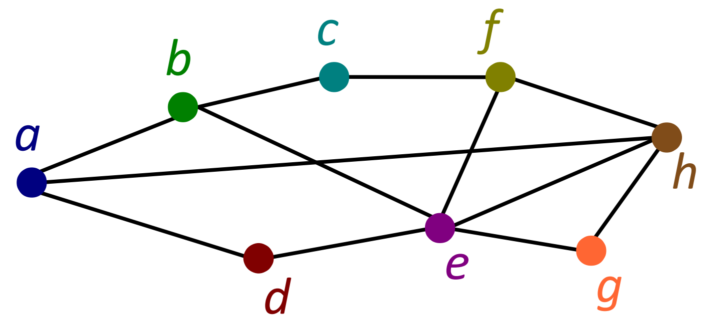
$\Longrightarrow k = 2$

$\Longrightarrow \mathrm{cc}(G) \leq 2$

closed contiguity of G, cc(G): smallest k / G has a closed-k-interval model

Contiguity

a graph *G*



a **closed-*k*-interval model** of *G*

→ order σ where the closed neighborhood of each vertex is the union of at most *k* intervals of σ

... *a d b c e f h g*

⟹ $k = 2$

⟹ $cc(G) \leq 2$

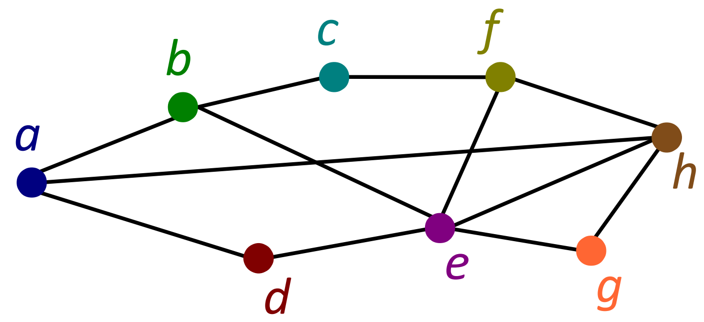
A **min-max parameter**:

$$cc(G) = \min_{\sigma} \max_{v} cc_{G,\sigma}(v)$$

closed contiguity of *G*, cc(*G*): smallest *k* / *G* has a closed-*k*-interval model

Contiguity

a graph *G*



a **closed-*k*-interval model** of *G*

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⟹ $k = 2$

→ order σ where the closed neighborhood of each vertex is the union of at most *k* intervals of σ

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A **min-max parameter**:

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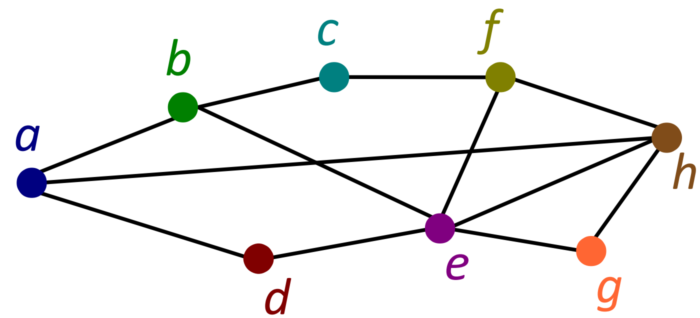
closed contiguity of *G*, cc(*G*): smallest *k* / *G* has a closed-*k*-interval model

➡ **compact representation of *G***

Contiguity and consecutives ones

a graph G



a **closed-k-interval model** of G

→ order σ where the closed neighborhood of each vertex is the union of at most k intervals of σ

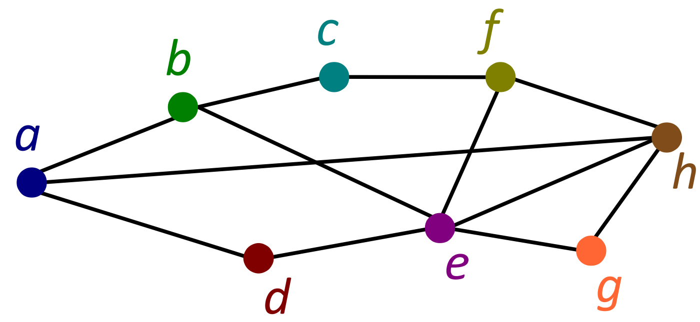
	a	*d*	*b*	*c*	*e*	*f*	*h*	*g*
a	1	1	1	0	0	0	1	0
d	1	1	0	0	1	0	0	0
b	1	0	1	1	1	0	0	0
c	0	0	1	1	0	1	0	0
e	0	1	1	0	1	1	1	1
f	0	0	0	1	1	1	1	0
h	1	0	0	0	1	1	1	1
g	0	0	0	0	1	0	1	1

adjacency matrix A of G

G has **closed contiguity** $\leq k$ if the lines and columns of the adjacency matric A of G can be reordered so that they contain at most k blocks of consecutives ones.

Contiguity and consecutives ones

a graph *G*



a **closed-*k*-interval model** of *G*

→ order σ where the closed neighborhood of each vertex is the union of at most k intervals of σ

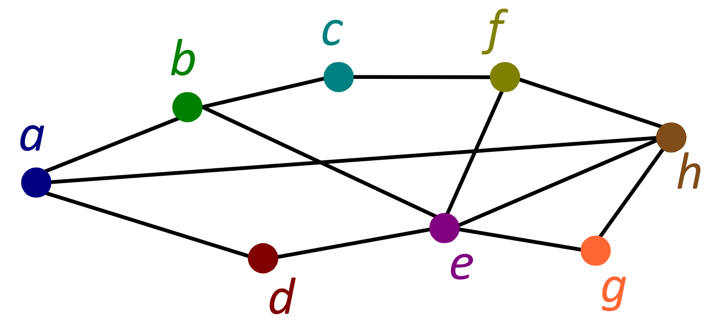
	a	*d*	*b*	*c*	*e*	*f*	*h*	*g*
a	**1**	**1**	**1**	0	0	0	**1**	0
d	**1**	**1**	0	0	**1**	0	0	0
b	**1**	0	**1**	**1**	**1**	0	0	0
c	0	0	**1**	**1**	0	**1**	0	0
e	0	**1**	**1**	0	**1**	**1**	**1**	**1**
f	0	0	0	**1**	**1**	**1**	**1**	0
h	**1**	0	0	0	**1**	**1**	**1**	**1**
g	0	0	0	0	**1**	0	**1**	**1**

$\Longrightarrow$ cc(G) ≤ 2

G has **closed contiguity** ≤*k* if the lines and columns of the adjacency matric *A* of *G* can be reordered so that they contain at most *k* blocks of consecutives ones.

Contiguity

a graph G



an **open-k-interval model** of G

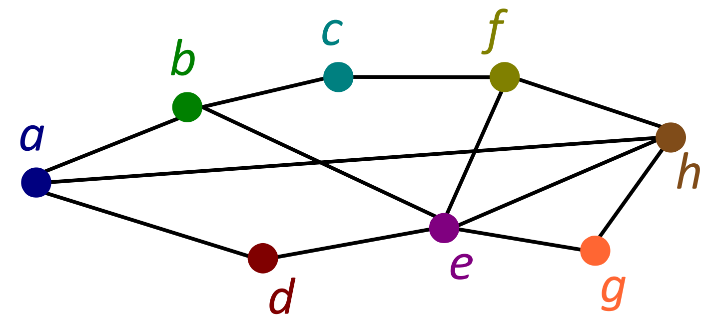
→ order σ where the open neighborhood of each vertex is the union of at most k intervals of σ

$N(a)$: a d b c e f h g $\Longrightarrow k \geq 2$

open contiguity of G, $oc(G)$: smallest k / G has an open-k-interval model

Contiguity

a graph G



an **open-k-interval model** of G

→ order σ where the open neighborhood of each vertex is the union of at most k intervals of σ

$N(h)$: $\underline{a}\ d\ b\ c\ \underline{e\ f}\ h\ \underline{g}$

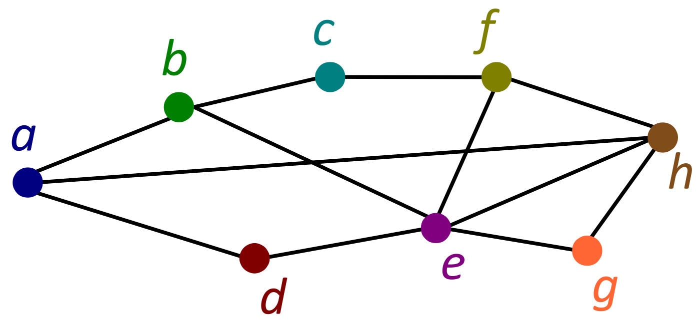
$\Longrightarrow k \geq 3$

$\Longrightarrow \mathrm{oc}(G) \leq 3$

open contiguity of G, $oc(G)$: smallest k / G has an open-k-interval model

Contiguity

a graph G



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→ order σ where the open neighborhood of each vertex is the union of at most k intervals of σ

$N(h)$: a d b c e f h g

$\Longrightarrow k \geq 3$

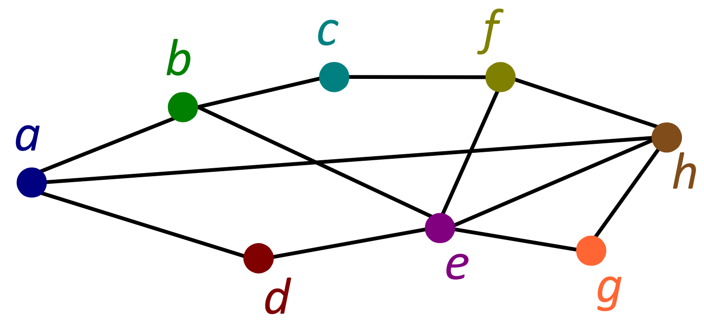
$\Longrightarrow \mathrm{oc}(G) \leq 3$

Remark:
$\mathrm{oc}(G) \leq \mathrm{cc}(G)+1$; $\mathrm{cc}(G) \leq \mathrm{oc}(G)+1$

open contiguity of G, $oc(G)$: smallest k / G has an open-k-interval model

Linearity

a graph *G*



a **closed-*k*-line model** of *G*

→ *k* orders where the closed neighborhood of each vertex is the union of one interval per order

N[*a*]: *a d b c e f h g*

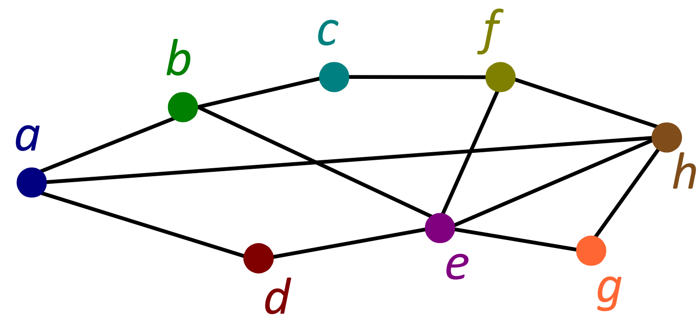
a d b c e f g h

⟹ k ≥ 2

closed linearity of *G*, *cl*(*G*): smallest *k* / *G* has a closed-*k*-line model

Linearity

a graph G



an **open-*k*-line model** of G

→ k orders where the open neighborhood of each vertex is the union of one interval per order

$N(h)$: $a\ d\ b\ c\ e\ f\ h\ g$

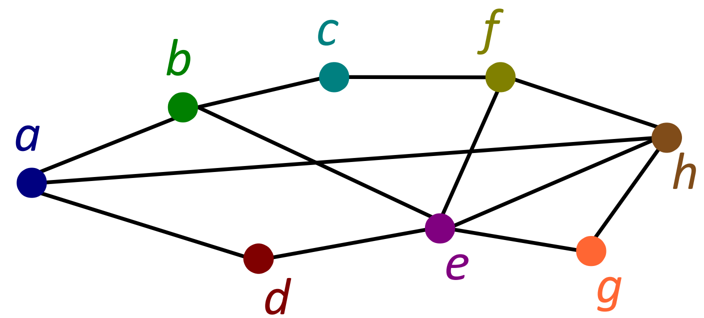
$a\ d\ b\ c\ e\ f\ g\ h$

$\Longrightarrow k \geq 2$

open linearity of G, $ol(G)$: smallest k / G has an open-k-line model

Linearity

a graph G



an **open-k-line model** of G

→ k orders where the open neighborhood of each vertex is the union of one interval per order

$N(h)$: $\underline{a}\ d\ b\ c\ e\ f\ h\ g$

$a\ d\ b\ c\ \underline{e\ f\ g}\ h$

$\Longrightarrow$ k ≥ 2

Remark:
ol(G) ≤ cl(G)+1 ; cl(G) ≤ ol(G)+1
cl(G) ≤ cc(G) (replicate same order)

open linearity of G, $ol(G)$: smallest k / G has an open-k-line model

Computing the contiguity/linearity

cc(G) = 1 $\Leftrightarrow$ cl(G) = 1 $\Leftrightarrow$ G **unit interval** graph

oc(G) = 1 $\Leftrightarrow$ ol(G) = 1 $\Leftrightarrow$ G **biconvex** graph

Given a fixed $k \geq 2$, cc(G) = k ? oc(G) = k ? **NP-complete**

Wang, Lau & Zhao, *DAM*, 2007

Bounds:

For any graph G, cl(G) ≤ cc(G) ≤ $n/4+O(\sqrt{n \log n})$

Gavoille & Peleg, *SIAM JoDM*, 1999

There exist interval graphs and permutation graphs with n vertices and with closed contiguity at least O(log n) / closed linearity at least O(log n / log log n)

Crespelle & Gambette, IWOCA 2009

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➡ upper bound ?

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Cographs

cograph *G* = graph without induced P_4

graph built by series composition (**s**eries operation) and disjoint union (**p**arallel operation)

→ cotree *T*

Two vertices adjacent in *G*
iff their lowest common ancestor in *T* is a series node

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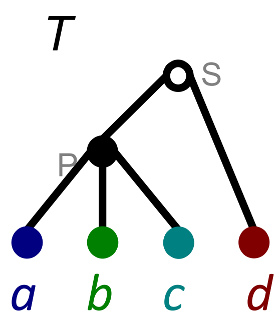

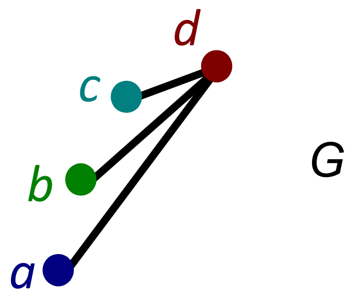

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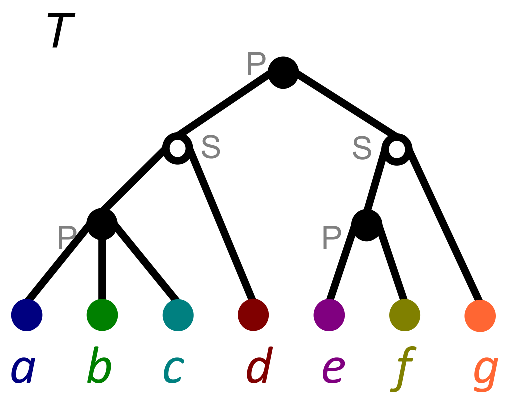

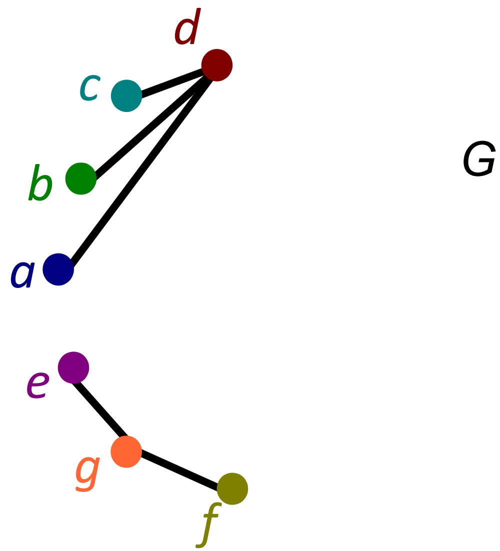

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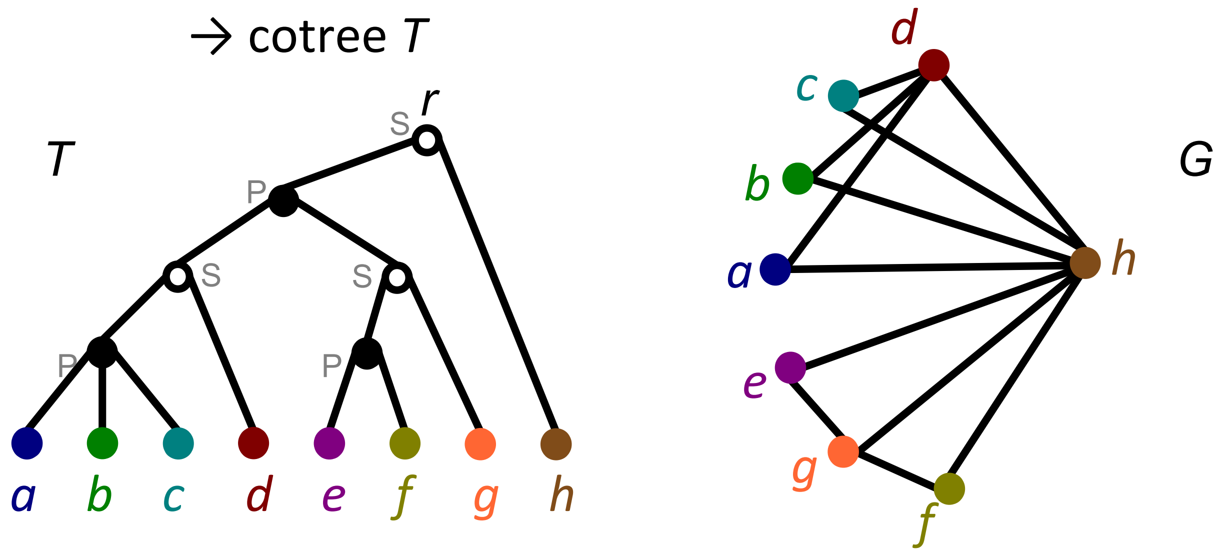

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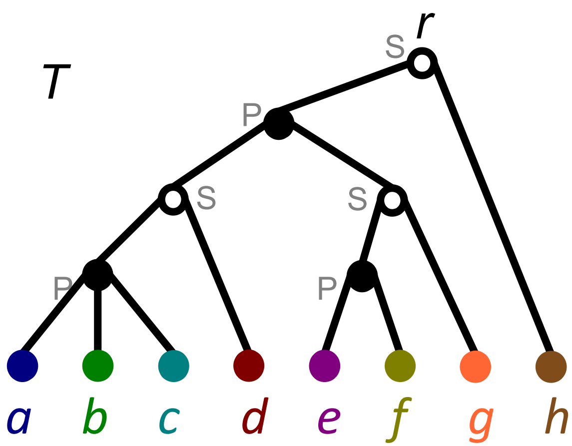
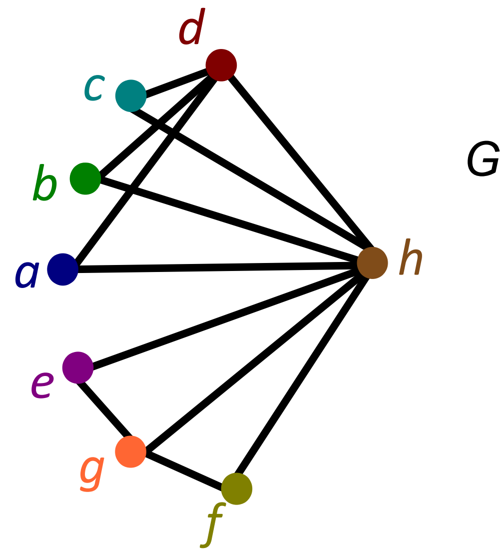




Useful properties:

Complement: For any graph *G*, cc($\overline{G}$) ≤ cc(*G*)+1

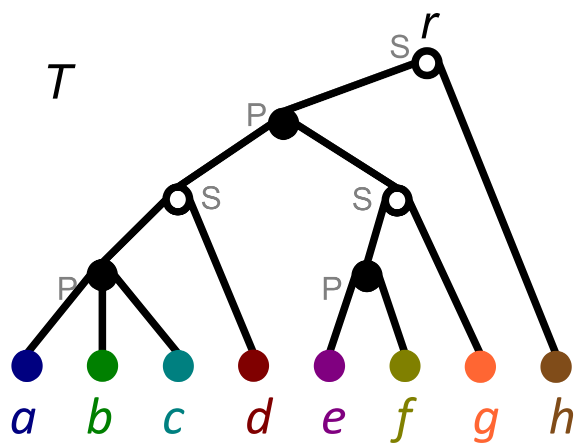
Series & parallel operation: For any graphs *G* and *H*,
cc(**s**(*G*,*H*) ≤ max(cc(*G*),cc(*H*))+1, cc(**p**(*G*,*H*) ≤ max(cc(*G*),cc(*H*))
cl(**s**(*G*,*H*) ≤ max(cl(*G*),cl(*H*))+1, cl(**p**(*G*,*H*) ≤ max(cl(*G*),cl(*H*))

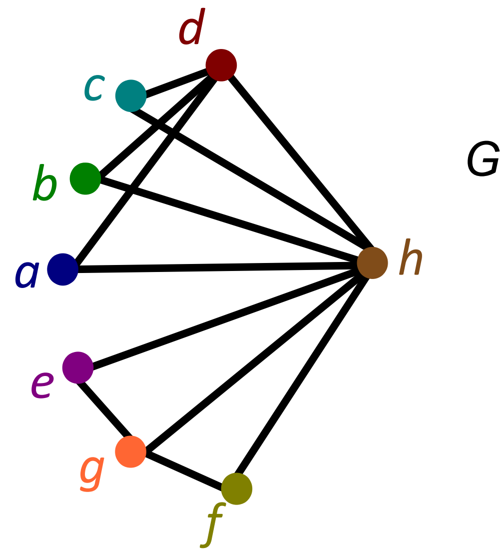
Cographs

cograph *G* = graph without induced P_4

graph built by series composition (**s**eries operation) and disjoint union (**p**arallel operation)

→ cotree *T*





Useful properties:

Complement: For any graph *G*, $cc(\overline{G}) \leq cc(G)+1$

Series & parallel operation: For any graphs *G* and *H*,

$cc(\mathbf{s}(G,H) \leq \max(cc(G),cc(H))+1$, $cc(\mathbf{p}(G,H) \leq \max(cc(G),cc(H))$

$cl(\mathbf{s}(G,H) \leq \max(cl(G),cl(H))+1$, $cl(\mathbf{p}(G,H) \leq \max(cl(G),cl(H))$

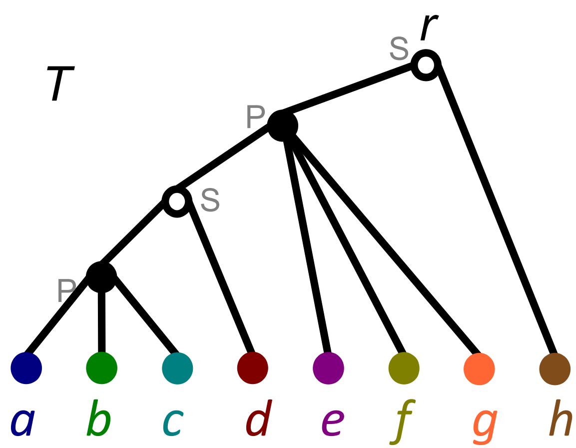
Contiguity of cographs

Useful properties:

Complement: For any graph G, $cc(\overline{G}) \leq cc(G)+1$

Series & parallel operation: For any graphs G and H,
$cc(\mathbf{s}(G,H) \leq \max(cc(G),cc(H))+1$, $cc(\mathbf{p}(G,H) \leq \max(cc(G),cc(H))$
$cl(\mathbf{s}(G,H) \leq \max(cl(G),cl(H))+1$, $cl(\mathbf{p}(G,H) \leq \max(cl(G),cl(H))$
} $cc(G) \leq \text{height}(T)$

Cograph with caterpillar cotree: $cc(G) \leq 2$



$\sigma = d\ h\ a\ b\ c\ e\ f\ g$

Contiguity of cographs

Useful properties:

Complement: For any graph G, $\mathrm{cc}(\overline{G}) \leq \mathrm{cc}(G)+1$

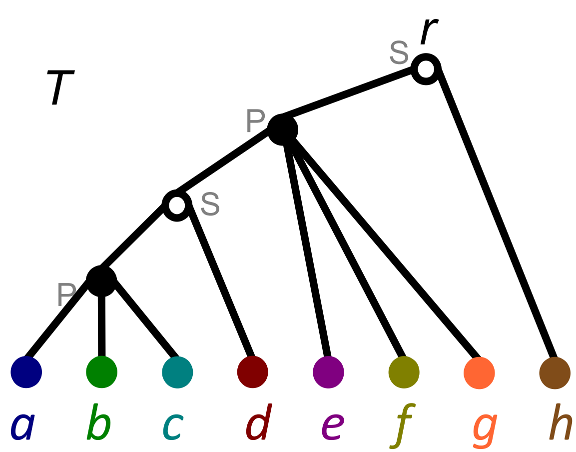
Series & parallel operation: For any graphs G and H,

$\mathrm{cc}(\mathbf{s}(G,H) \leq \max(\mathrm{cc}(G),\mathrm{cc}(H))+1$, $\mathrm{cc}(\mathbf{p}(G,H) \leq \max(\mathrm{cc}(G),\mathrm{cc}(H))$

$\mathrm{cl}(\mathbf{s}(G,H) \leq \max(\mathrm{cl}(G),\mathrm{cl}(H))+1$, $\mathrm{cl}(\mathbf{p}(G,H) \leq \max(\mathrm{cl}(G),\mathrm{cl}(H))$

$\mathrm{cc}(G) \leq \mathrm{height}(T)$

Cograph with caterpillar cotree: $\mathrm{cc}(G) \leq 2$



$\sigma = d\ h\ a\ b\ c\ e\ f\ g$

Combine both to get an upper bound for general cographs?

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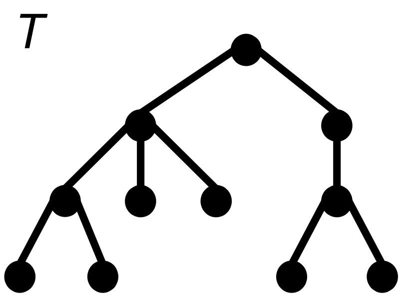
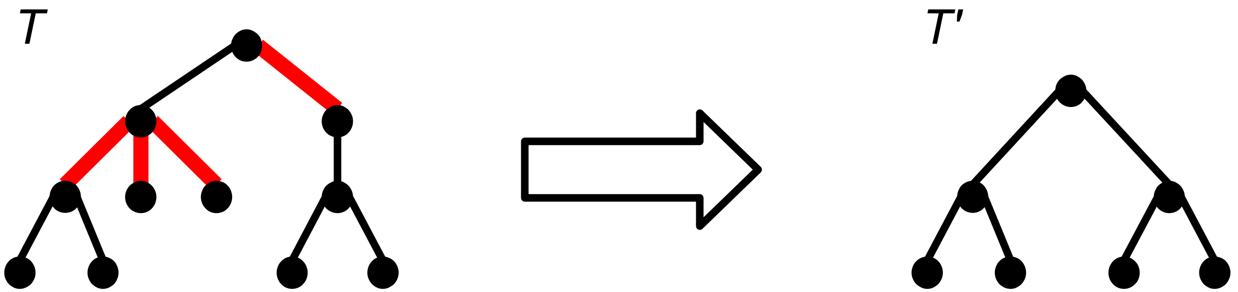
Rank of a tree

rank(T) = maximal height of a complete binary tree T' obtained from T by edge contractions



Rank of a tree

rank(T) = maximal height of a complete binary tree T' obtained from T by **edge contractions**

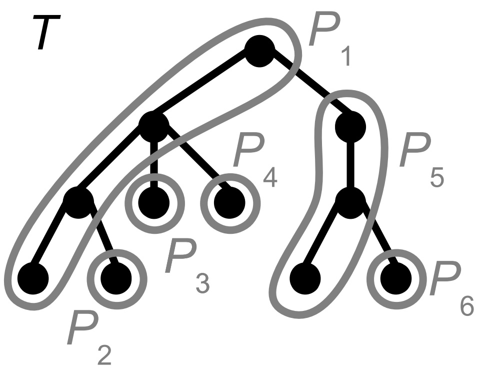


$\Longrightarrow$ rank(T)=2

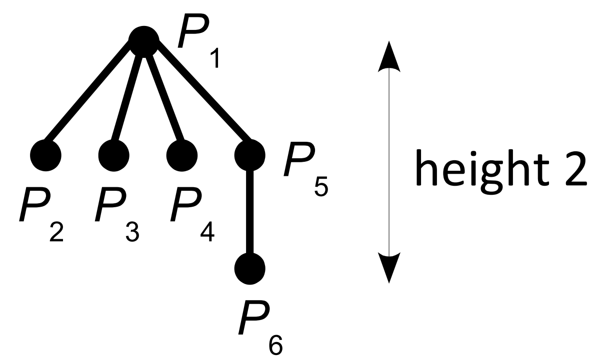
Rank and path partition of a tree

For any rooted tree T, **rank**(T) = maximum height of its **path partitions**

A path partition $\{P_1, P_2, P_3, P_4, P_5, P_6\}$ of T



A path partition tree of T



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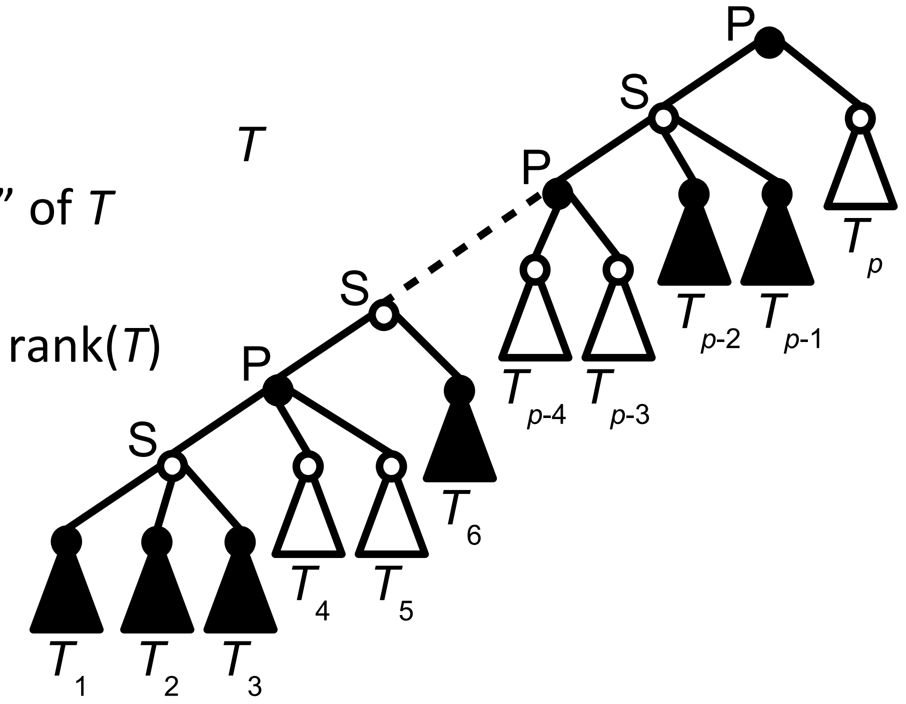
Upper bound on the contiguity / linearity

For a cograph G and its cotree T, cc(G) ≤ 2 rank(T) + 1 ≤ 2 (log n) + 1

Idea of the proof:

1 Consider a "root-path decomposition" of T

$\forall\ i \in [1..p]$, rank(T_i) +1 ≤ rank(T)



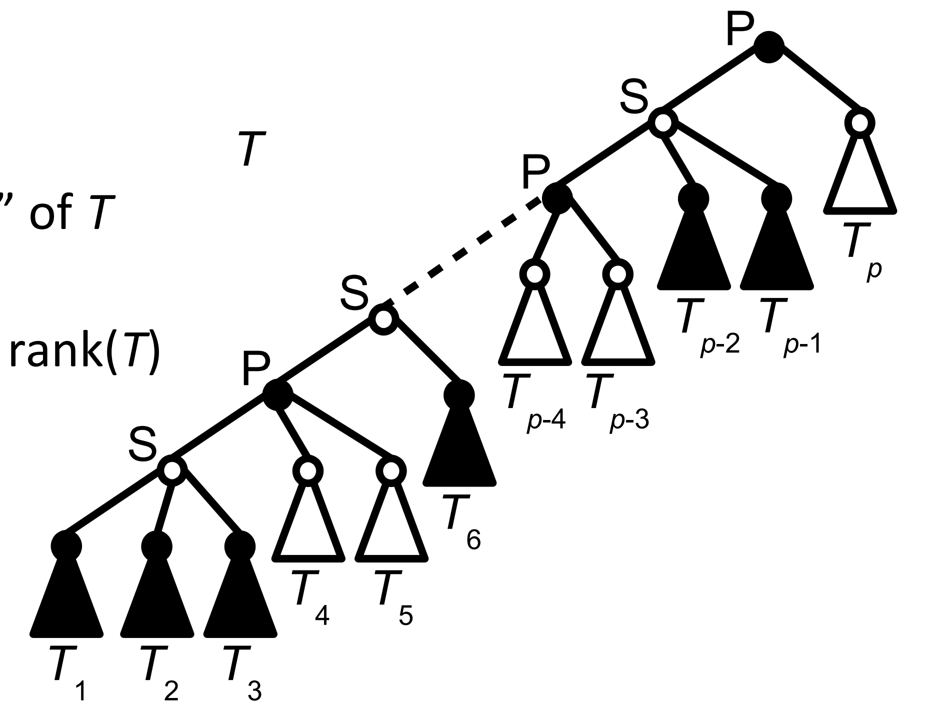
Upper bound on the contiguity / linearity

For a cograph G and its cotree T, $cc(G) \leq 2\ \mathrm{rank}(T) + 1 \leq 2\ (\log n) + 1$

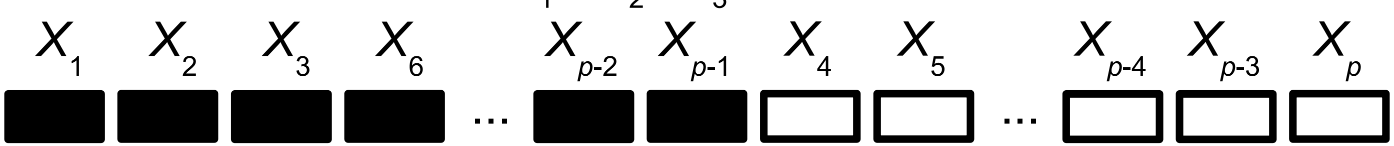
Idea of the proof:

1 Consider a "root-path decomposition" of T

$\forall\ i \in [1..p]$, $\mathrm{rank}(T_i) + 1 \leq \mathrm{rank}(T)$



2 Build the order

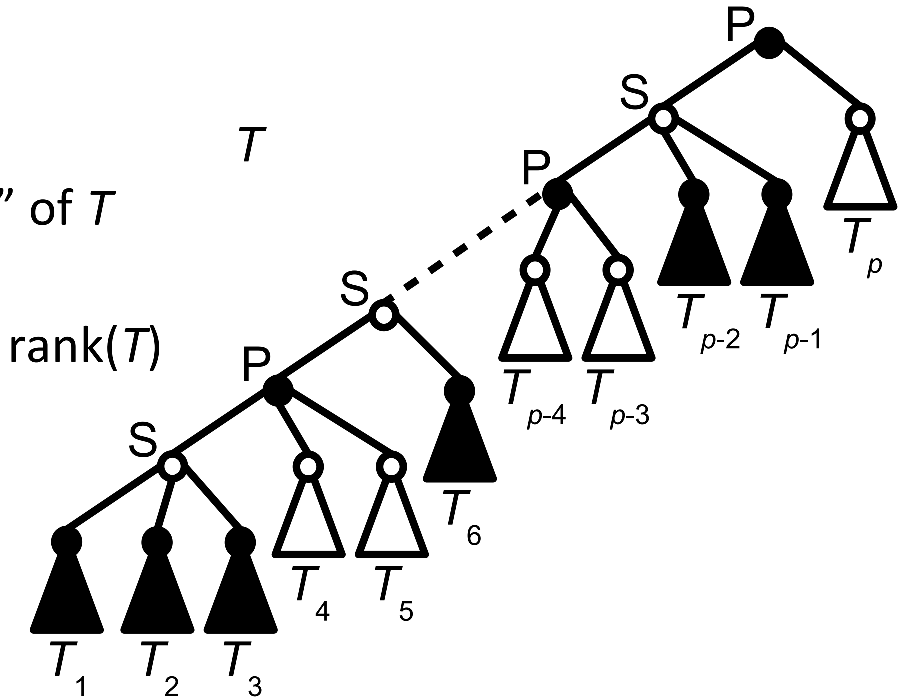


$\rightarrow cc(G) \leq 2 + \max_{i \in [1..p]} cc(G[X_i])$

Upper bound on the contiguity / linearity

For a cograph G and its cotree T, $\mathrm{cc}(G) \leq 2\ \mathrm{rank}(T) + 1 \leq 2\ (\log n) + 1$

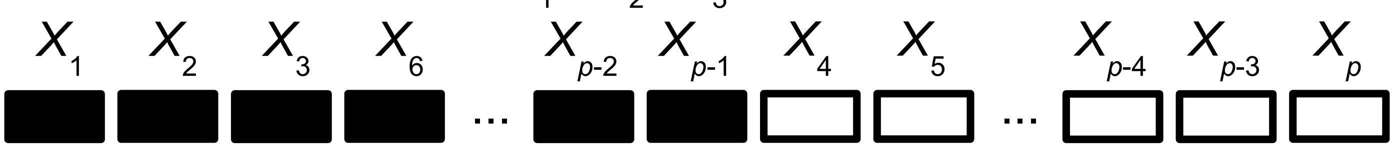
Idea of the proof:



① Consider a “root-path decomposition” of T

$\forall\ i \in [1..p]$, $\mathrm{rank}(T_i) + 1 \leq \mathrm{rank}(T)$

② Build the order



→ $\mathrm{cc}(G) \leq 2 + \max_{i \in [1..p]} \mathrm{cc}(G[X_i])$

③ Refine the order by recursively treating each T_i

Outline

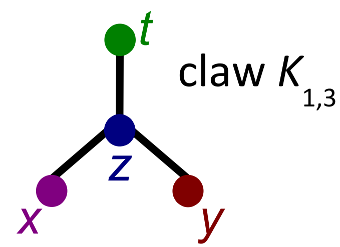
- Contiguity and linearity
- Cographs and cotrees
- A min-max theorem on the rank of a tree
- Upper bounds with caterpillar decompositions
- Lower bounds with claws
- Perspectives

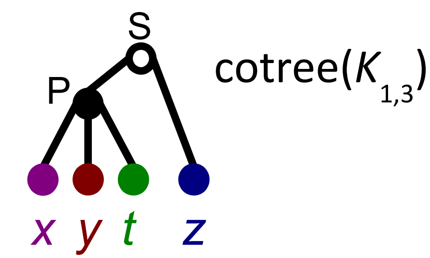
Lower bound on the contiguity

For a cograph G with complete binary cotree, $cc(G) \geq ((\log n)-5)/4$

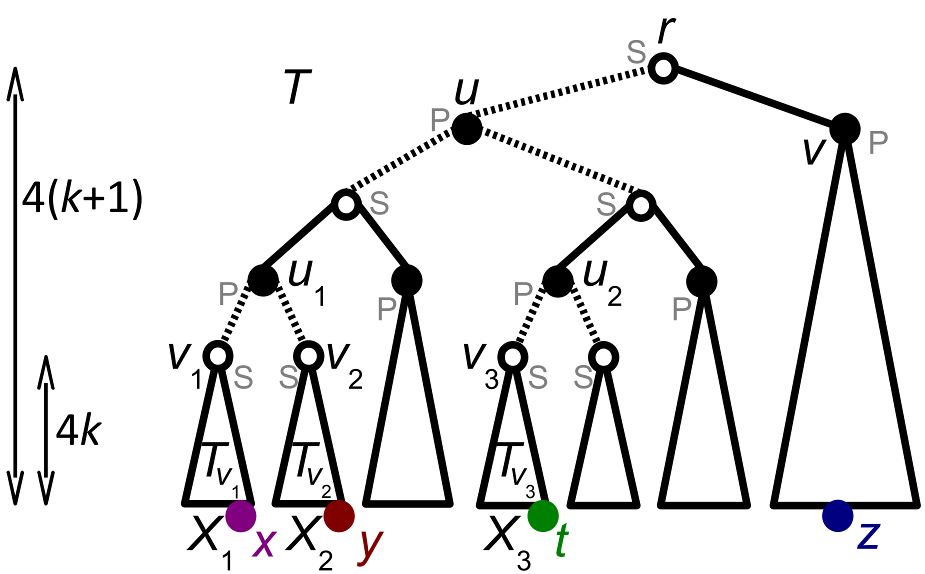
Idea of the proof:

Base case: Vertices x y & t all need to be adjacent to z in σ so that their neighborhood is one interval $\rightarrow cc(K_{1,3}) \geq 2$





Induction:



Lower bound on the linearity

For a cograph G with complete binary cotree, $\mathrm{cl}(G) \geq O((\log n)/(\log \log n))$

Idea of the proof:

Base case is star $K_{1,2k+1}$ (bigger than $K_{1,3}$)

→ need a bigger complete binary cotree, of height $\geq 2k\lceil\log(2k+1)\rceil+1$

Tightness of the bounds

For a cograph G with complete binary cotree, cc(G) = (log n)/2 +1

Oreste Manoussakis, 2012

Idea of the proof:

Careful analysis of the result of the root-path decomposition algorithm for the upper bound.

Analysis based on C_4-cycles for the lower bound.

Linearity open: O(log n) or O((log n)/(log log n)) ?

Tightness of the bounds

For any cograph G, there is a linear time constant-factor approximation algorithm to compute its contiguity.

Crespelle & Gambette, WALCOM 2013

Idea of the proof:

Approximate value given by the root-path decomposition algorithm.

Lower and upper bounds on the contiguity depending on the height of the biggest complete binary tree which is a minor of the cotree T of G, i.e. the **rank** of T:

$$cc(G) \leq 2 \text{ rank}(T) + 1$$

$$cc(G) \geq (\text{rank}(T) - 7) / 4$$

→ approximation ratio 23

Outline

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Perspectives

Open problems

• Linearity of cographs?

• Gap between linearity and contiguity?

• Linearity or contiguity of graphs classes generalizing cographs

Practical applications of linearity and contiguity

• practical approaches to get upper bounds?

• use in algorithmic contexts? Solving problems on graphs with bounded linearity or contiguity.

• use for some graph classes arising from applications:
→ express a complexity value for phylogenetic networks (*min. spread*)
Asano, Jansson, Sadakane, Uehara & Valiente, 2010

Thank you for your attention

Any questions?

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Christophe Crespelle & Philippe Gambette (2009), *Efficient Neighbourhood Encoding for Interval Graphs and Permutation Graphs and O(n) Breadth-First Search*, IWOCA'09, LNCS 5874, p. 146-157.

Christophe Crespelle & Philippe Gambette (2013), *Linear-time Constant-ratio Approximation Algorithm and Tight Bounds for the Contiguity of Cographs*, WALCOM'13, LNCS, to appear.