Orientations bipolaires et chemins tandem

Éric Fusy (CNRS/LIX)
Travaux avec Mireille Bousquet-Mélou et Kilian Raschel

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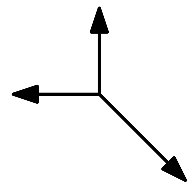
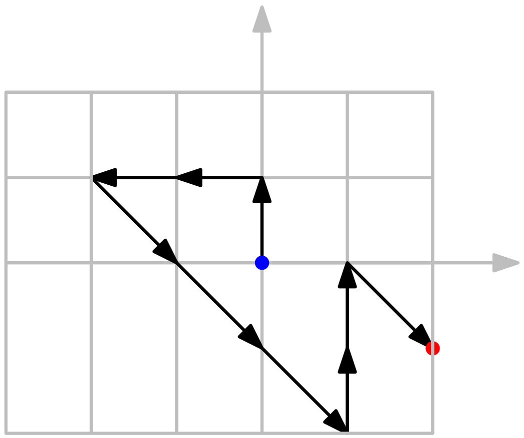
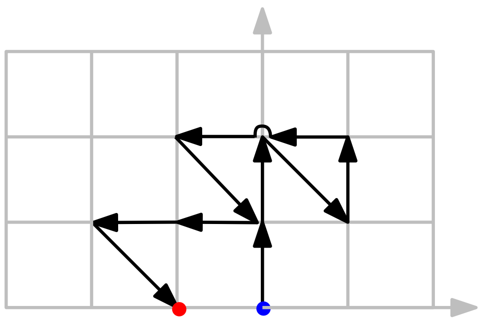
Tandem walks

A **tandem-walk** is a walk in $\mathbb{Z}^2$ with step-set $\{N, W, SE\}$

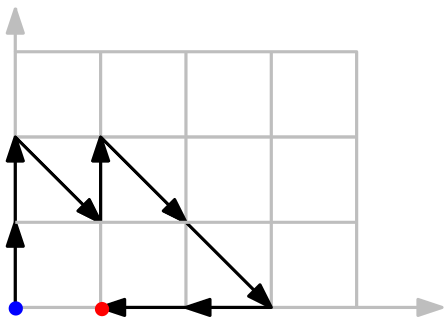




in the plane $\mathbb{Z}^2$

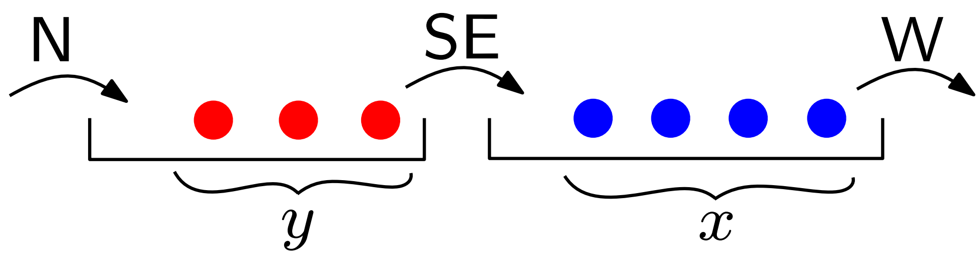


in the half-plane $\{y \geq 0\}$



in the quarter plane $\mathbb{N}^2$

cf 2 queues in series



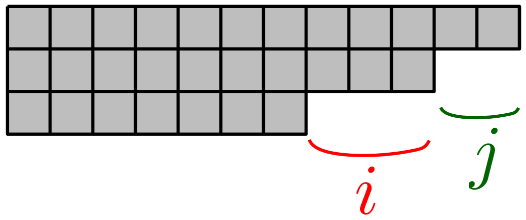
Link to Young tableaux of height ≤ 3

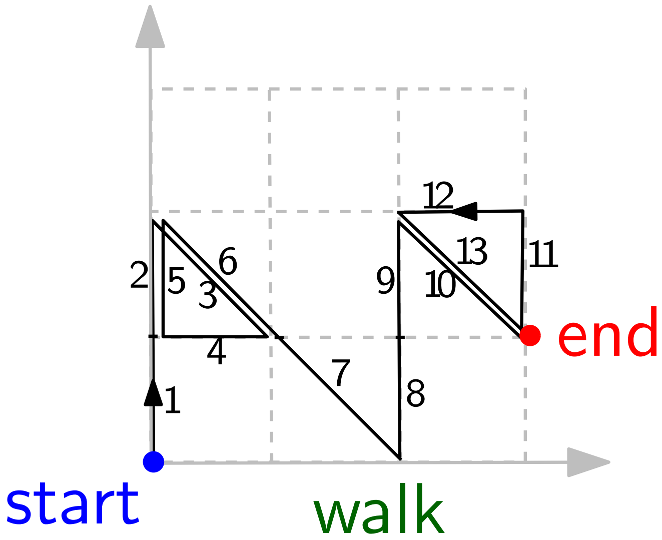
- There is a bijection between:

tandem walks of length n **staying in the quadrant** $\mathbb{N}^2$, ending at (i, j)

$\Updownarrow$

Young tableaux of size n and height ≤ 3, of shape





$\longleftrightarrow$

N	1	2	5	8	9	11
SE	3	6	7	10	13	
W	4	12				

tableau

(after k steps, current $y = \#N - \#SE$, current $x = \#SE - \#W$)

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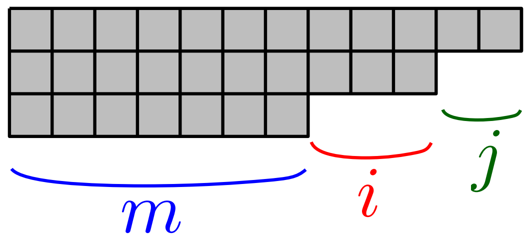

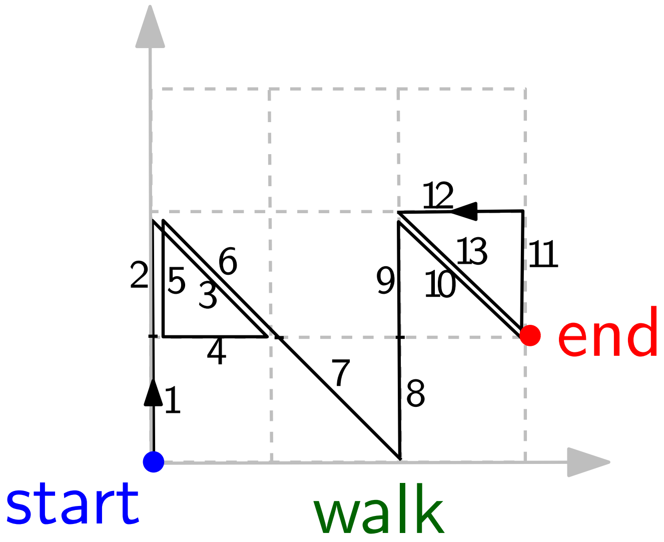

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- Let $q[n;i,j] := \#$ tandem walks of length n in $\mathbb{N}^2$, ending at (i,j)

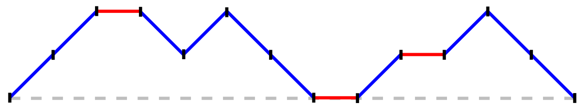
Hook-length formula: for n of the form $n = 3m + 2i + j$ we have

$$q[n;i,j] = \frac{(i+1)(j+1)(i+j+2)n}{m!(m+i+1)!(m+i+j+2)!}$$

Algebraicity when the endpoint is free

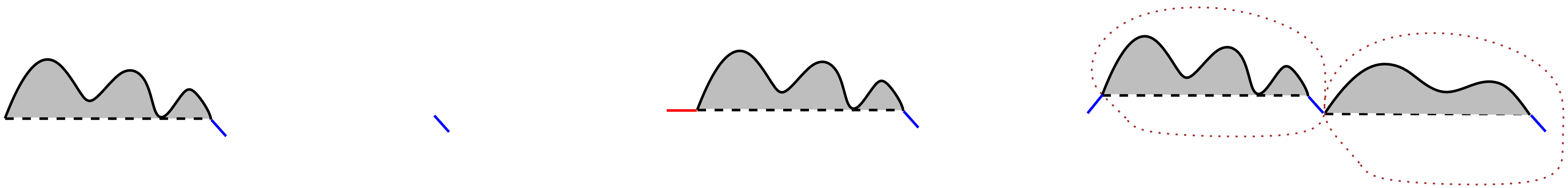
Let $Q(t;x,y) = \sum_{n,i,j} q[n;i,j] t^n x^i y^j$

Theorem: [Gouyou-Beauchamps'89], [Bousquet-Mélou,Mishna'10]
Then $Q(t,1,1)$ is the series counting Motzkin walks,



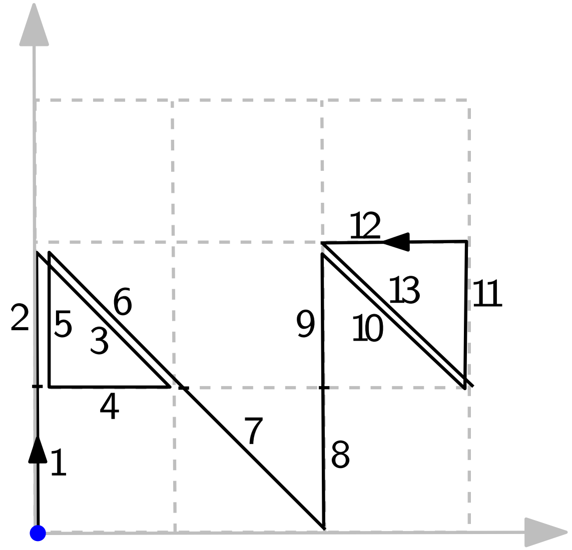
i.e., $Y \equiv t\,Q(t,1,1)$ satisfies $\boxed{Y = t \cdot (1 + Y + Y^2)}$

$$Y \quad = \quad t \quad + \quad t \cdot Y \quad + \quad t \cdot Y^2$$



Bijection with Motzkin walks

[Gouyou-Beauchamps'89]



tandem walk in $\mathbb{N}^2$

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Young tableau of height ≤ 3

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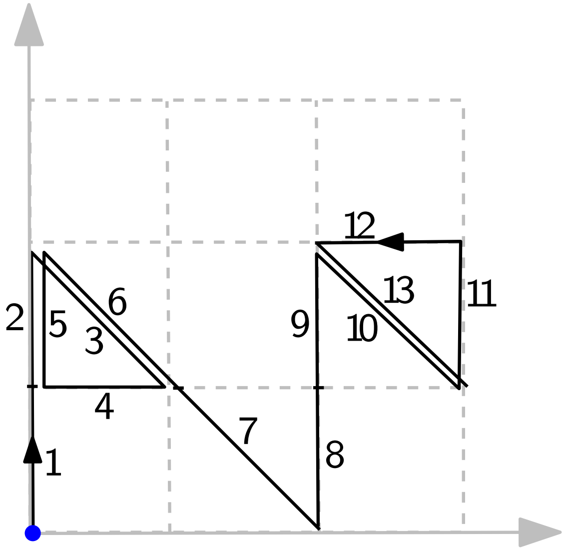

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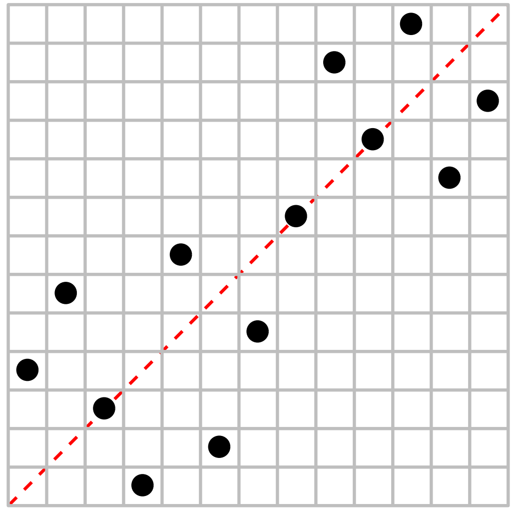
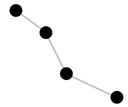
Robinson Schensted



involution with no 

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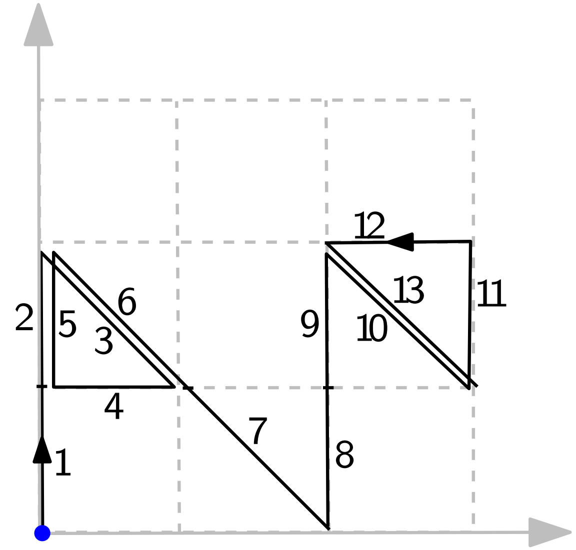

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↔

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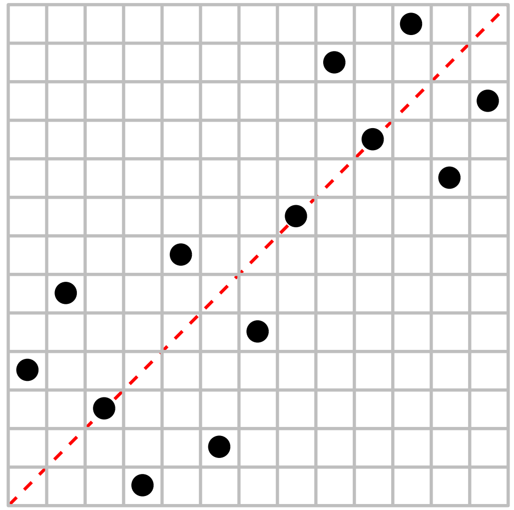
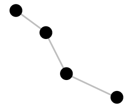
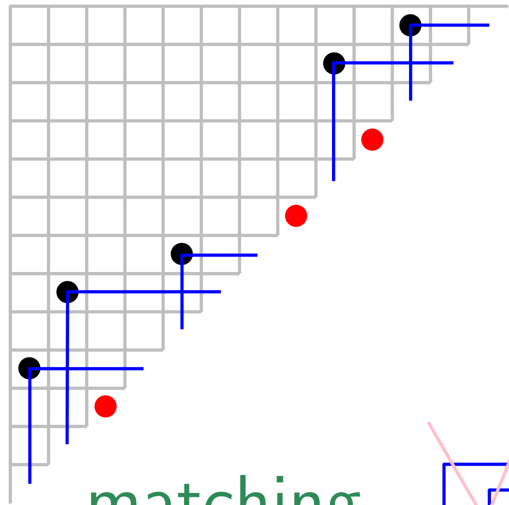
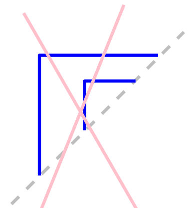


involution with no 

↕



matching with no nesting 

Bijection with Motzkin walks

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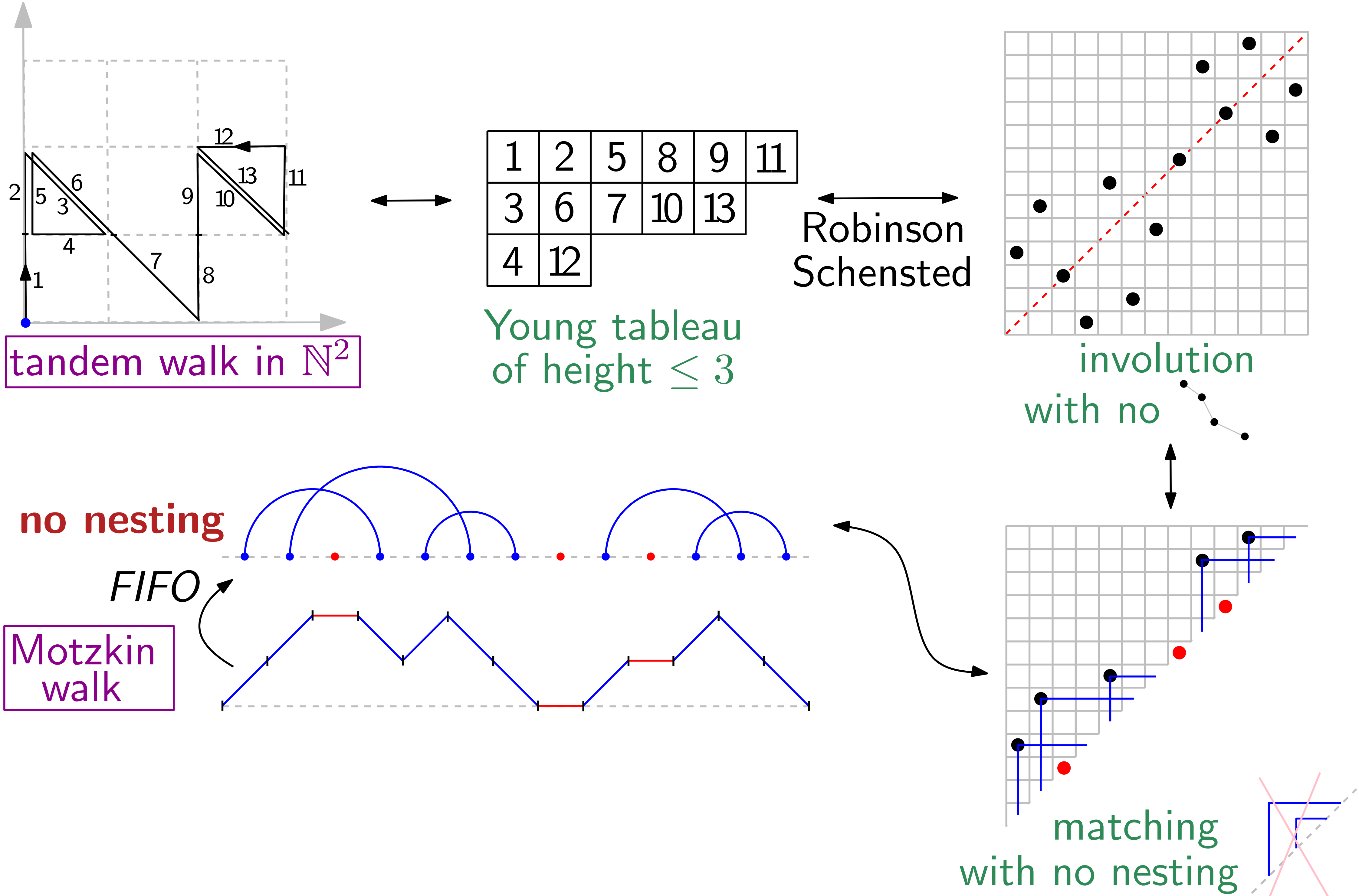

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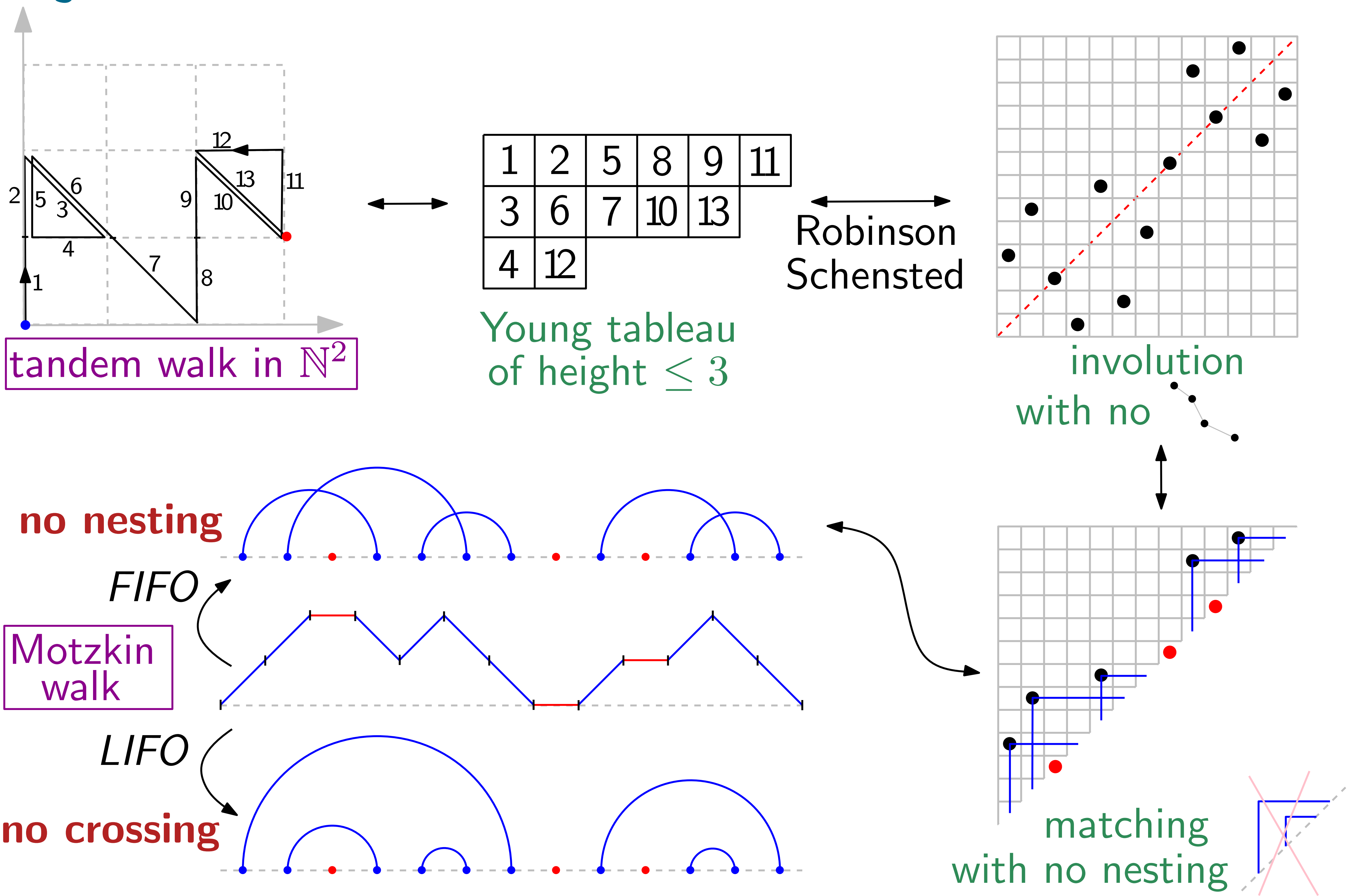
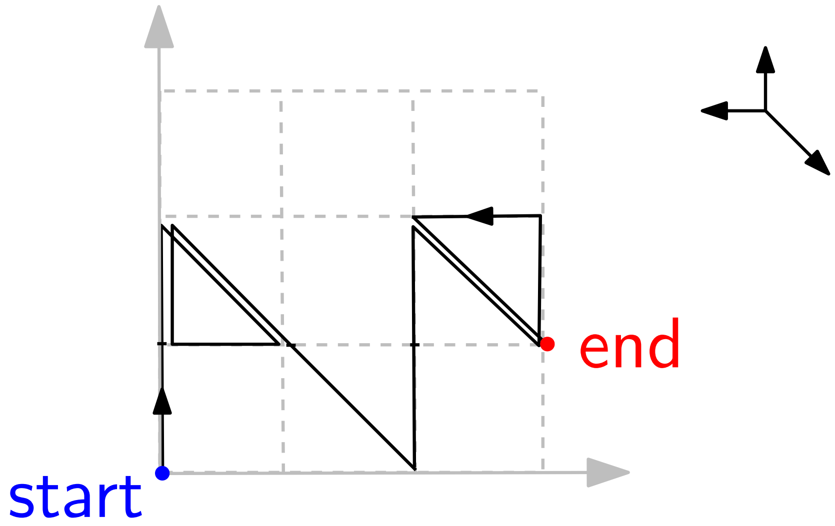


Reformulation with half-plane tandem-walks

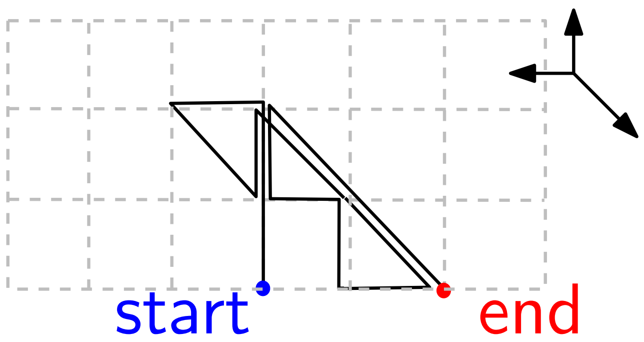
There is a bijection between:

- tandem walks of length n
  staying in the quarter plane $\mathbb{N}^2$

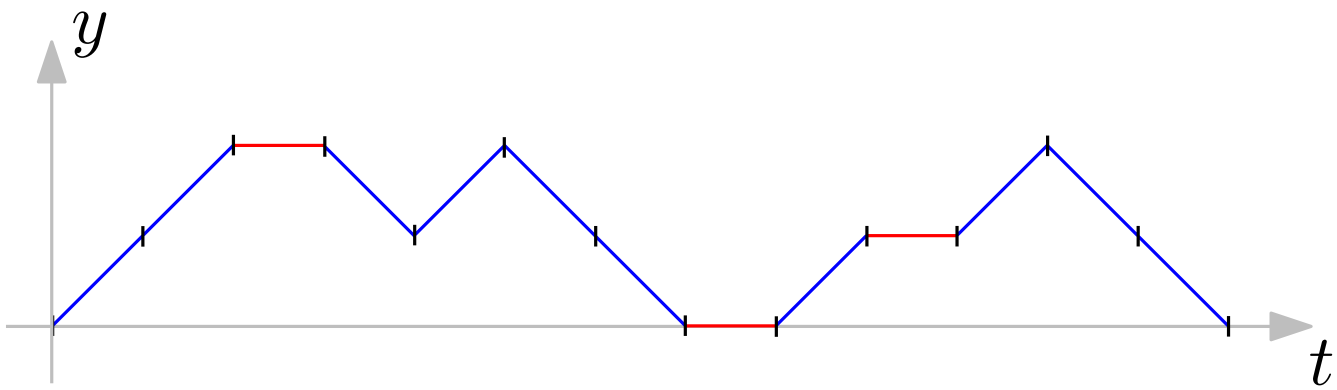


$\Updownarrow$

- tandem walks of length n
  staying in the half-plane $\{y \geq 0\}$
  and ending at $y = 0$



$\Leftrightarrow$



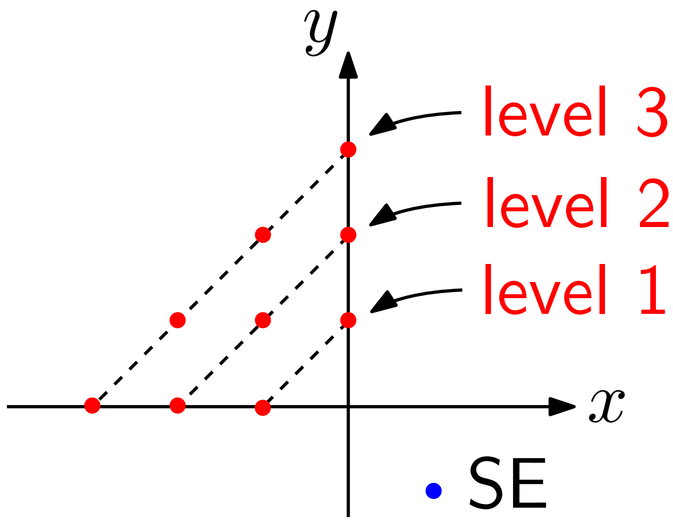
Rk: The bijection **preserves the number of SE steps**

An extension of the walk model

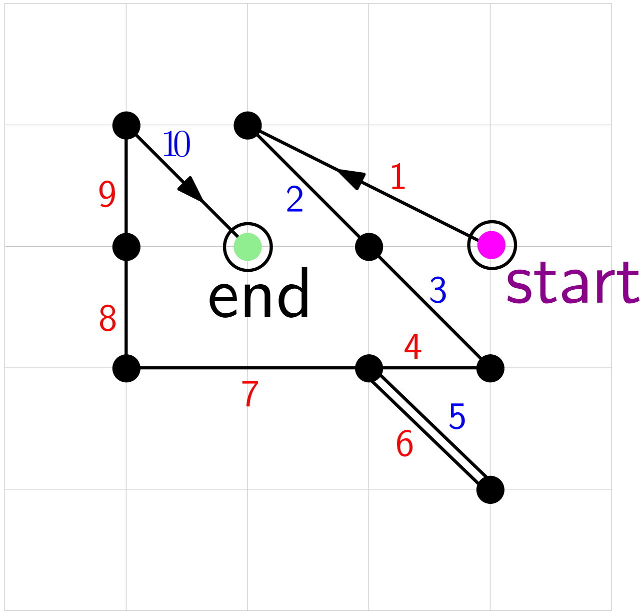
General model:

step-set: • the SE step
• every step $(-i, j)$ (with $i, j \geq 0$)

level:$= i + j$



Example:



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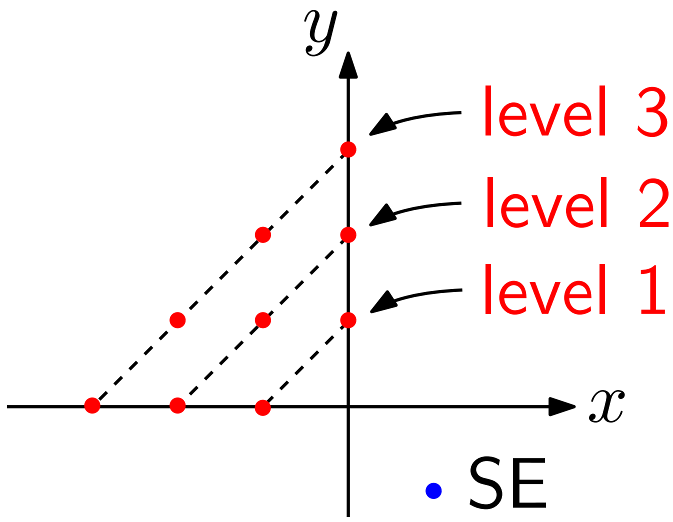

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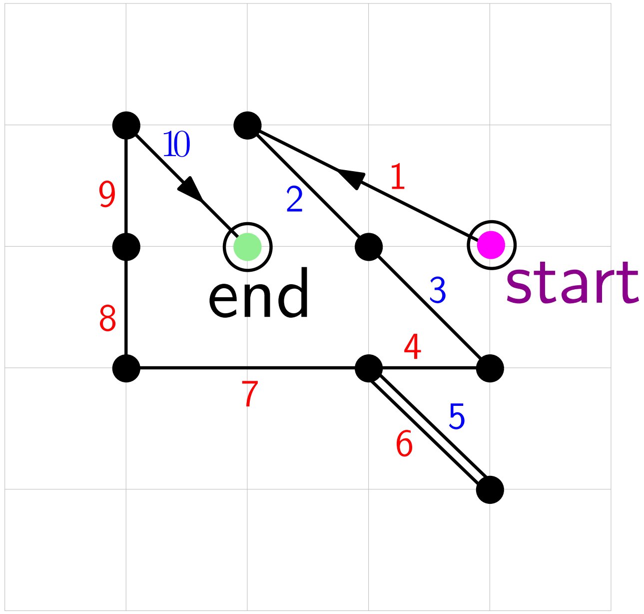


There is **still a bijection** between:

- general tandem walks of length n in the quarter plane $\mathbb{N}^2$
- general tandem walks of length n in $\{y \geq 0\}$ ending at $y = 0$

The bijection **preserves** the number of SE-steps
and the number of steps in each level $p \geq 1$

Bipolar and marked bipolar orientations

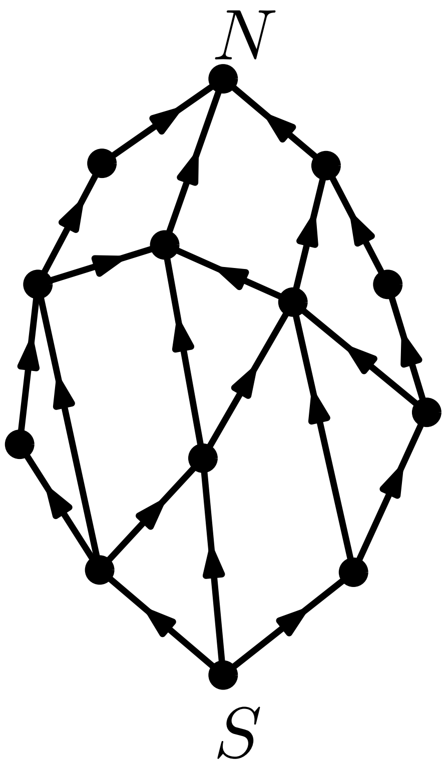
bipolar orientation:

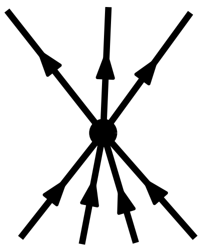
(on planar maps)

= acyclic orientation

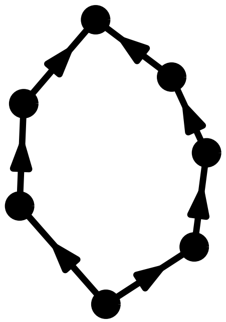
with a unique source S
and a unique sink N

with S, N incident to the outer face





inner vertex



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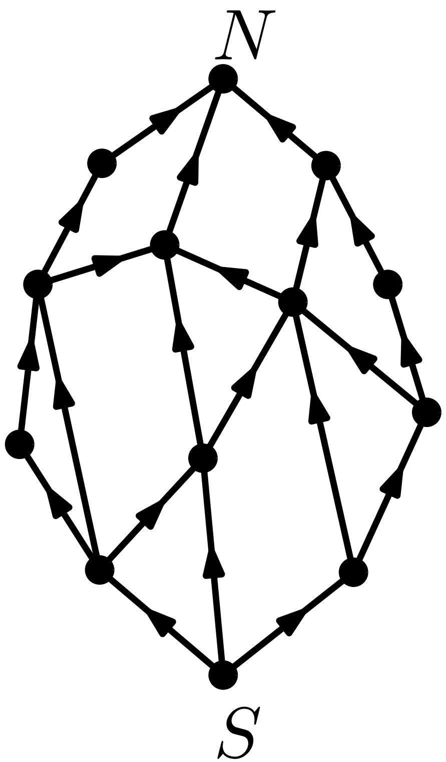
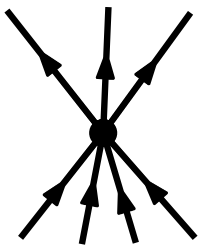
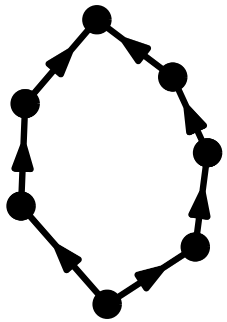




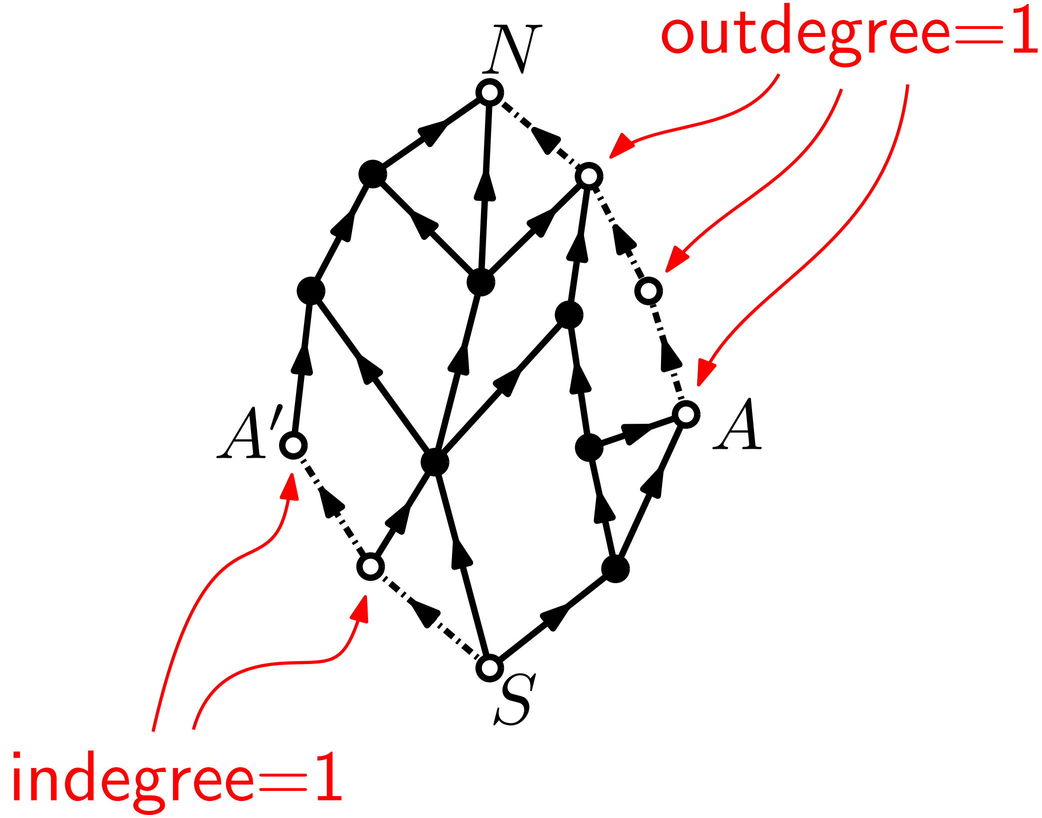
inner vertex



inner face

marked bipolar orientation:

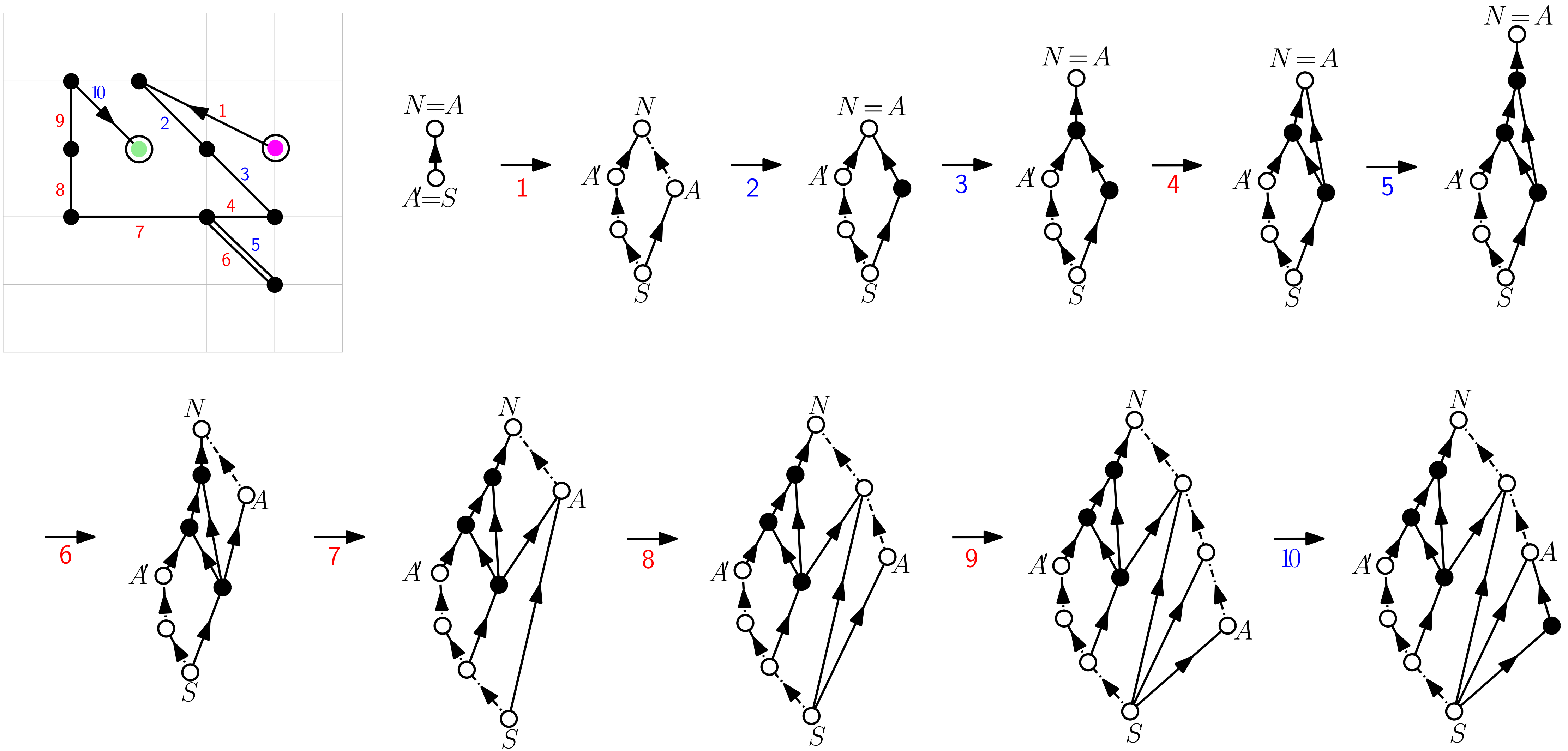
a marked vertex $A' \neq N$ on left boundary
a marked vertex $A \neq S$ on right boundary



The Kenyon et al. bijection

[Kenyon, Miller, Sheffield, Wilson'16]

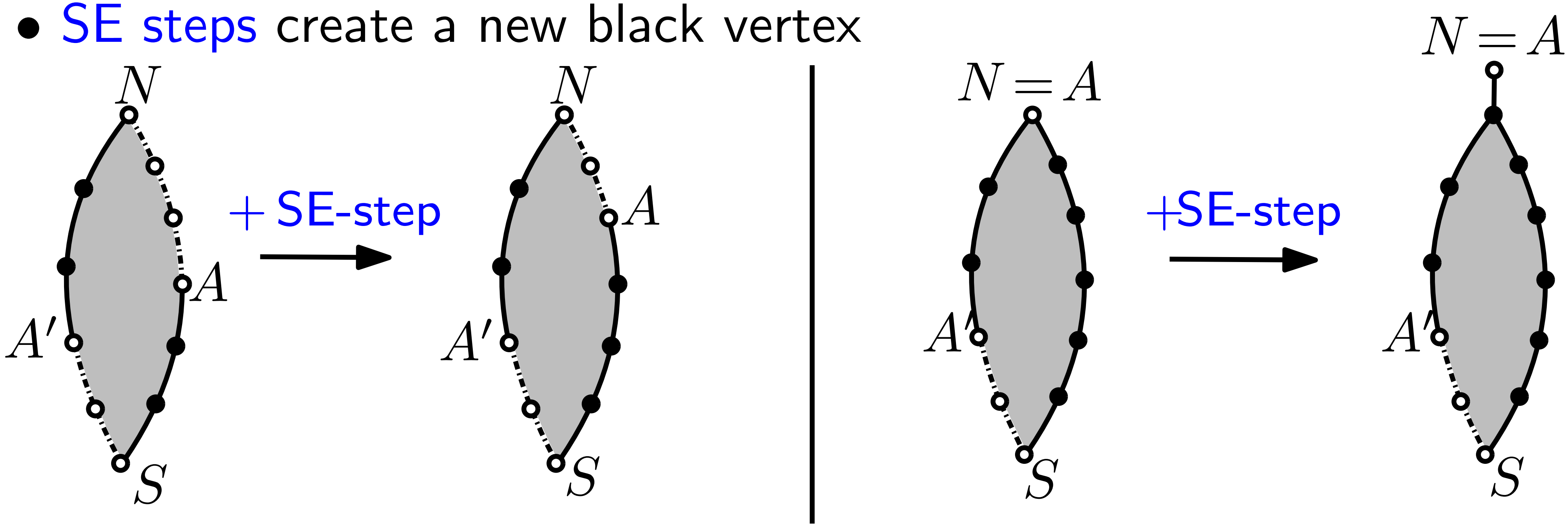
general tandem-walk (in $\mathbb{Z}^2$)	$\xleftrightarrow{\textit{bijection}}$	marked bipolar orientation
SE step	$\longleftrightarrow$	black vertex
step $(-i, j)$	$\longleftrightarrow$	inner face of degree $i+j+2$



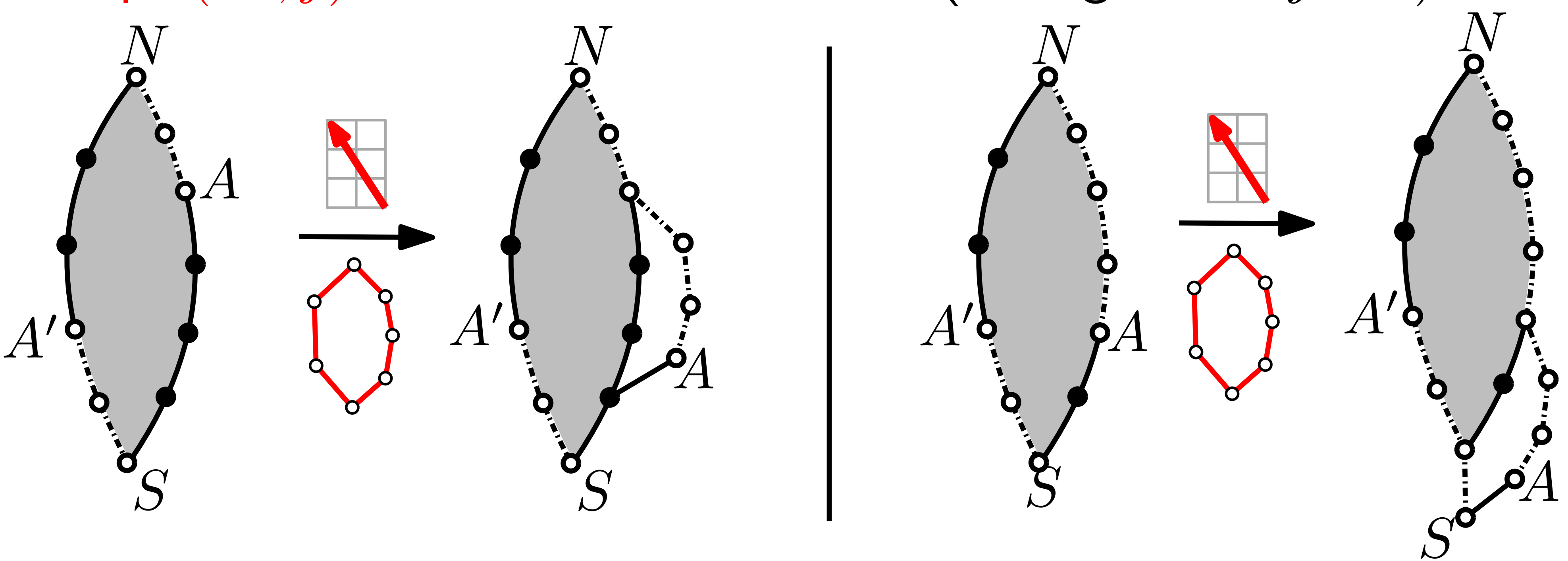
The Kenyon et al. bijection

[Kenyon, Miller, Sheffield, Wilson'16]

- SE steps create a new black vertex

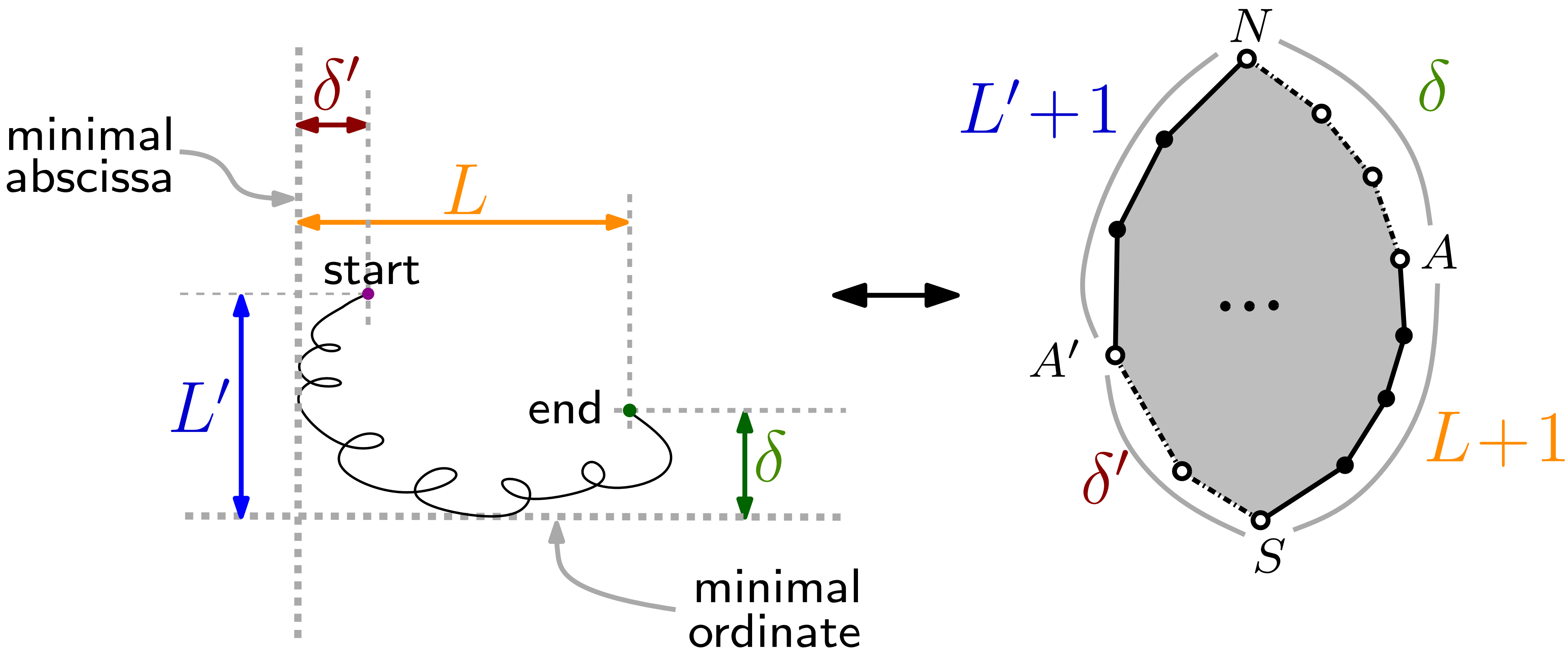


- steps $(-i, j)$ create a new inner face (of degree $i + j + 2$)

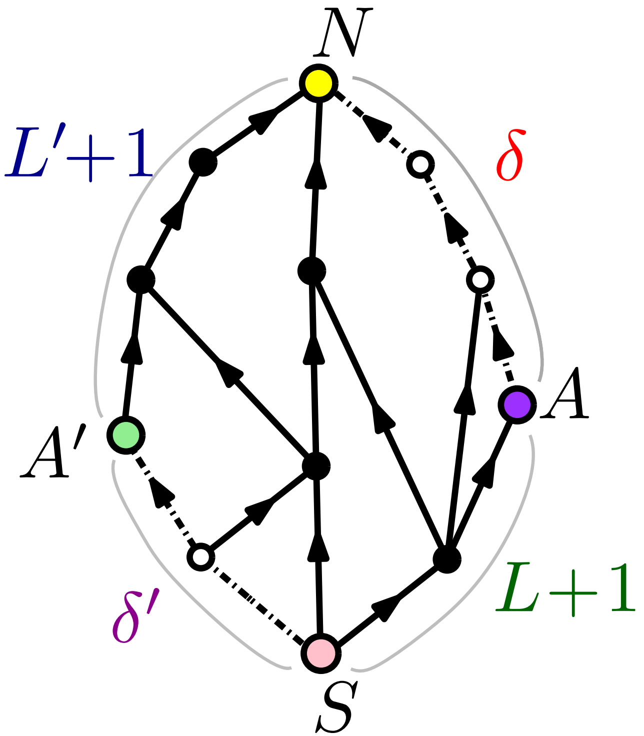


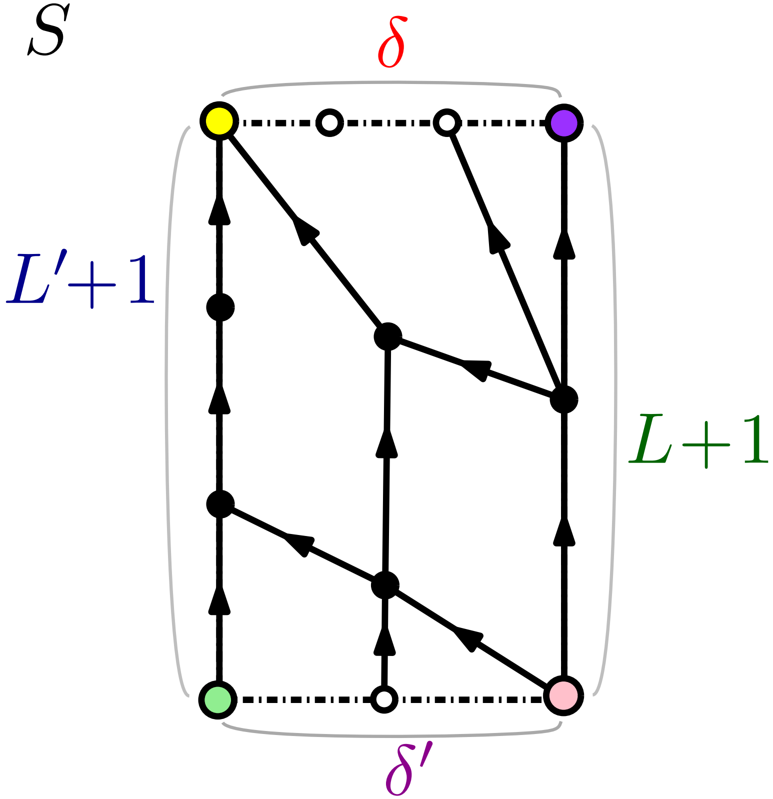
Parameter-correspondence in the bijection

# "face-steps" of level p	$\longleftrightarrow$	# inner faces of degree $p+2$
# SE-steps	$\longleftrightarrow$	# black vertices
1 + # steps	$\longleftrightarrow$	# plain edges (not dashed)



An involution on marked bipolar orientations





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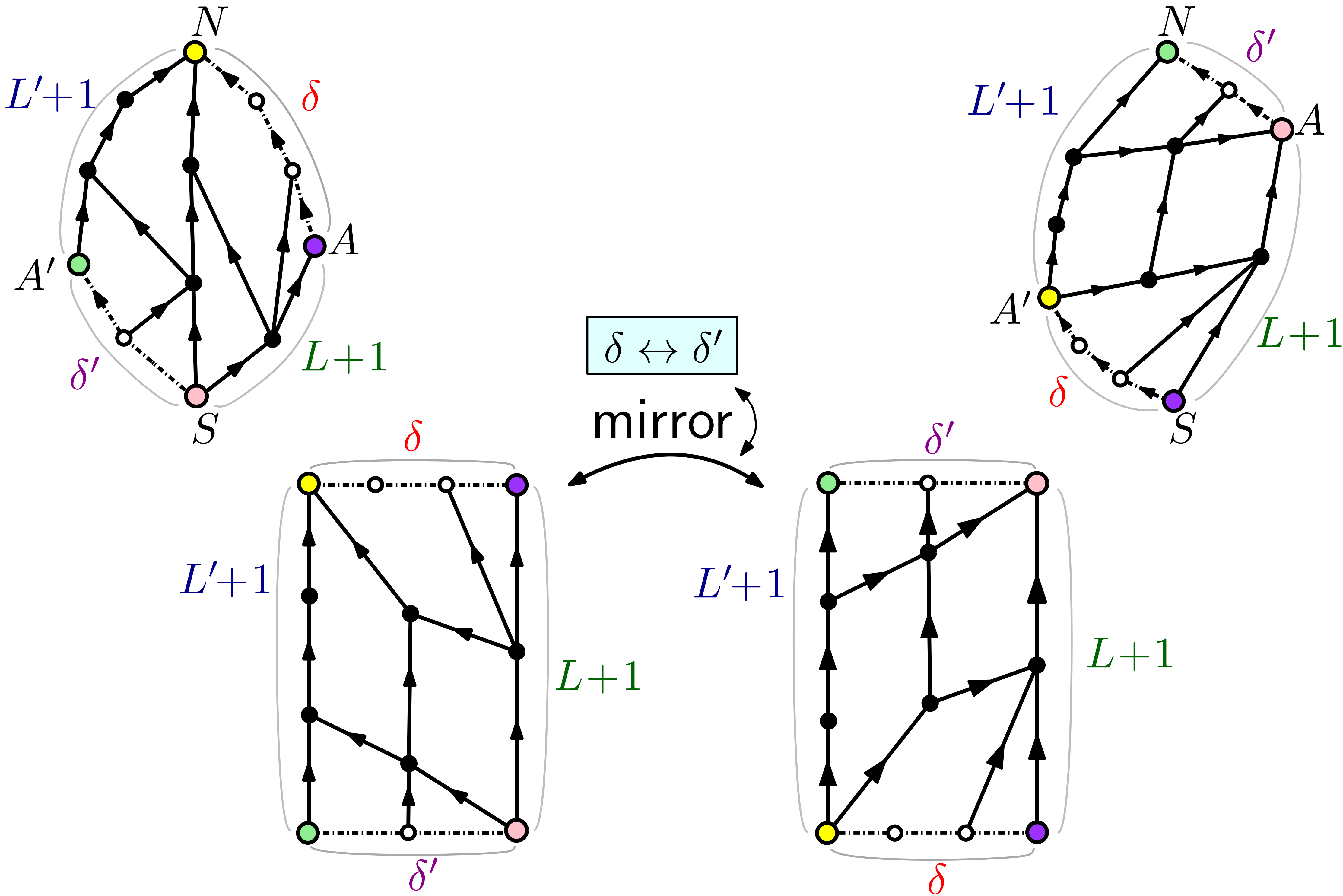
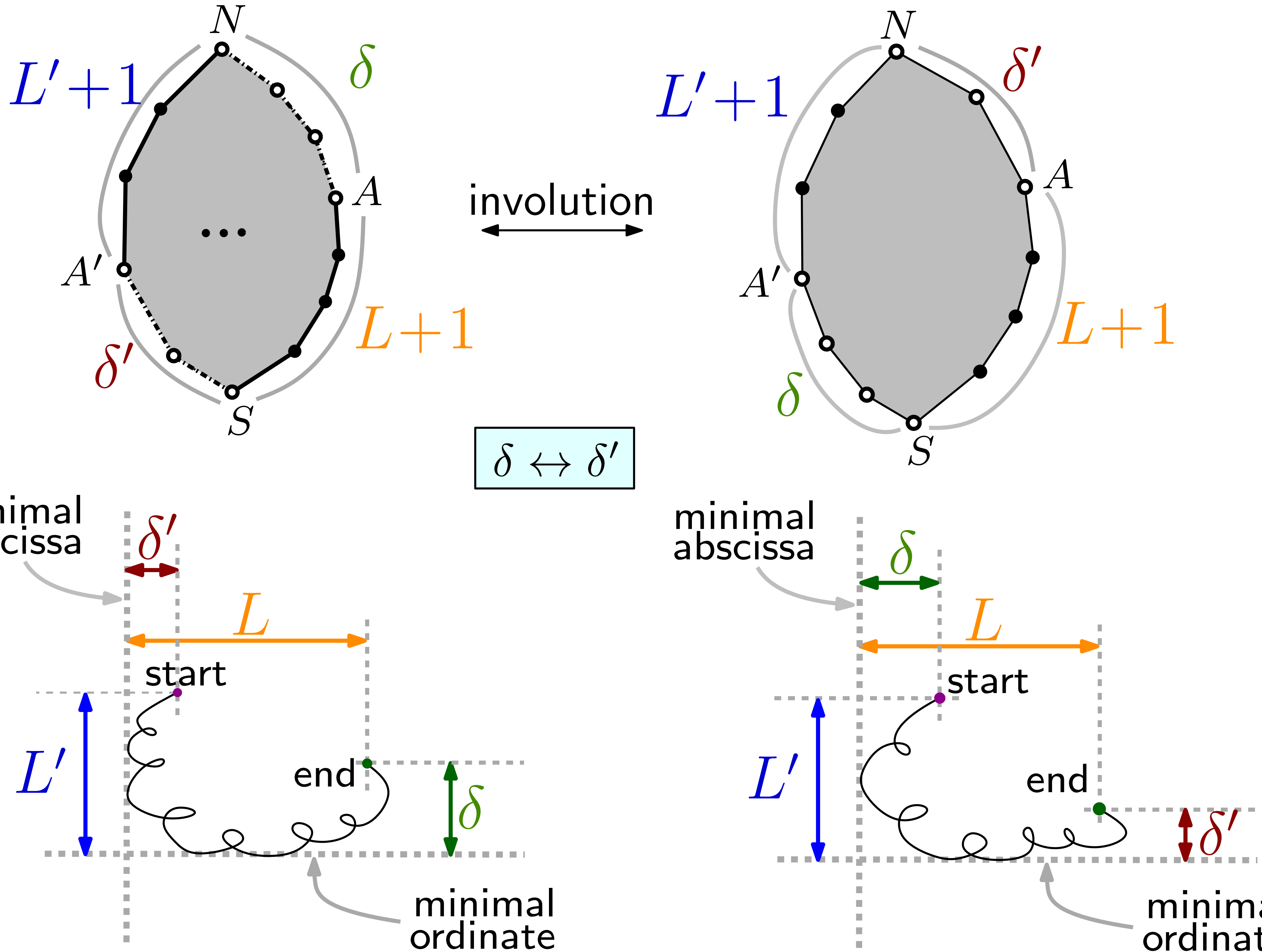
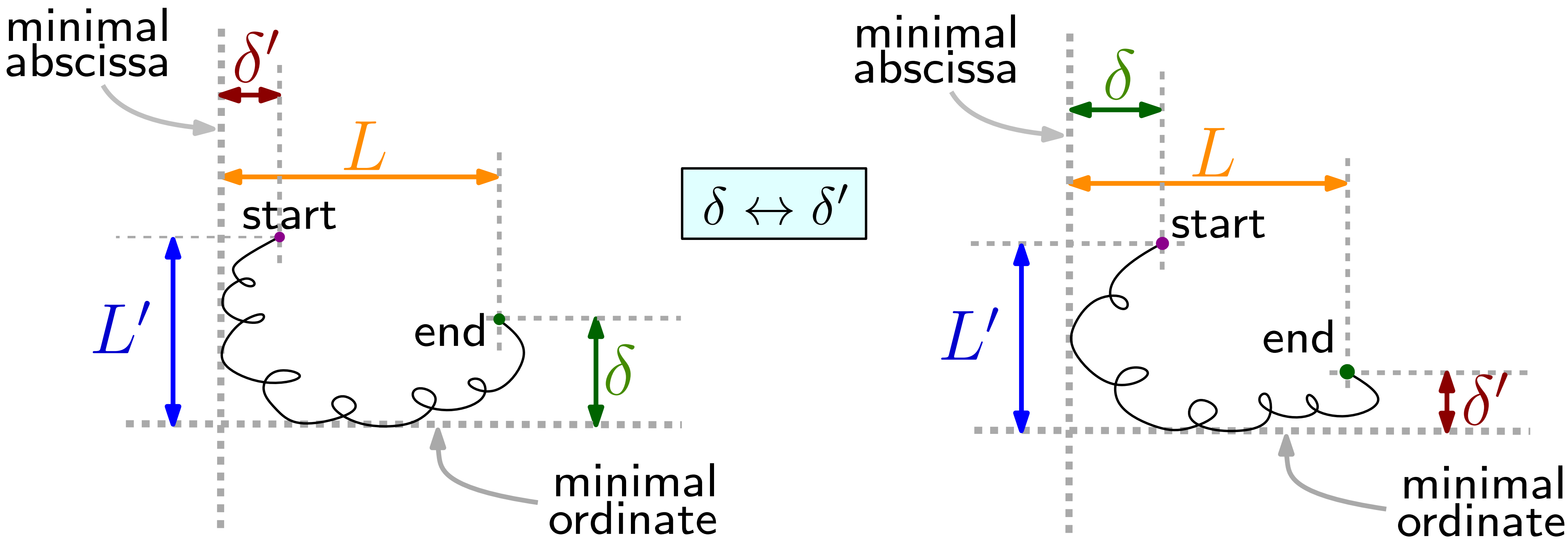


Effect of the involution on walks



Quarter plane walks $\leftrightarrow$ half-plane walks ending at $y = 0$



- Specialize the involution at $\{L' = 0, \delta' = 0\}$

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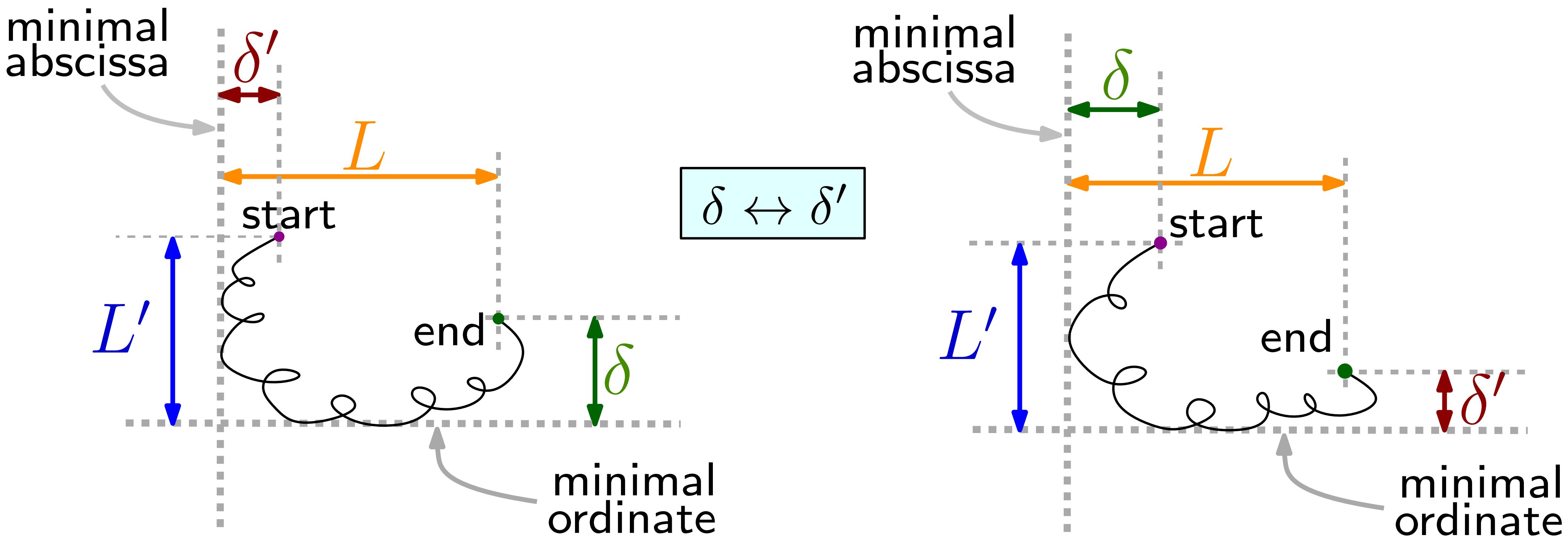
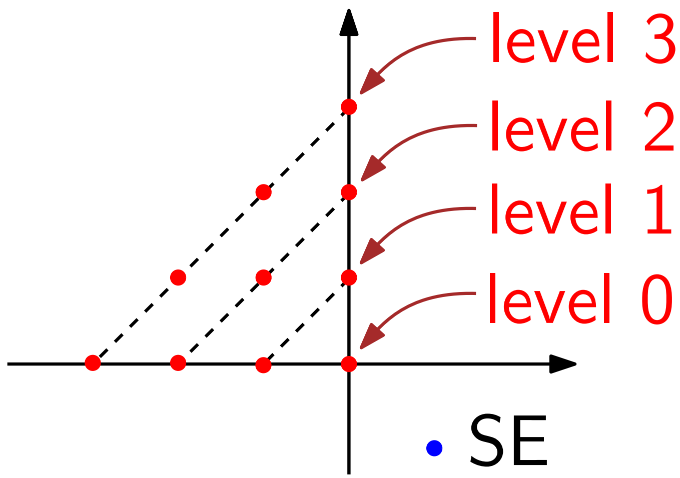
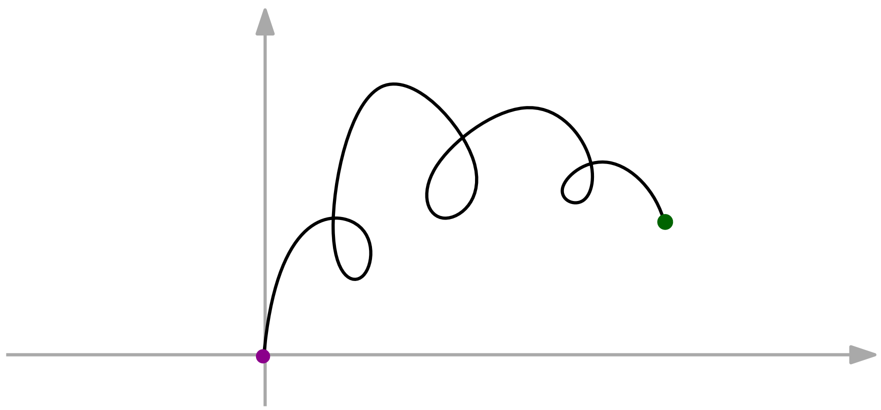


- Specialize the involution at $\{L' = 0, \delta' = 0\}$

- Specialize at $\{\delta' \leq a, L' \leq b\} \Rightarrow$ quarter plane walks starting at (a, b)

Generating function expressions





Let $Q(t)$ be the generating function of general tandem-walks in $\mathbb{N}^2$

- counted w.r.t. the length (variable t)
- with a weight z_i for each "face-step" of level i

Then $Y \equiv t\,Q(t)$ is given by $\quad Y = t \cdot (1 + w_0 Y + w_1 Y^2 + w_2 Y^3 + \cdots)$

where $w_i = z_i + z_{i+1} + z_{i+2} + \cdots$

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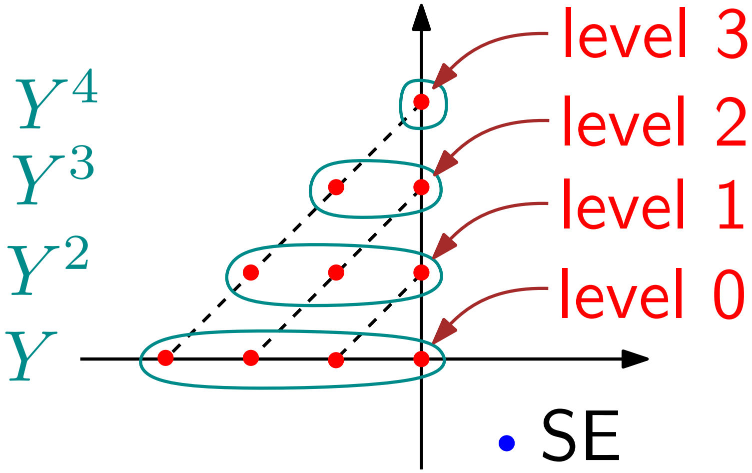

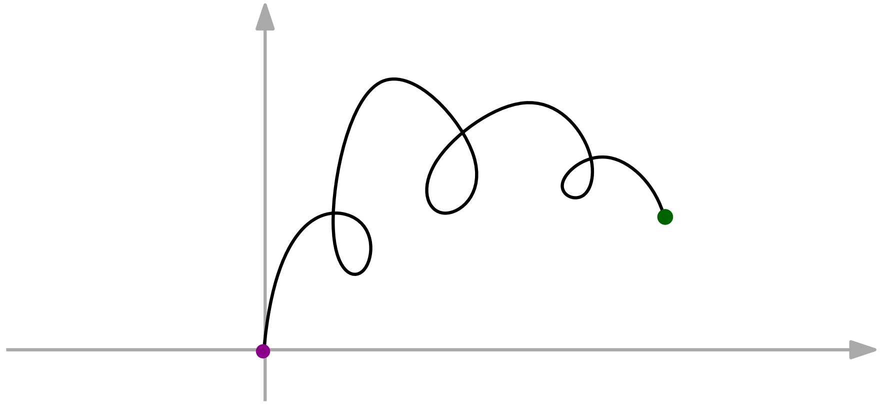

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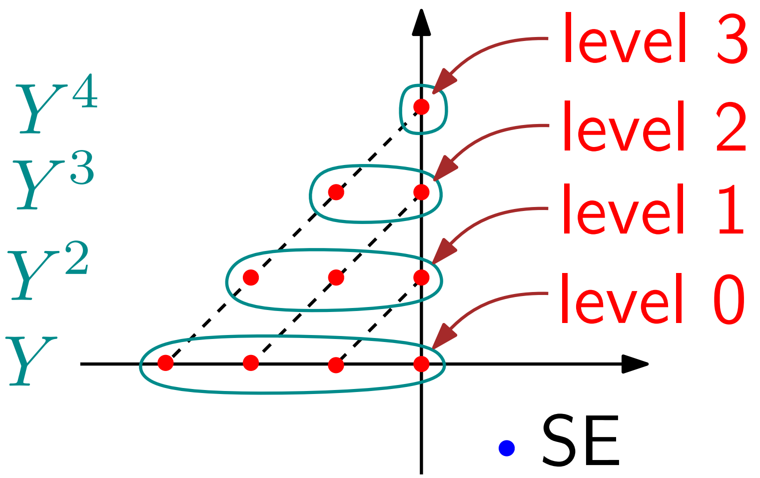

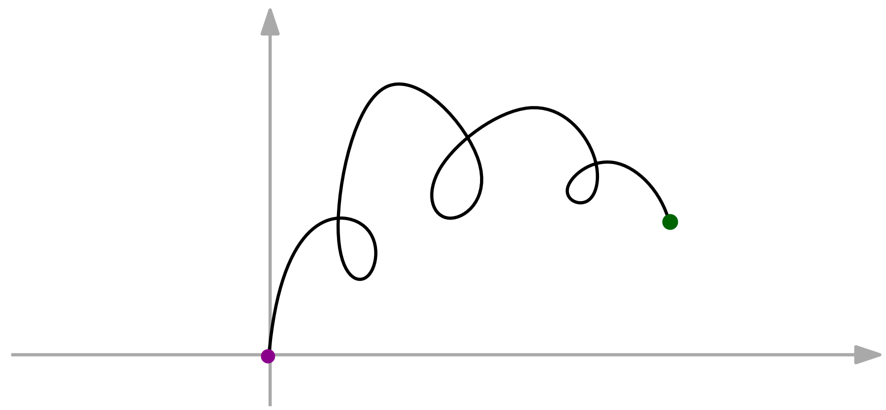

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Rk: alternative proof (earlier!) with obstinate kernel method

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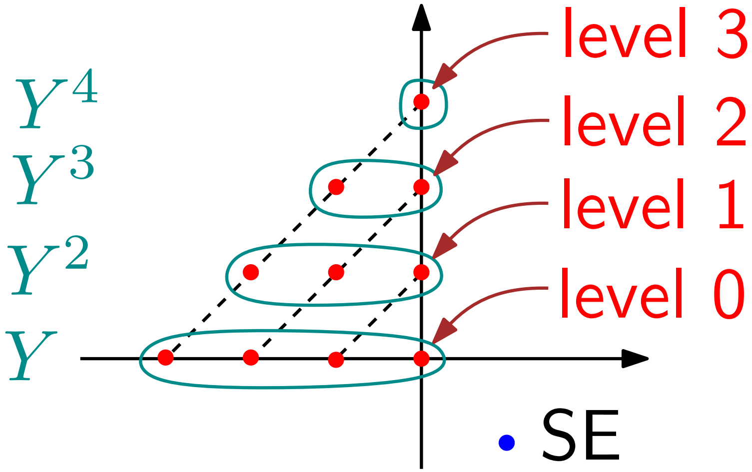

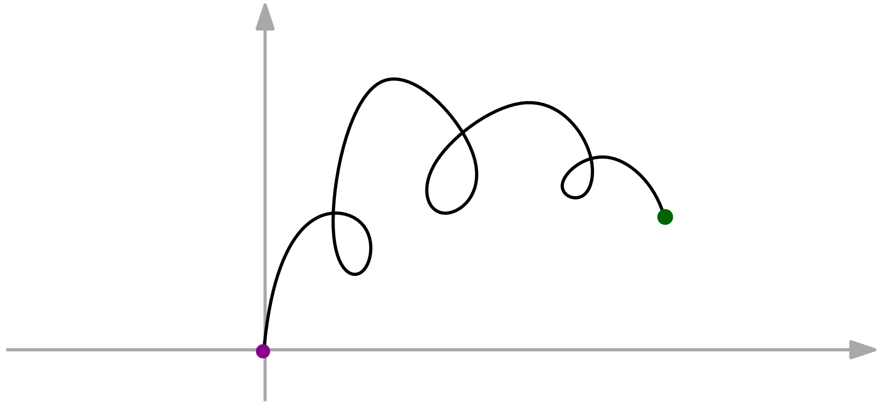

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where $w_i = z_i + z_{i+1} + z_{i+2} + \cdots$

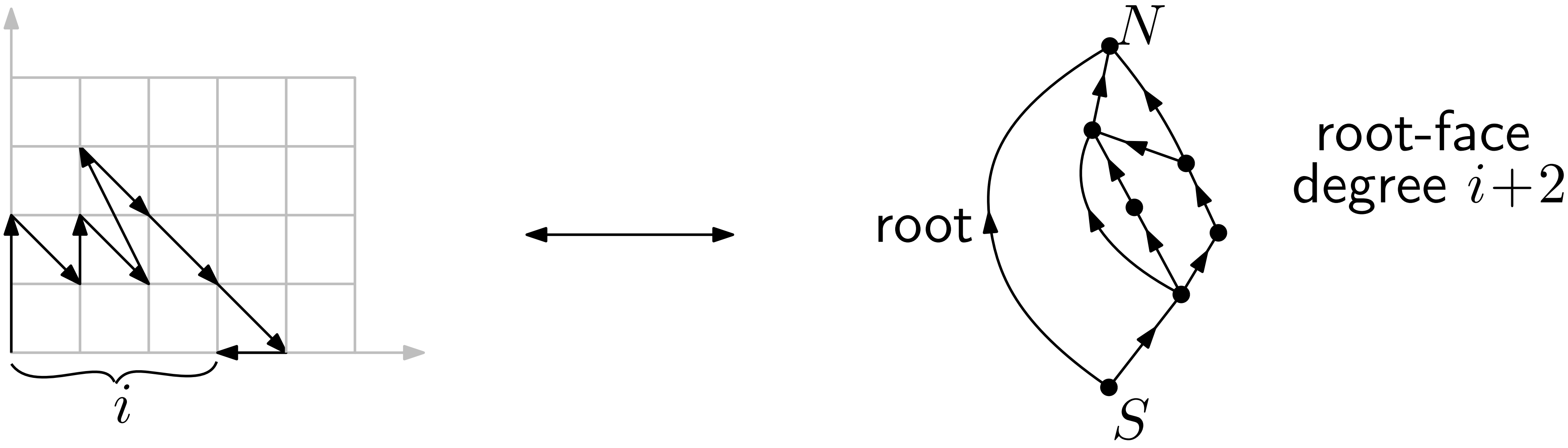
Rk: alternative proof (earlier!) with obstinate kernel method

Rk: Let $Q^{(a,b)}(t) :=$ GF of general tandem walks in $\mathbb{N}^2$ starting at (a,b)

Then $t\,Q^{(a,b)}(t) =$ explicit polynomial in Y (with positive coefficients)

Quarter plane walks ending at $(i, 0)$

The series $F_i(t) := \sum_n q[n; i, 0]t^n$ counts bipolar orientation of the form



with t for # edges, and weight z_r for each inner face of degree $0 \leq r \leq p$

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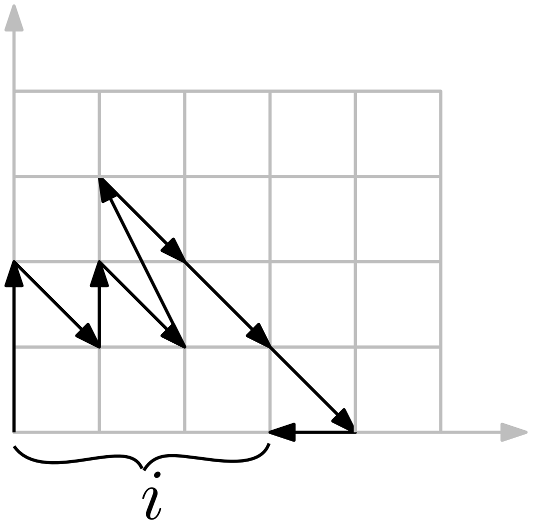

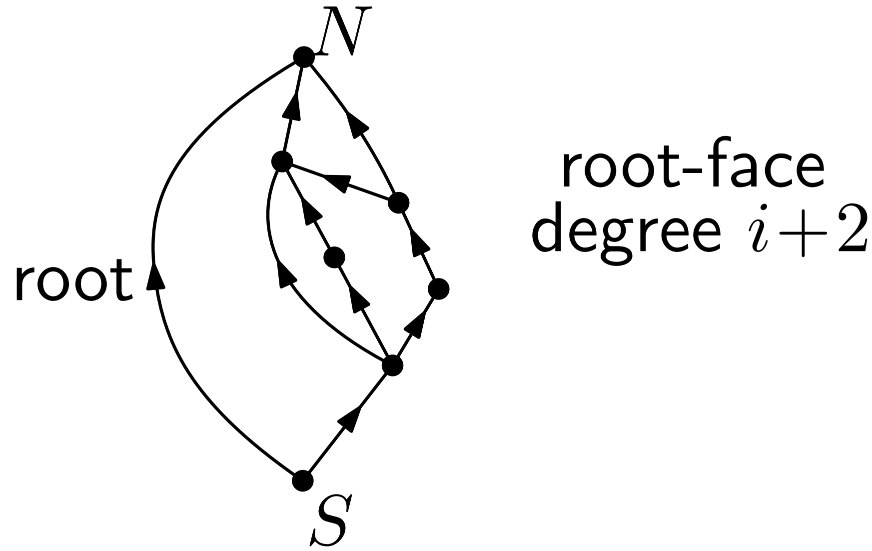

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Asymptotic enumeration

$$q[n;i,0] \sim_{n\to\infty} C_i \cdot \gamma^n \cdot n^{-4}$$

where $C_i = c \cdot \alpha^i (i+1)(i+2)$

from **[Denisov-Wachtel'11]**

Rk: For **undirected rooted maps**

$$M[n;i] \sim_{n\to\infty} C_i \cdot \gamma^n \cdot n^{-5/2}$$

where $C_i = c \cdot \alpha^i 4^{-i} i \binom{2i}{i}$

(with applications to **peeling**)

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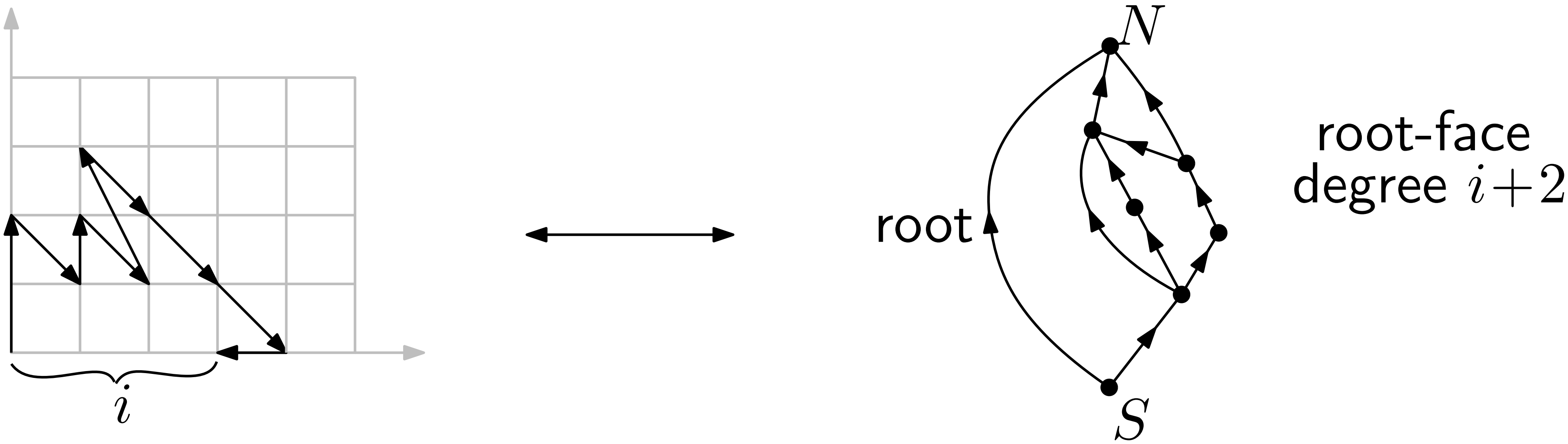

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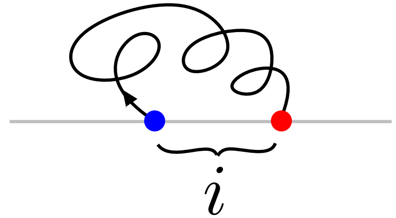
Exact enumeration

Let $H_i(t) =$ GF of



Then $F_i(t) = H_i(t) - \frac{1}{t} H_{i+2}(t)$

$+ \sum_{r=0}^{p} (r+1) z_i H_{i+r+2}(t)$

Proof using obstinate kernel method

or from Kenyon et al. + local operations