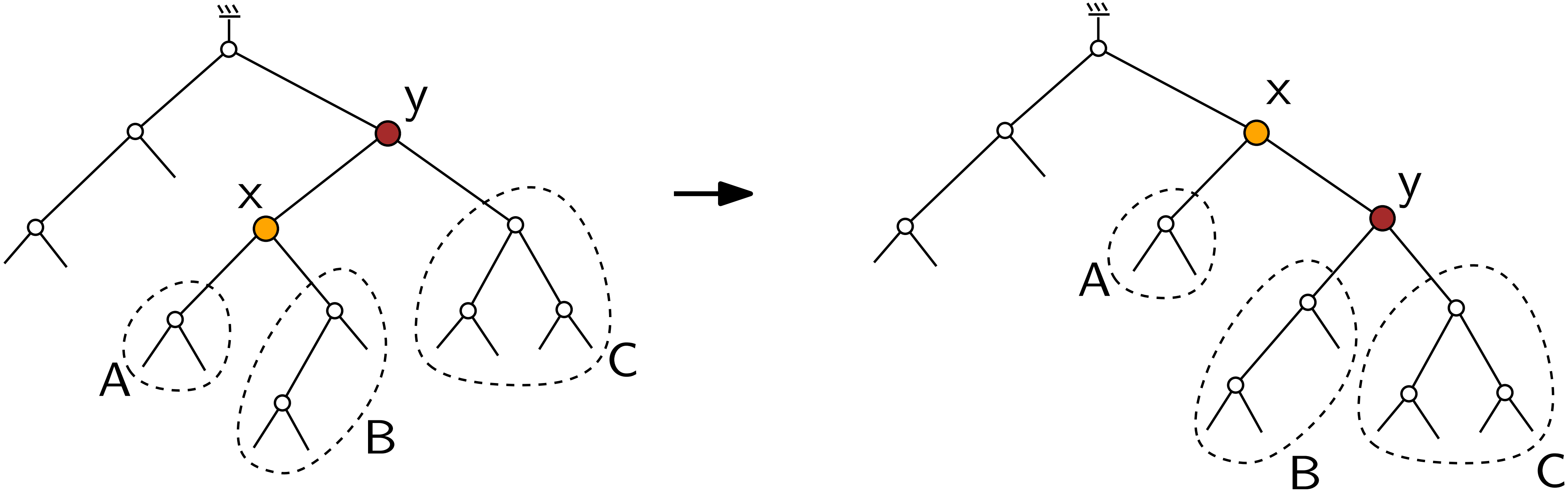
Tamari intervals and blossoming trees

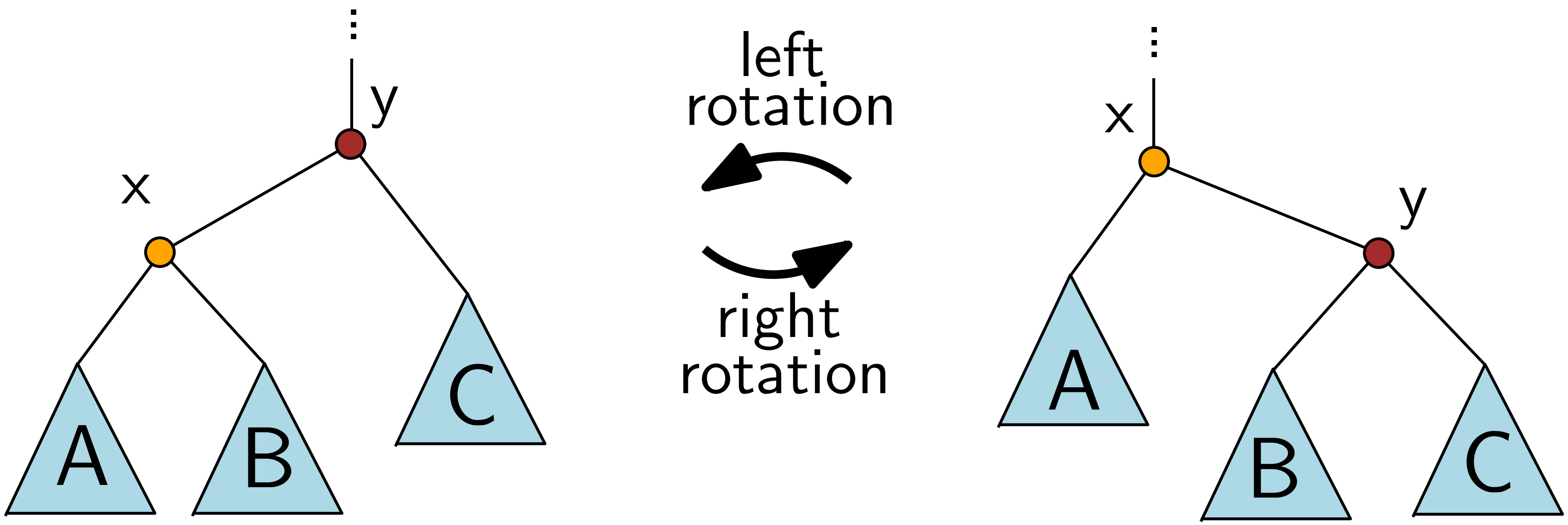
Éric Fusy (LIGM-CNRS Université Gustave Eiffel)

Joint work with Wenjie Fang and Philippe Nadeau

Brandeis, Jan 23 2024

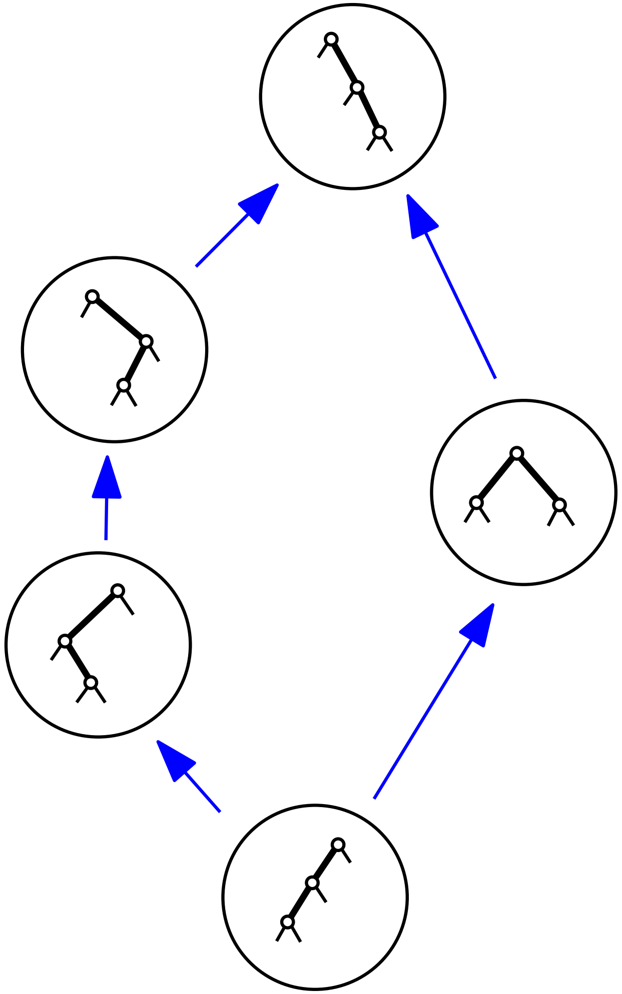
Rotation operations on binary trees



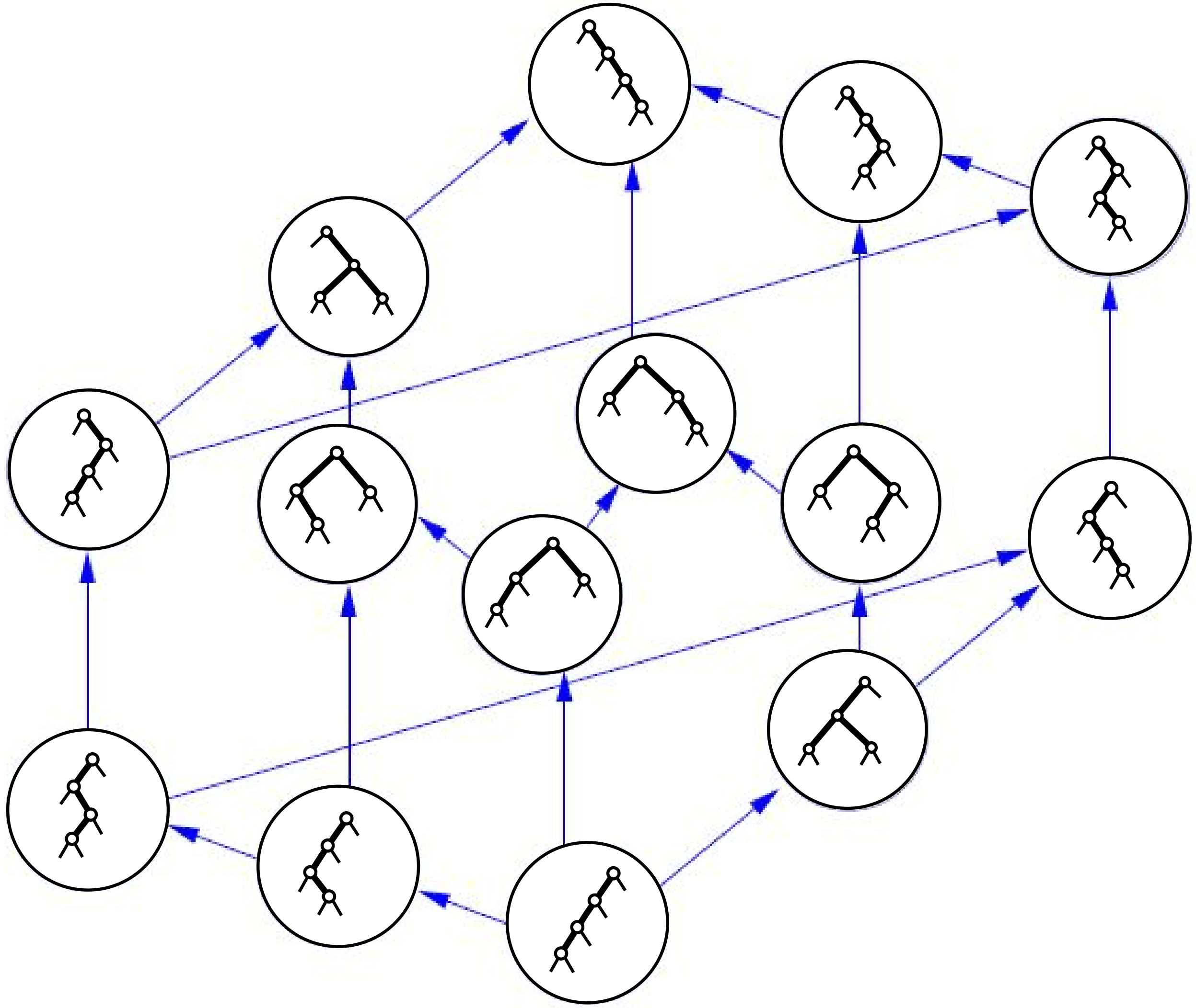


The Tamari lattice

The Tamari lattice $\mathcal{L}_n$ is the partial order on binary trees with n nodes where the covering relation corresponds to right rotation

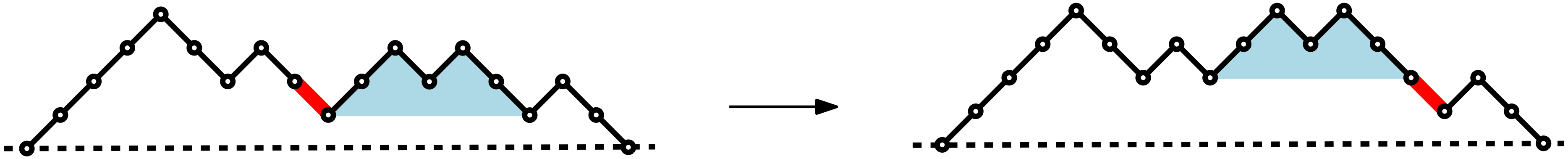


n=3

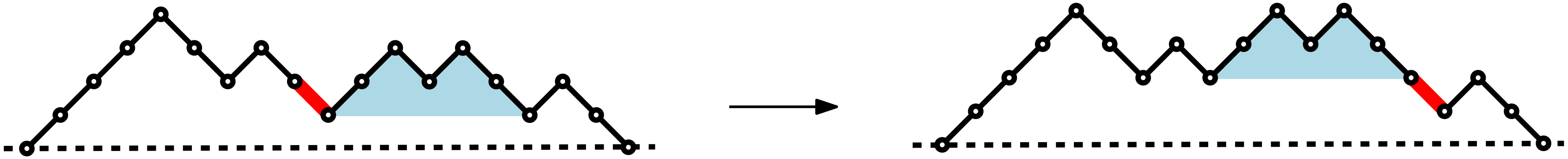


n=4

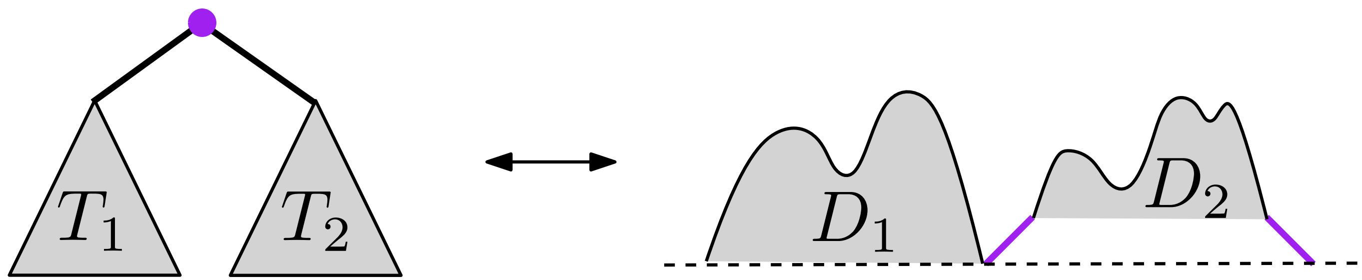
The covering relation for Dyck paths



The covering relation for Dyck paths

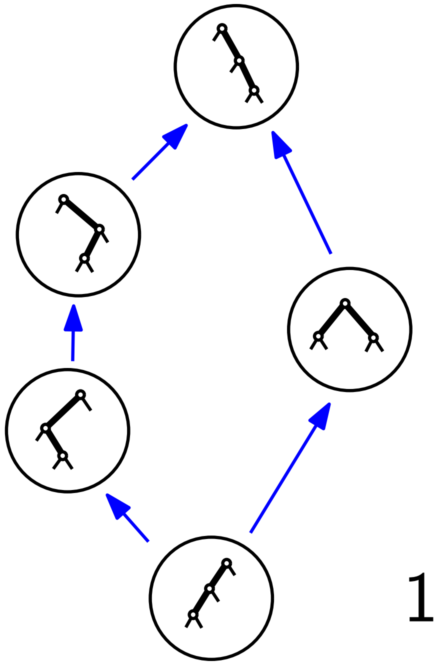


These correspond to right rotations, via the (recursive) bijection

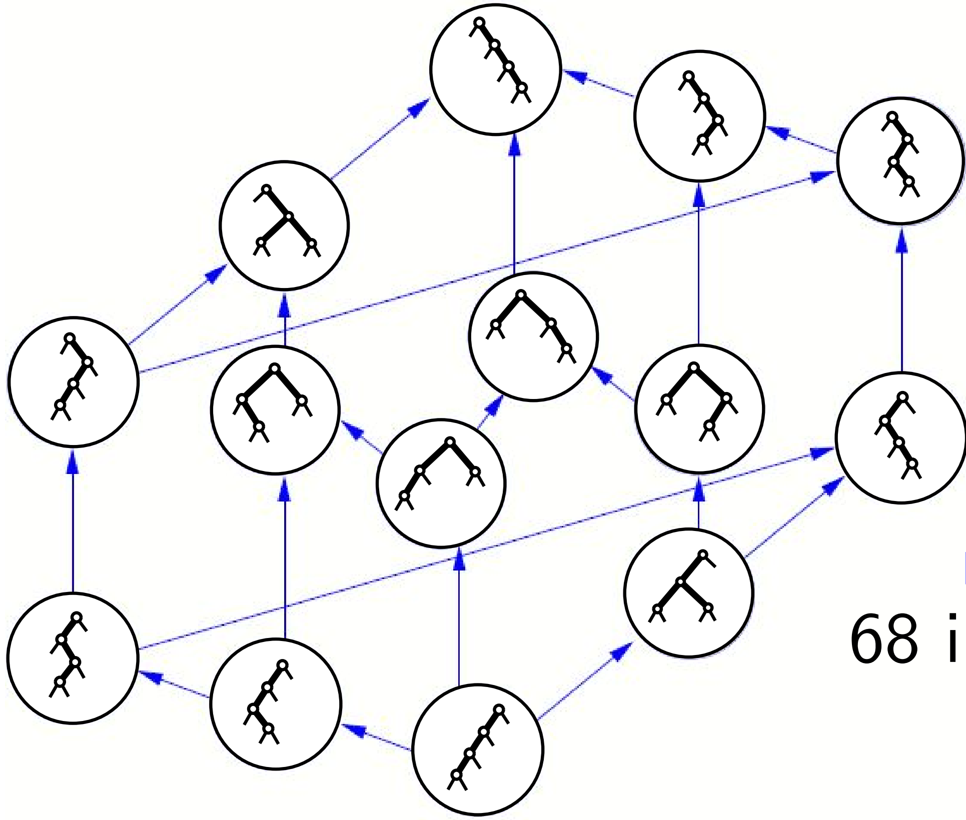


Tamari intervals

An interval in $\mathcal{L}_n$ is a pair (T, T') such that $T \leq T'$



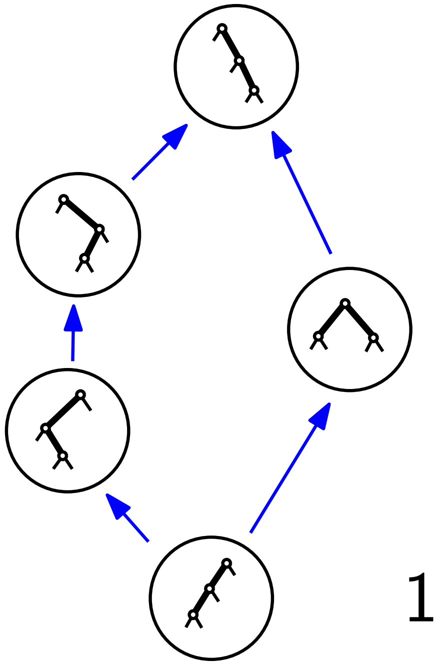
n=3
13 intervals



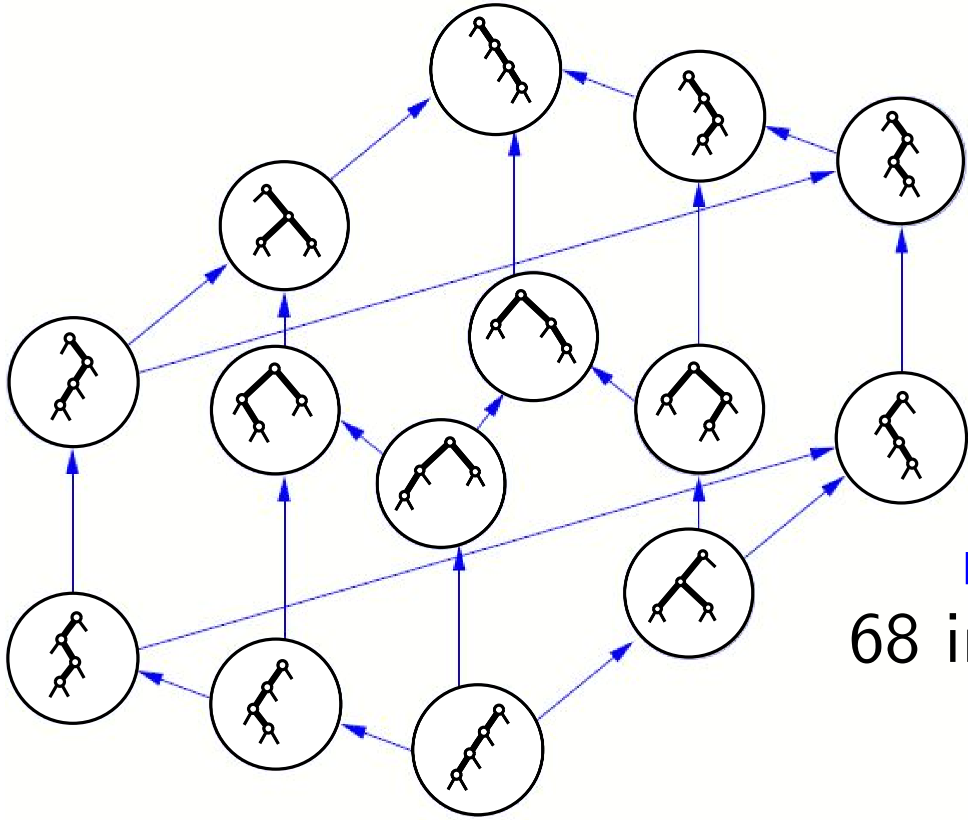
n=4
68 intervals

Tamari intervals

An interval in $\mathcal{L}_n$ is a pair (T, T') such that $T \leq T'$



n=3
13 intervals



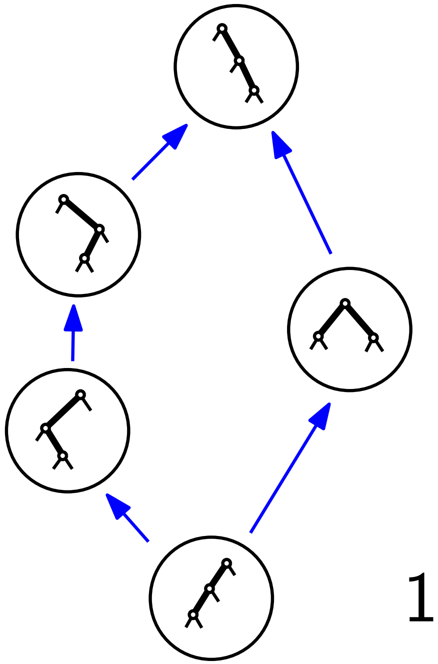
n=4
68 intervals

Theorem **[Chapoton'06]: there are** $\frac{2}{n(n+1)}\binom{4n+1}{n-1}$ **intervals in** $\mathcal{L}_n$

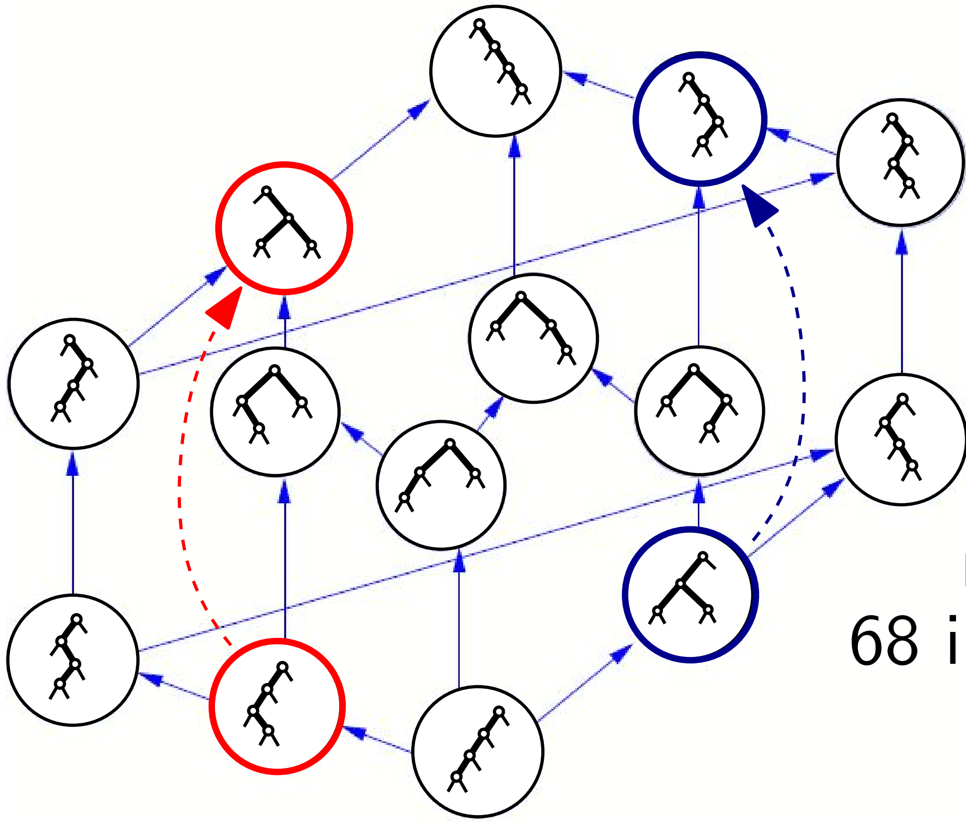
1, 3, 13, 68, 399,2530, 16965, 118668, ...

Tamari intervals

An interval in $\mathcal{L}_n$ is a pair (T, T') such that $T \leq T'$



n=3
13 intervals



n=4
68 intervals

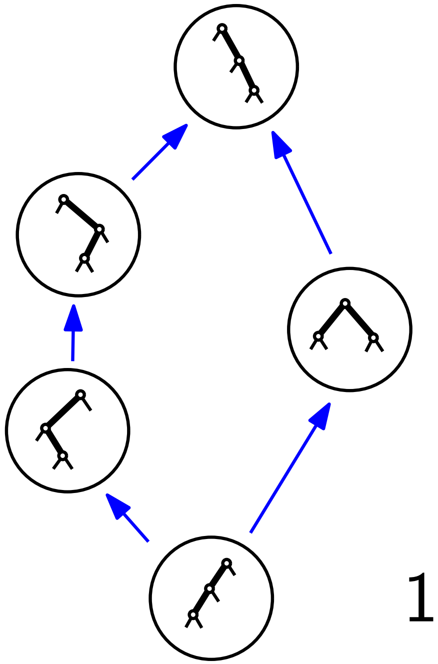
Theorem **[Chapoton'06]**: **there are** $\frac{2}{n(n+1)}\binom{4n+1}{n-1}$ **intervals in** $\mathcal{L}_n$

1, 3, 13, 68, 399,2530, 16965, 118668, ...

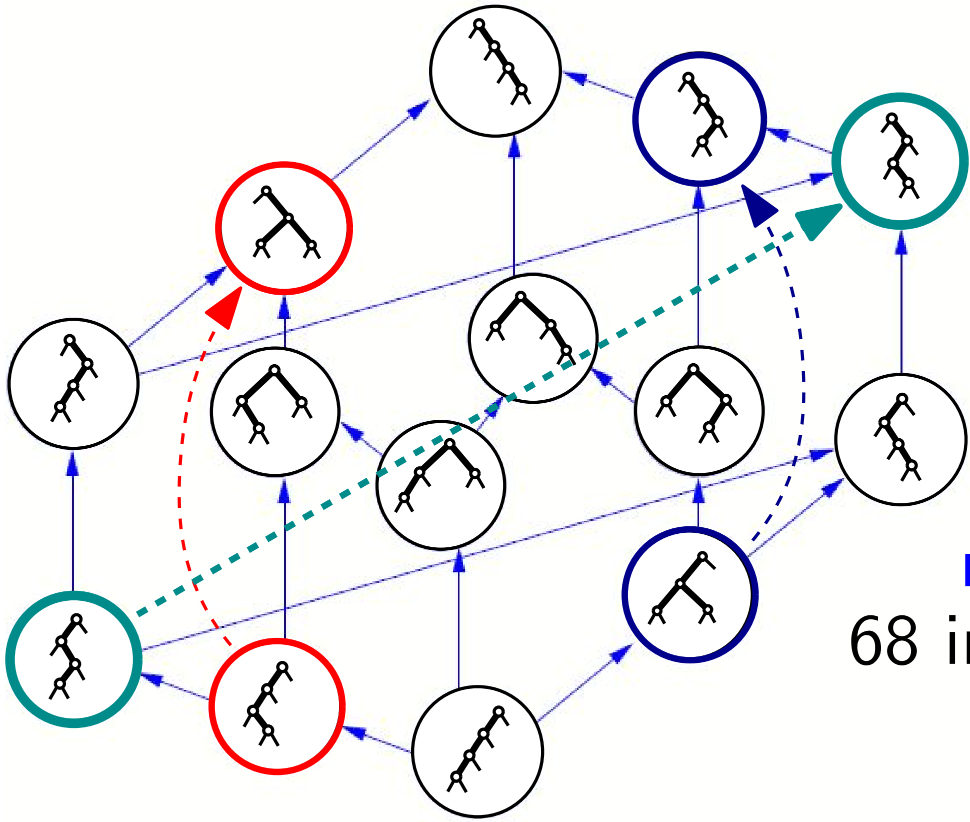
Rk: The **dual** of (T, T') is $(\mathrm{mir}(T'), \mathrm{mir}(T))$
It is an involution on Tamari intervals

Tamari intervals

An interval in $\mathcal{L}_n$ is a pair (T, T') such that $T \leq T'$



n=3
13 intervals



n=4
68 intervals

Theorem **[Chapoton'06]: there are** $\frac{2}{n(n+1)}\binom{4n+1}{n-1}$ **intervals in** $\mathcal{L}_n$

1, 3, 13, 68, 399,2530, 16965, 118668, ...

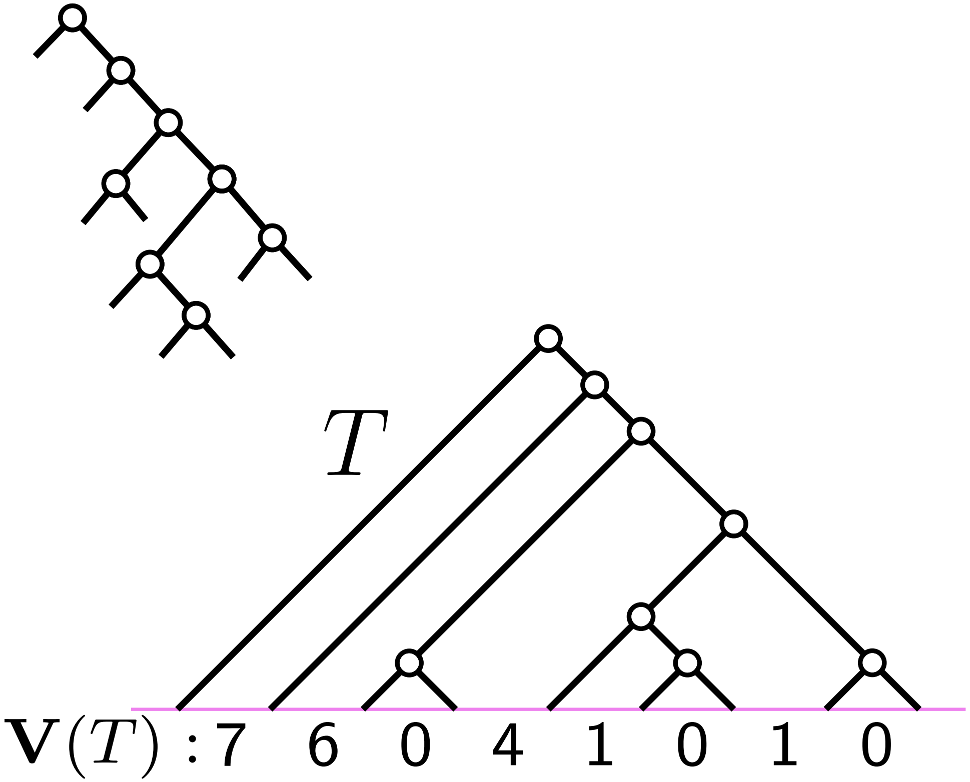
Rk: The **dual** of (T, T') is $(\mathrm{mir}(T'), \mathrm{mir}(T))$
It is an involution on Tamari intervals

A Tamari interval equal to its dual is called **self-dual**

1, 1, 3, 4, 15, 22, 91, 140, ...

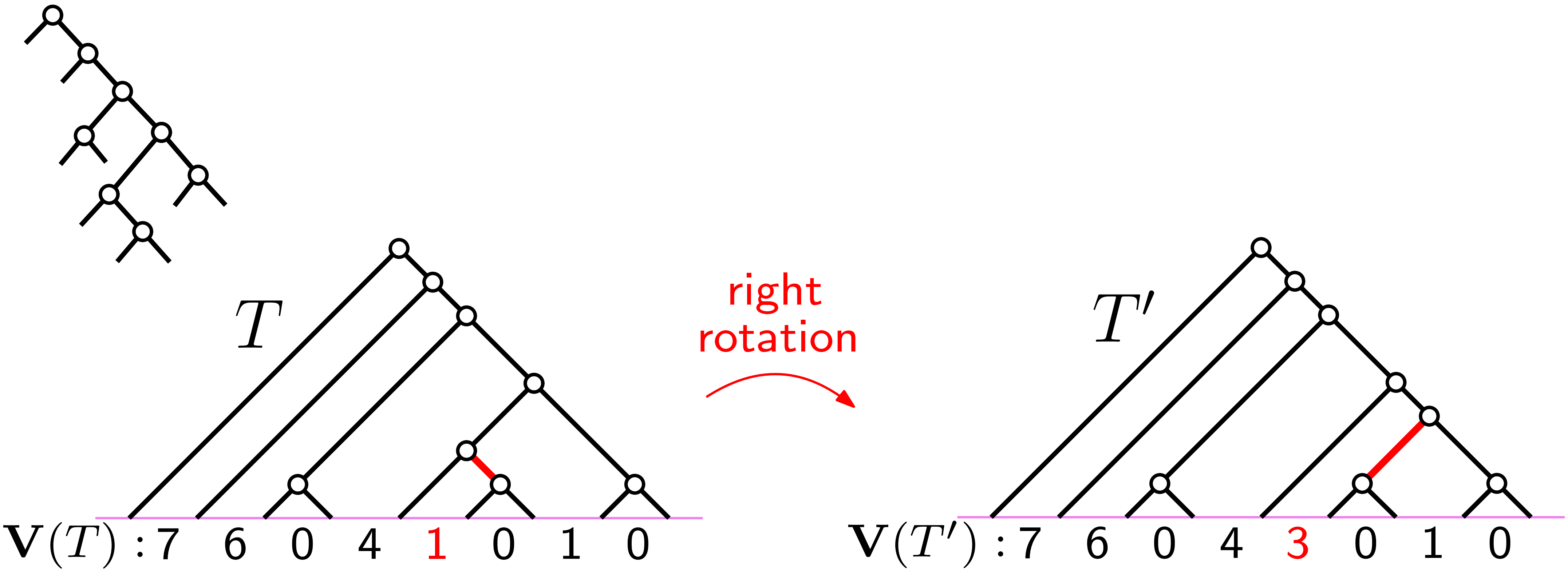
Bracket-vectors

Bracket-vector = vector of sizes of right subtrees of nodes (in infix order)



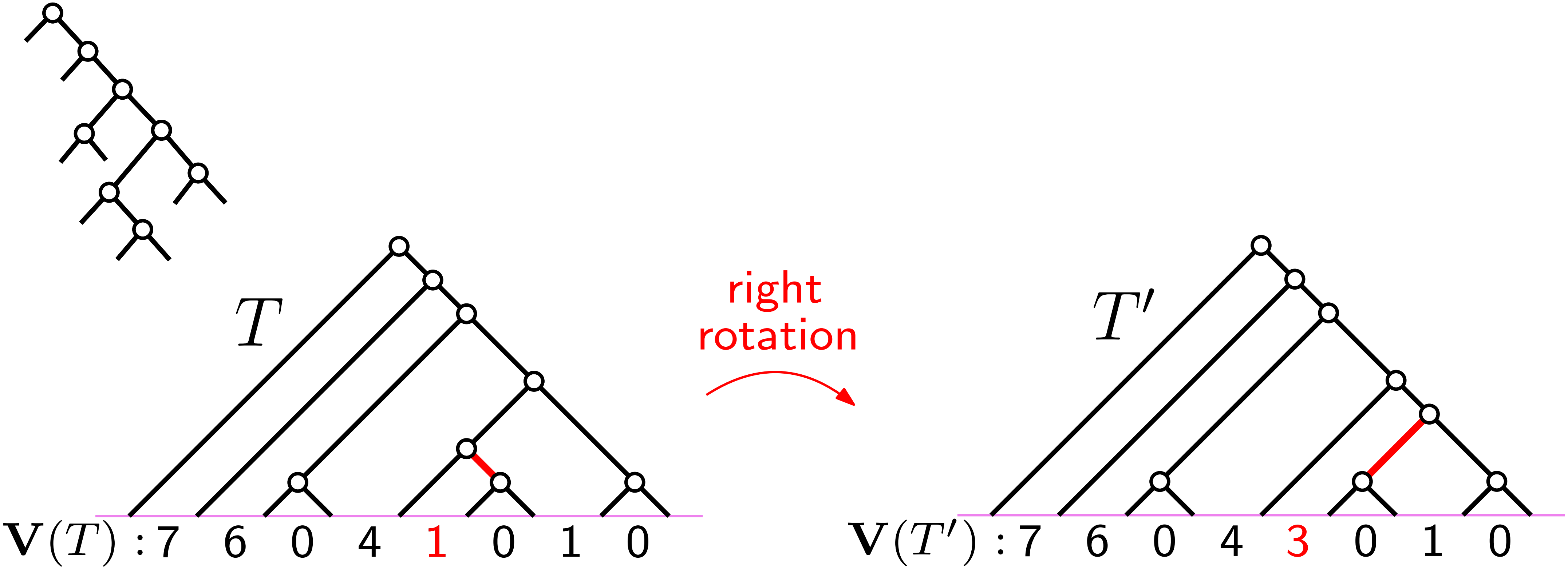
Bracket-vectors

Bracket-vector = vector of sizes of right subtrees of nodes (in infix order)



Bracket-vectors

Bracket-vector $=$ vector of sizes of right subtrees of nodes (in infix order)



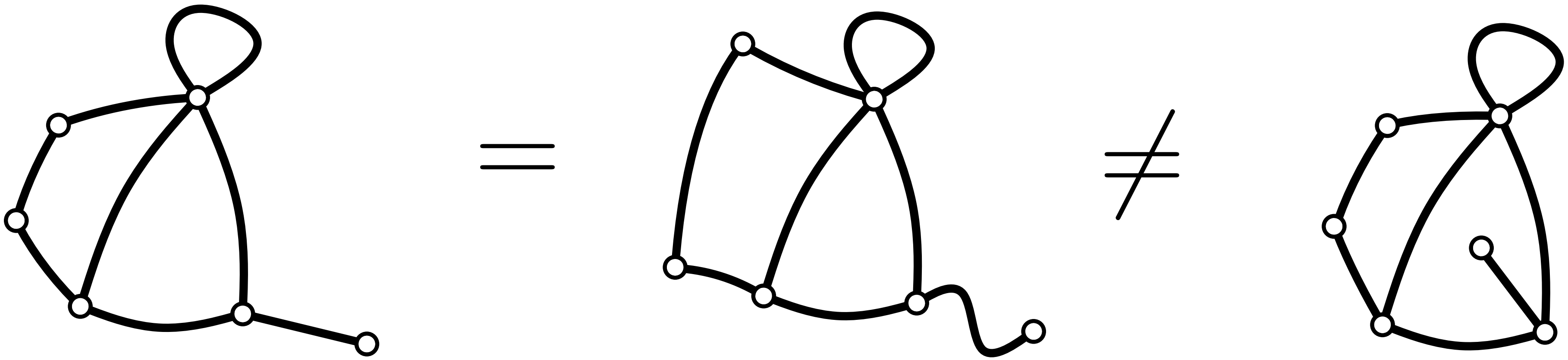
Prop: For two binary trees T, T' of size n **[Huang-Tamari'72]**

$$T \leq T' \text{ in } \mathcal{L}_n \text{ iff } \mathbf{V}(T) \leq \mathbf{V}(T')$$

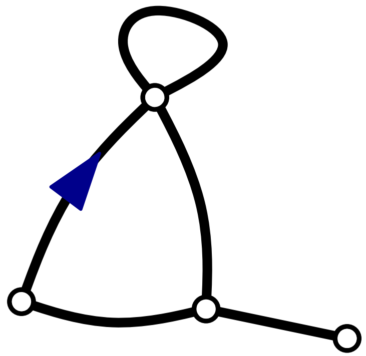
Tamari intervals and planar maps

Planar maps, triangulations

Planar map = connected graph embedded on the sphere up to isotopy

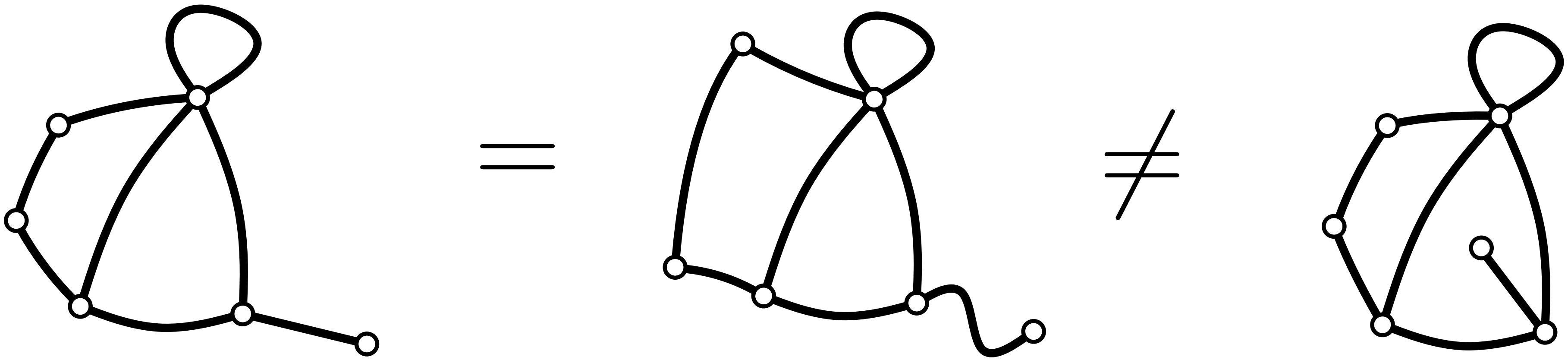


rooted map = map + marked directed edge
(outer face = face on its left)

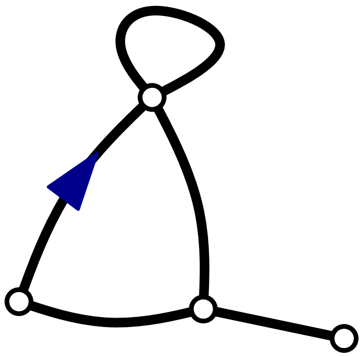


Planar maps, triangulations

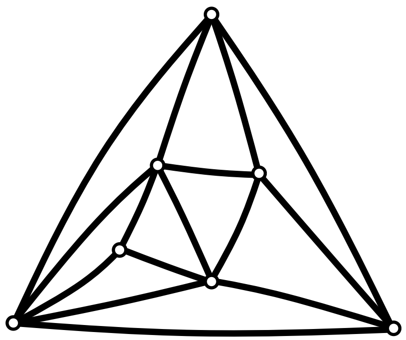
Planar map = connected graph embedded on the sphere up to isotopy



rooted map = map + marked directed edge
(outer face = face on its left)



Triangulation = simple planar map with all faces of degree 3



$n = 4$ internal vertices

[Tutte'62] #(triangulations on n internal vertices)$= \frac{2}{n(n+1)}\binom{4n+1}{n-1}$

Enumeration of Tamari interval families

family	formula	equinumerous map family
all intervals	$\frac{2}{n(n+1)}\binom{4n+1}{n-1}$ [Chapoton'06]	simple triangulations
synchronized / generalized	$\frac{2}{n(n+1)}\binom{3n}{n-1}$ [Fang-Préville-Ratelle'17]	simple quadrangulations
new/modern	$\frac{3\cdot 2^{n-2}}{n(n+1)}\binom{2n-2}{n-1}$ [Chapoton'06, Rognerud'18]	bipartite maps
m-Tamari	$\frac{m+1}{n(nm+1)}\binom{(m+1)^2 n+m}{n-1}$ [Bousquet-Mélou-F-Préville-Ratelle'11]	??
labeled	$2^n(n+1)^{n-2}$ [Bousquet-Mélou-Chapuy-Préville-Ratelle'12]	??

Bijective approach via maps

interval-family $\underset{(1)}{\longleftrightarrow}$ map family $\underset{(2)}{\longleftrightarrow}$ (unrooted) tree-family

$\downarrow$

counting formula

Bijective approach via maps

interval-family $\underset{(1)}{\longleftrightarrow}$ map family $\underset{(2)}{\longleftrightarrow}$ (unrooted) tree-family

$\downarrow$

counting formula

Bijections for step (2): closure of decorated trees [Poulalhon-Schaeffer'06] for simple triangulations

Bijective approach via maps

interval-family $\underset{(1)}{\longleftrightarrow}$ map family $\underset{(2)}{\longleftrightarrow}$ (unrooted) tree-family

$\downarrow$

counting formula

Bijections for step (2): closure of decorated trees [Poulalhon-Schaeffer'06]
for simple triangulations

Two types of bijections for step (1):

- Parallel decomposition with a catalytic variable
  $\Rightarrow$ recursive bijection
  (can be derecursified using dfs/description trees)
  [Fang-Préville-Ratelle'17], [Fang'18], [Fang'21]

Bijective approach via maps

interval-family $\overset{}{\underset{(1)}{\longleftrightarrow}}$ map family $\underset{(2)}{\longleftrightarrow}$ (unrooted) tree-family

$\downarrow$

counting formula

Bijections for step (2): closure of decorated trees [Poulalhon-Schaeffer'06]
for simple triangulations

Two types of bijections for step (1):

- Parallel decomposition with a catalytic variable
  $\Rightarrow$ recursive bijection
  (can be derecursified using dfs/description trees)
  [Fang-Préville-Ratelle'17], [Fang'18], [Fang'21]
- Specialize bijections between oriented maps and walk-systems
  [Bernardi-Bonichon'09] [F-Humbert'19]

Bijective approach via maps

this talk
(inspired by interval posets)

interval-family $\overset{}{\longleftrightarrow}$ (1) map family $\longleftrightarrow$ (2) (unrooted) tree-family

(unrooted) tree-family $\downarrow$ counting formula

Bijections for step (2): closure of decorated trees [Poulalhon-Schaeffer'06]
for simple triangulations

Two types of bijections for step (1):

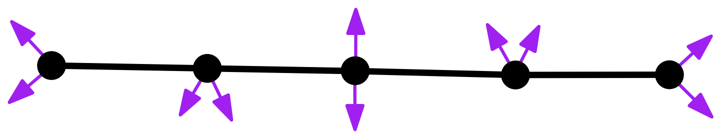
- Parallel decomposition with a catalytic variable
  $\Rightarrow$ recursive bijection
  (can be derecursified using dfs/description trees)
  [Fang-Préville-Ratelle'17], [Fang'18], [Fang'21]
- Specialize bijections between oriented maps and walk-systems
  [Bernardi-Bonichon'09] [F-Humbert'19]

The Poulalhon-Schaeffer bijection

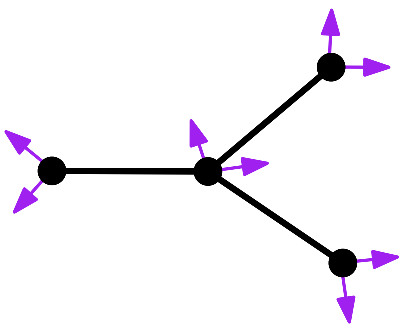
[Poulalhon-Schaeffer'06] [Albenque-Poulalhon'13]

Blossoming tree = (unrooted) plane tree with two buds per node
counted with multiplicity 2 if no half-turn symmetry

size $n = \#$ edges



multiplicity 1



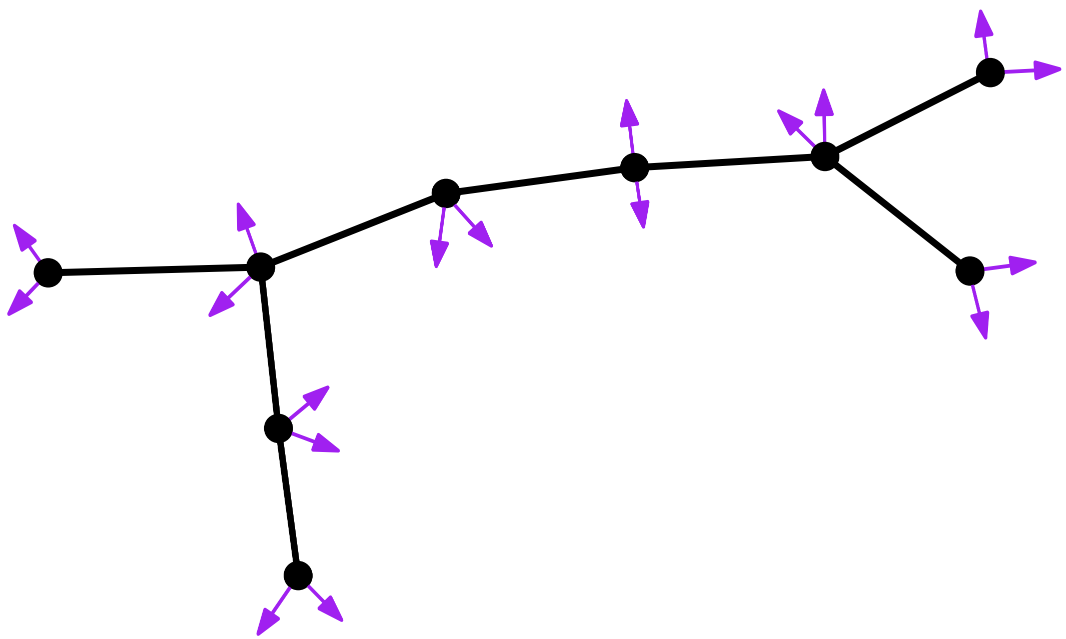
multiplicity 2

The (weighted) number of blossoming trees of size n is $\frac{2}{n(n+1)}\binom{4n+1}{n-1}$

The Poulalhon-Schaeffer bijection

[Poulalhon-Schaeffer'06] [Albenque-Poulalhon'13]

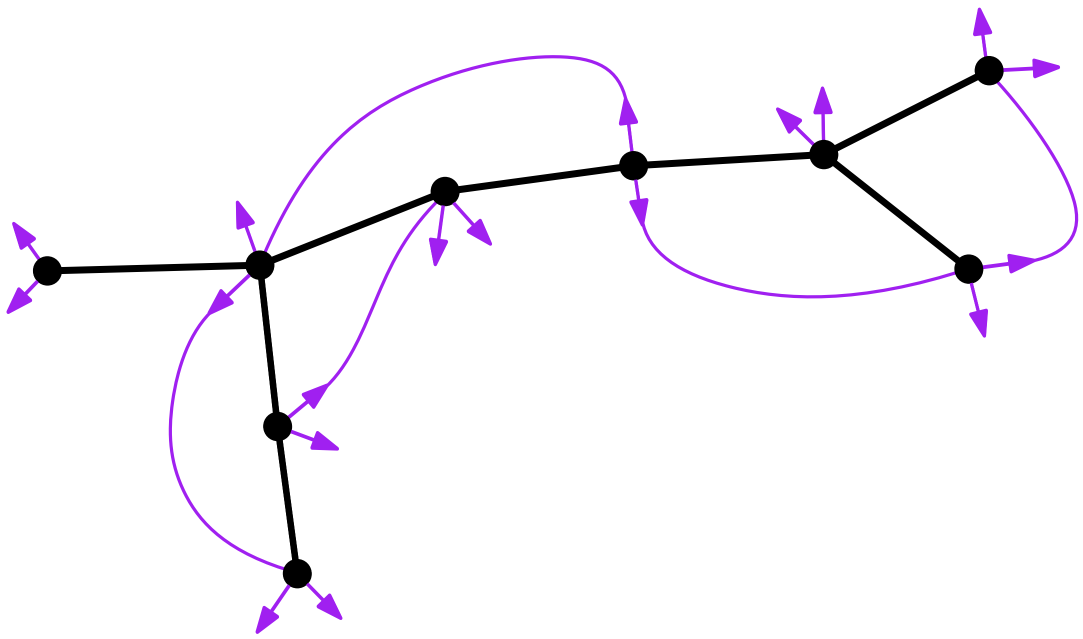
Blossoming tree → Simple triangulation
closure operations



The Poulalhon-Schaeffer bijection

[Poulalhon-Schaeffer'06] [Albenque-Poulalhon'13]

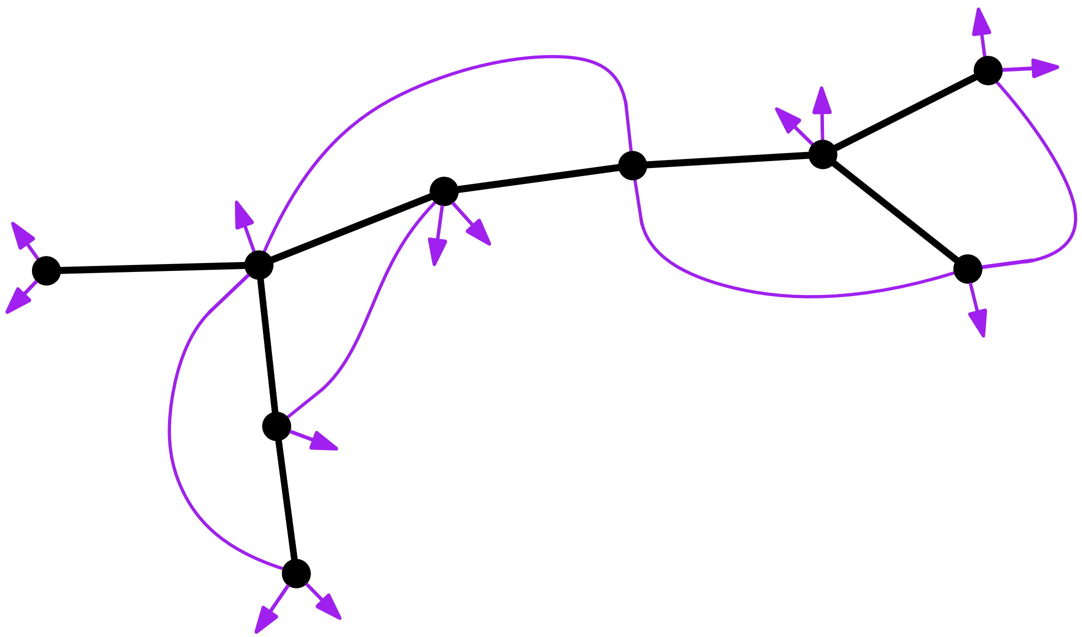
Blossoming tree → *closure operations* → Simple triangulation



The Poulalhon-Schaeffer bijection

[Poulalhon-Schaeffer'06] [Albenque-Poulalhon'13]

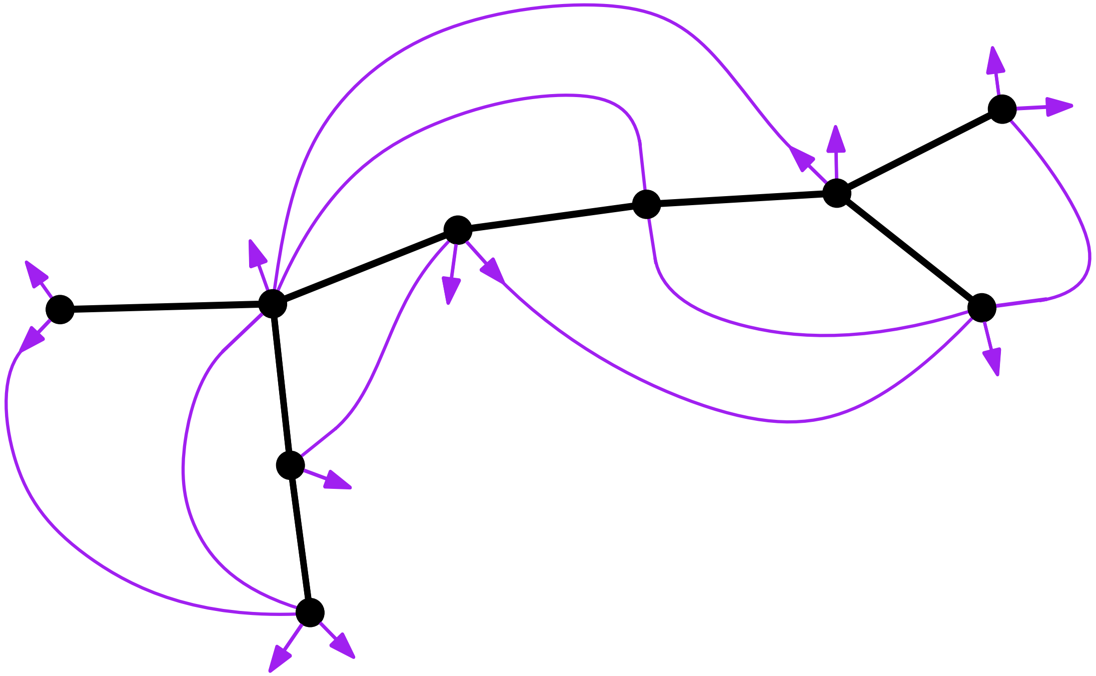
Blossoming tree → *closure operations* → Simple triangulation



The Poulalhon-Schaeffer bijection

[Poulalhon-Schaeffer'06] [Albenque-Poulalhon'13]

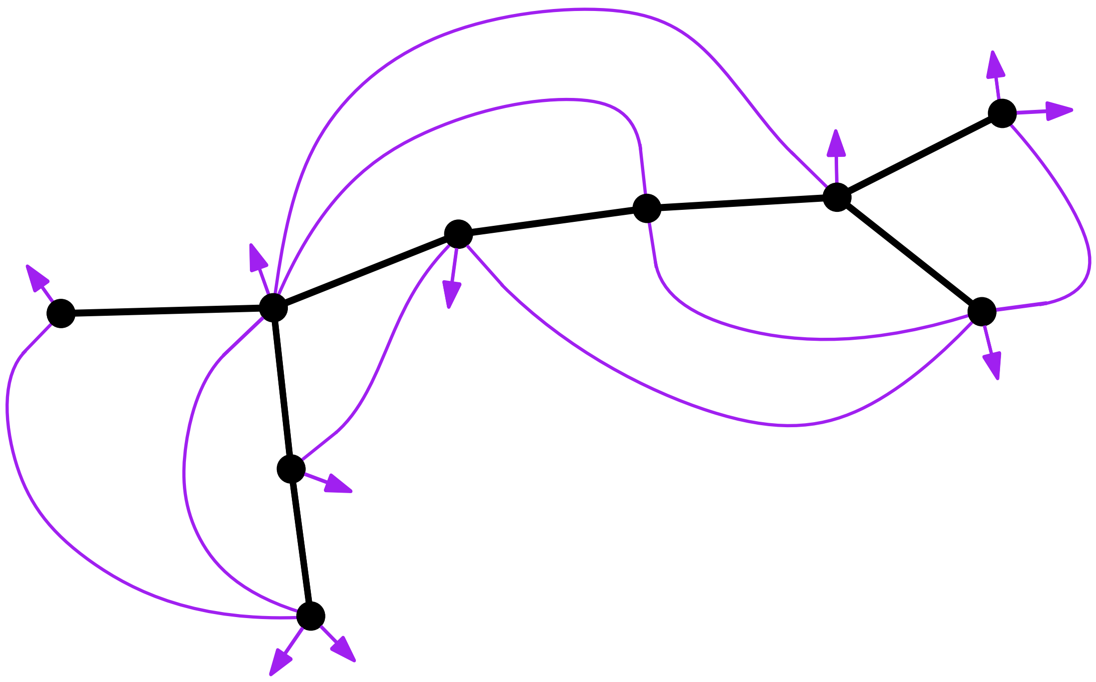
Blossoming tree $\xrightarrow{\textit{closure operations}}$ Simple triangulation



The Poulalhon-Schaeffer bijection

[Poulalhon-Schaeffer'06] [Albenque-Poulalhon'13]

Blossoming tree $\xrightarrow{\textit{closure operations}}$ Simple triangulation

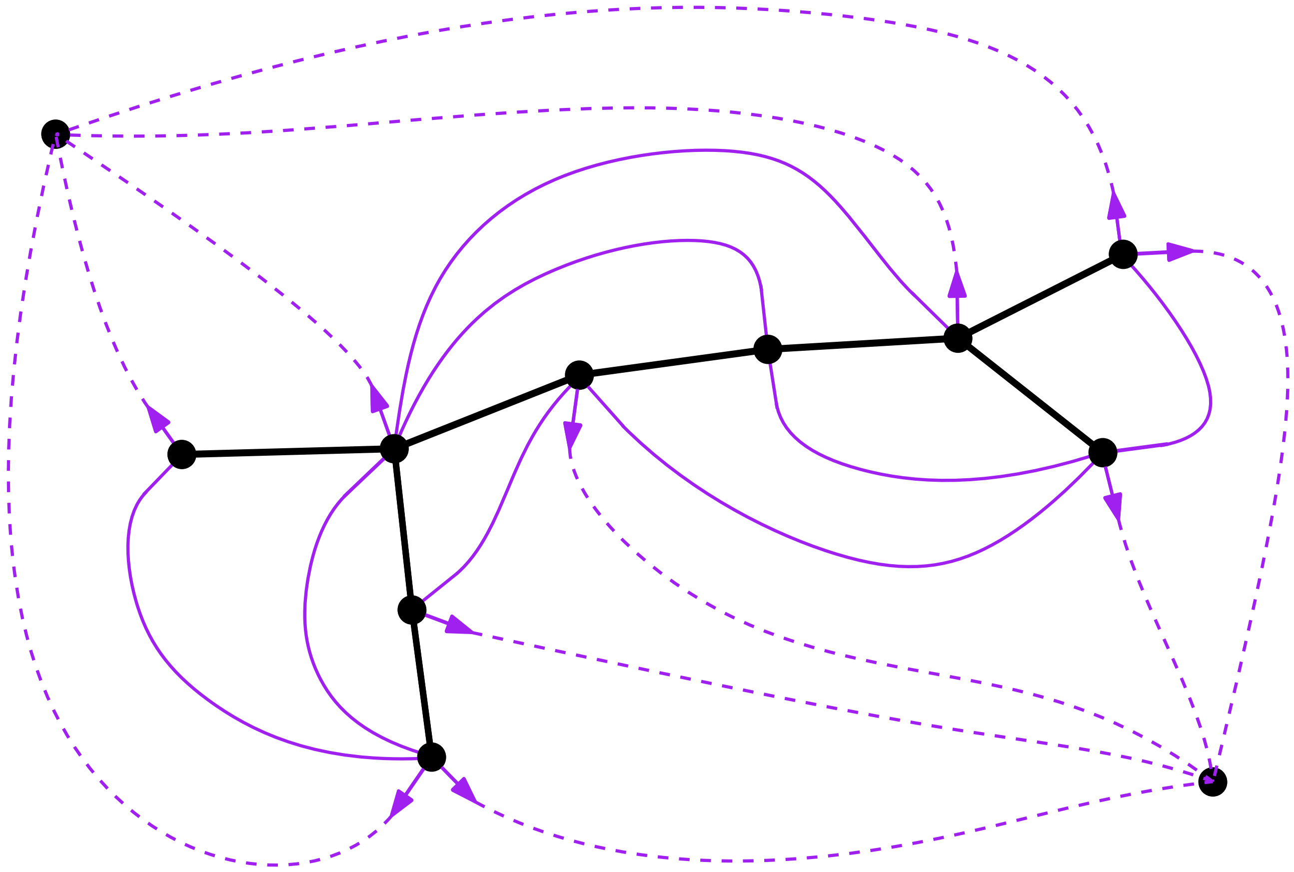


The Poulalhon-Schaeffer bijection

[Poulalhon-Schaeffer'06] [Albenque-Poulalhon'13]

Blossoming tree → Simple triangulation

closure operations

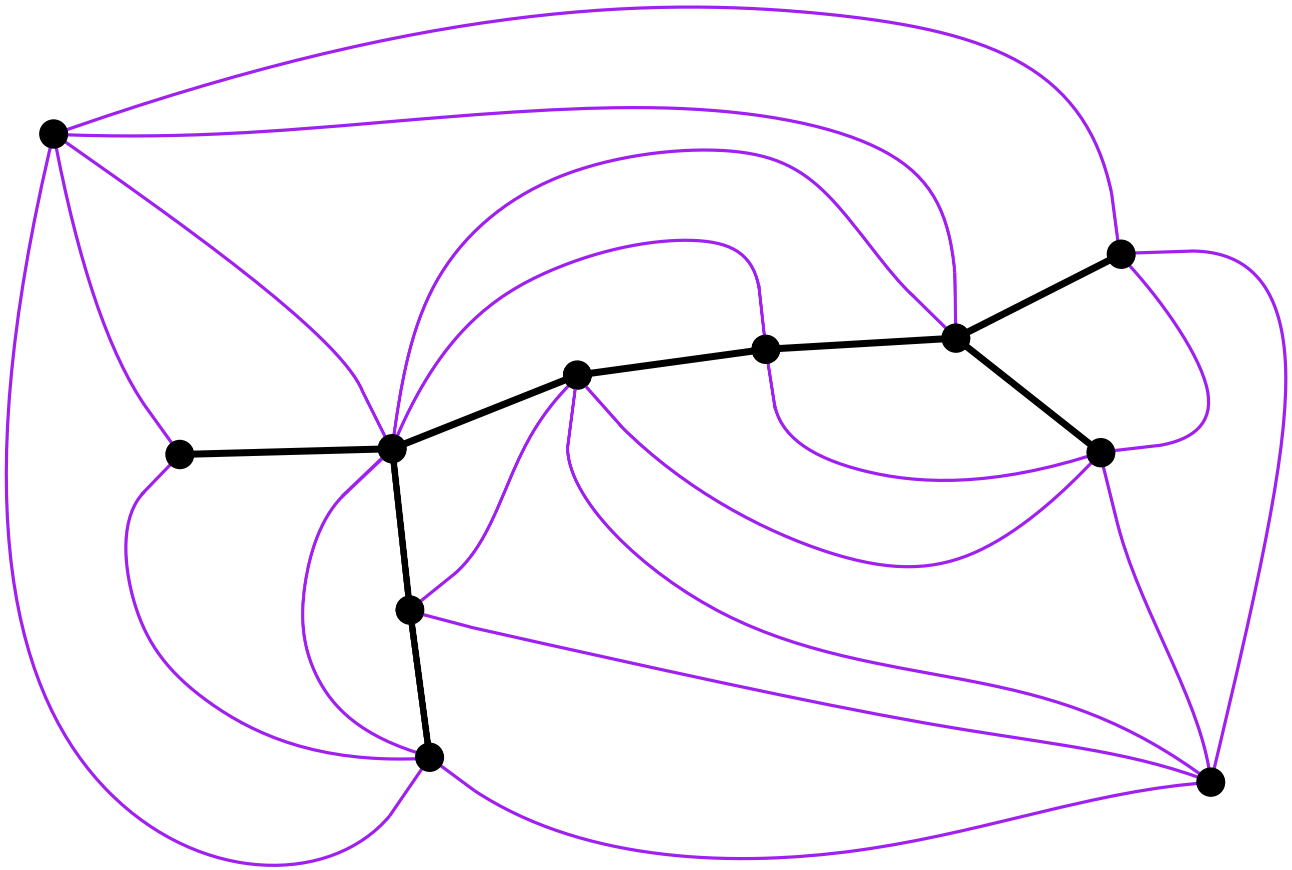


The Poulalhon-Schaeffer bijection

[Poulalhon-Schaeffer'06] [Albenque-Poulalhon'13]

Blossoming tree → Simple triangulation

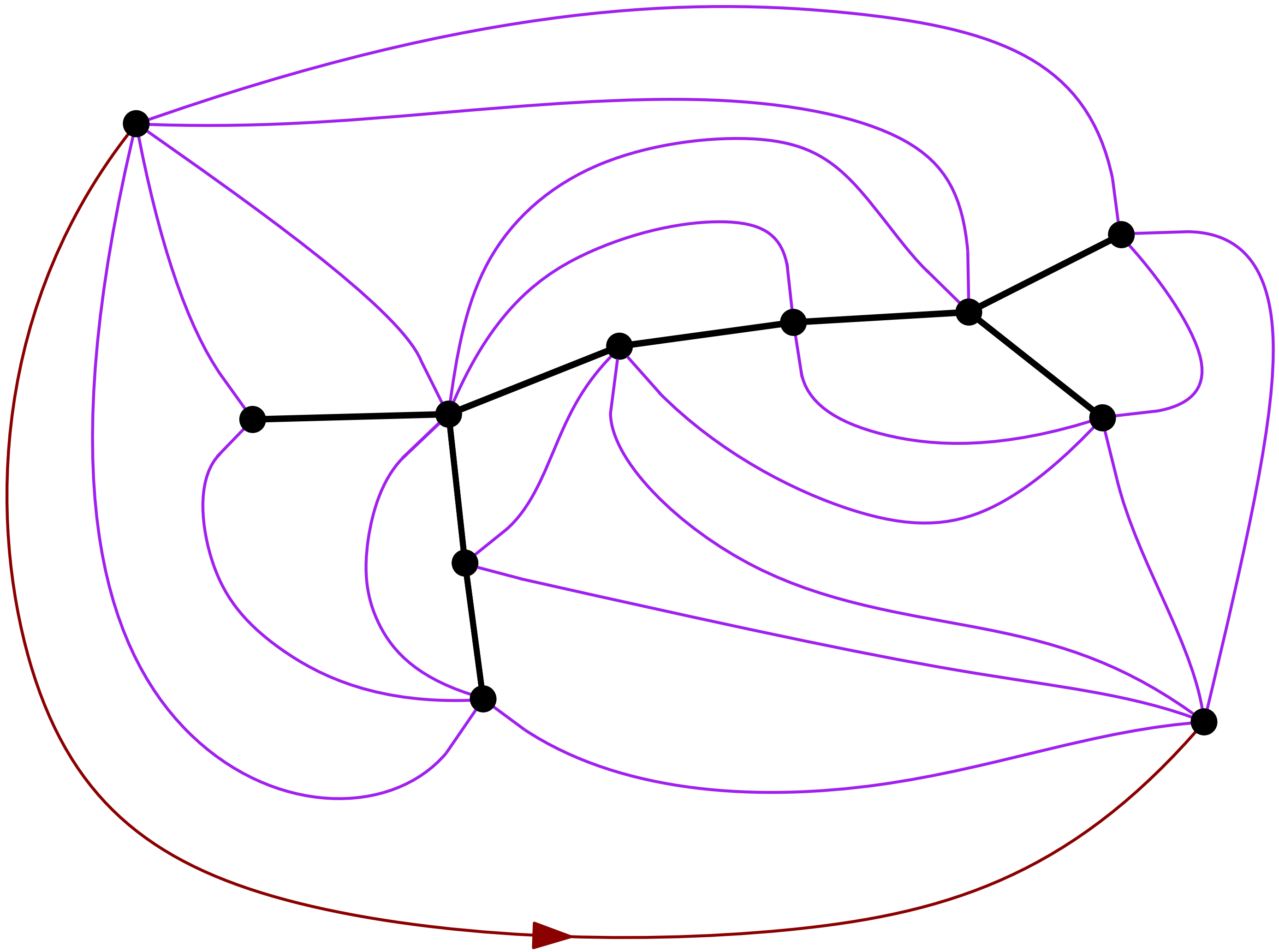
closure operations



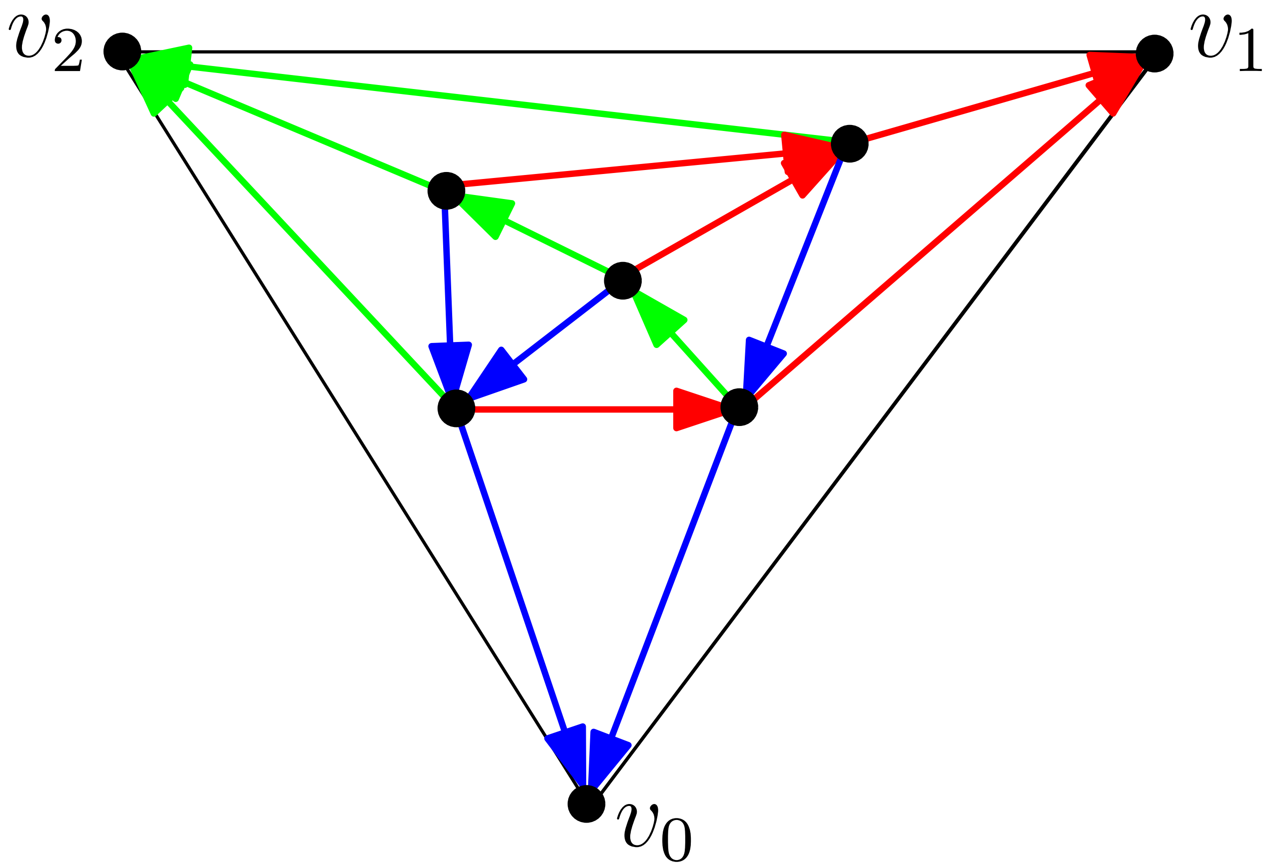
The Poulalhon-Schaeffer bijection

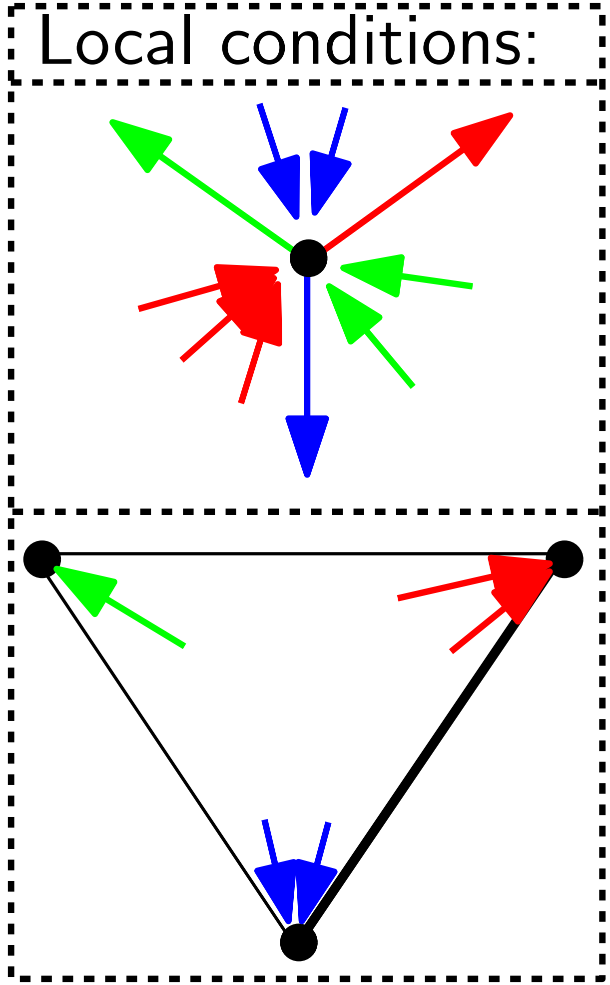
[Poulalhon-Schaeffer'06] [Albenque-Poulalhon'13]

Blossoming tree → Simple triangulation
closure operations

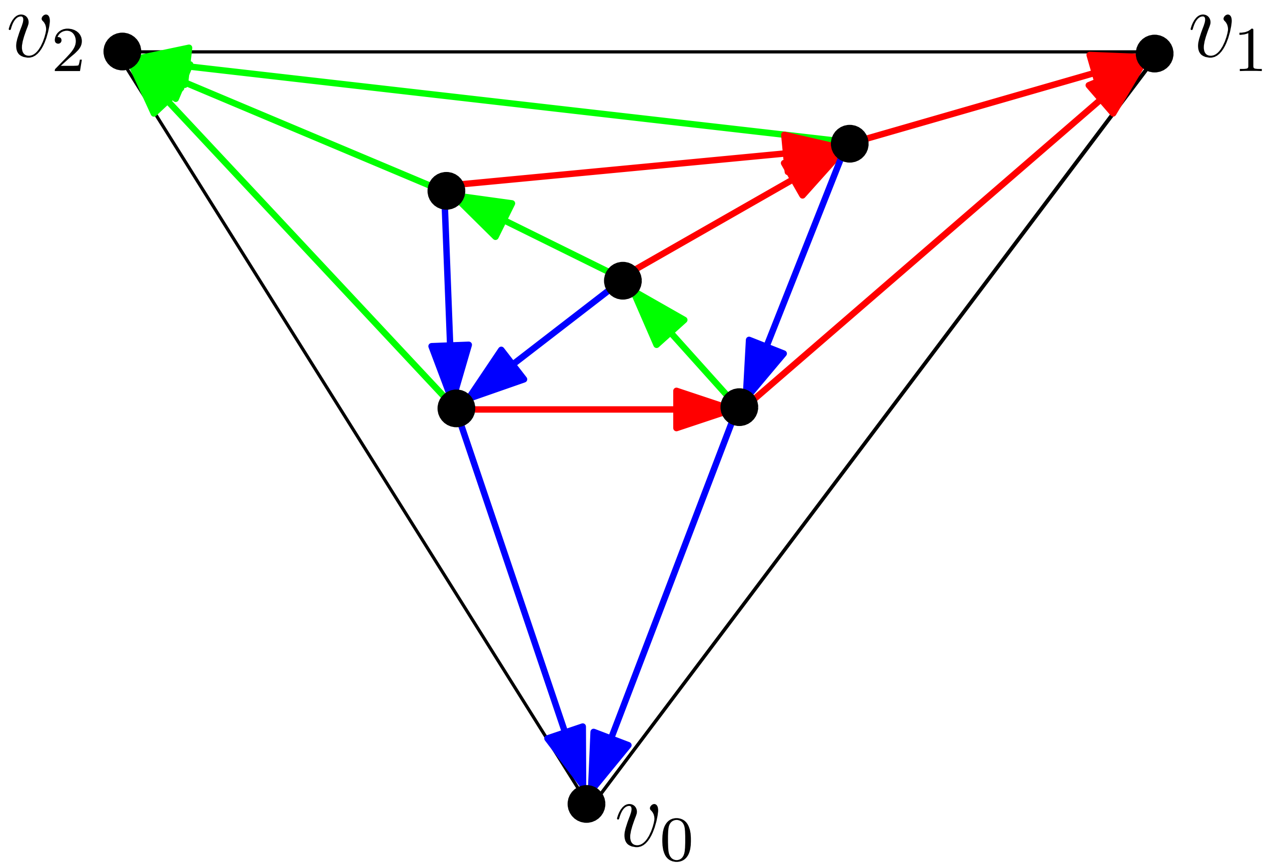


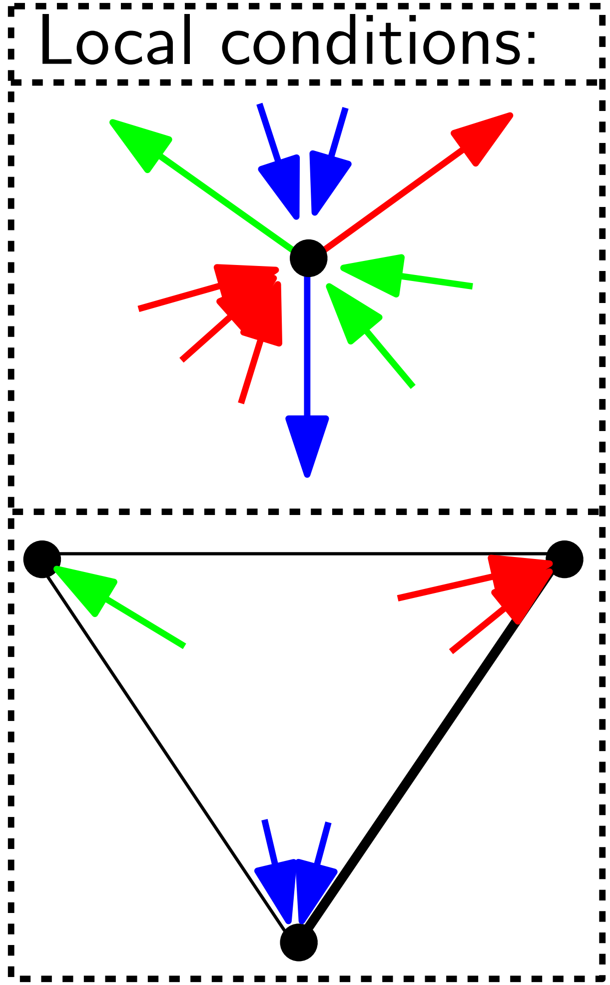
Schnyder woods





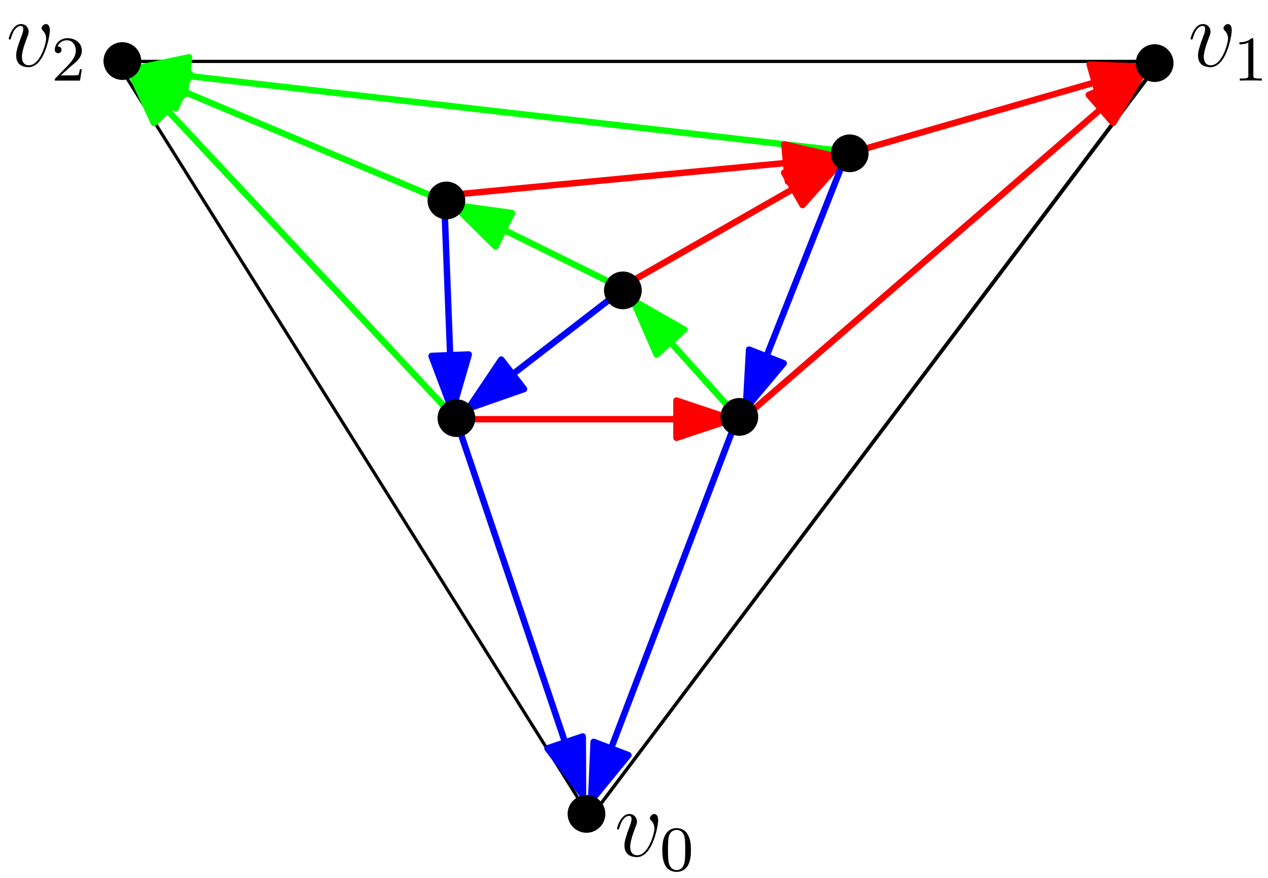
Schnyder woods

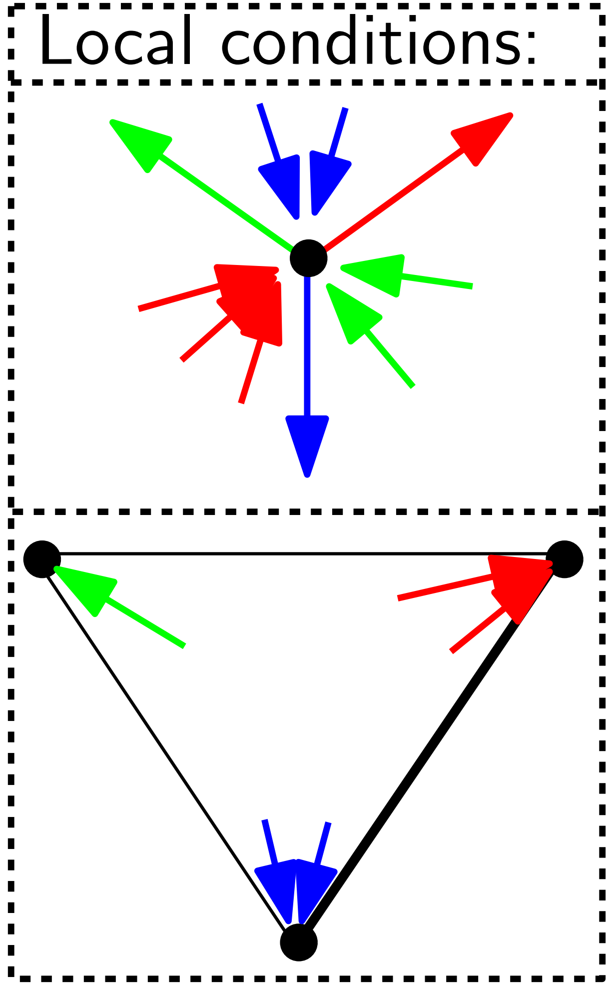




Theo: Any triangulation admits a Schnyder wood **[Schnyder'89]**

Schnyder woods

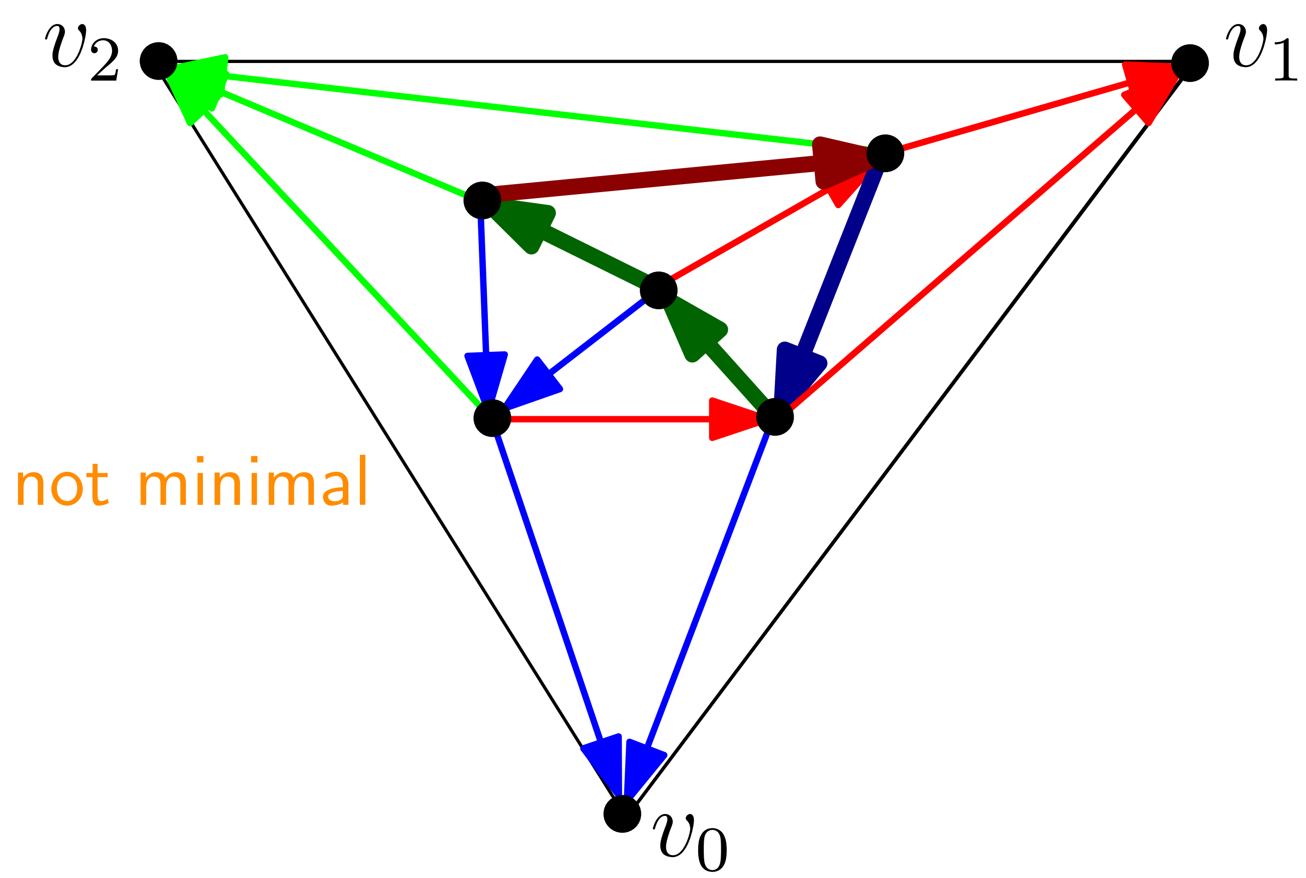


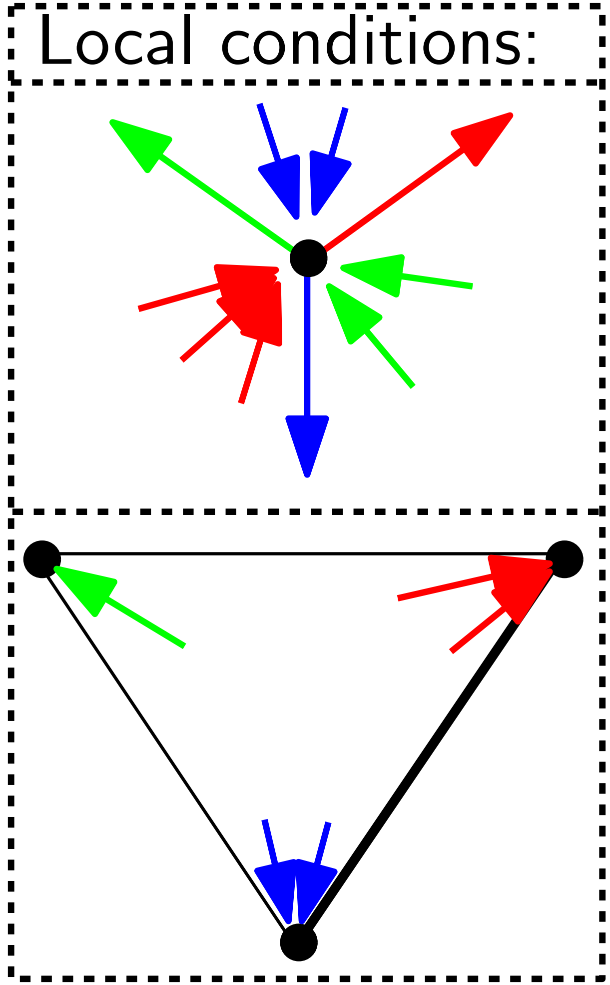


Theo: Any triangulation admits a Schnyder wood **[Schnyder'89]**

- A Schnyder wood with no cw circuit is called **minimal**

Schnyder woods

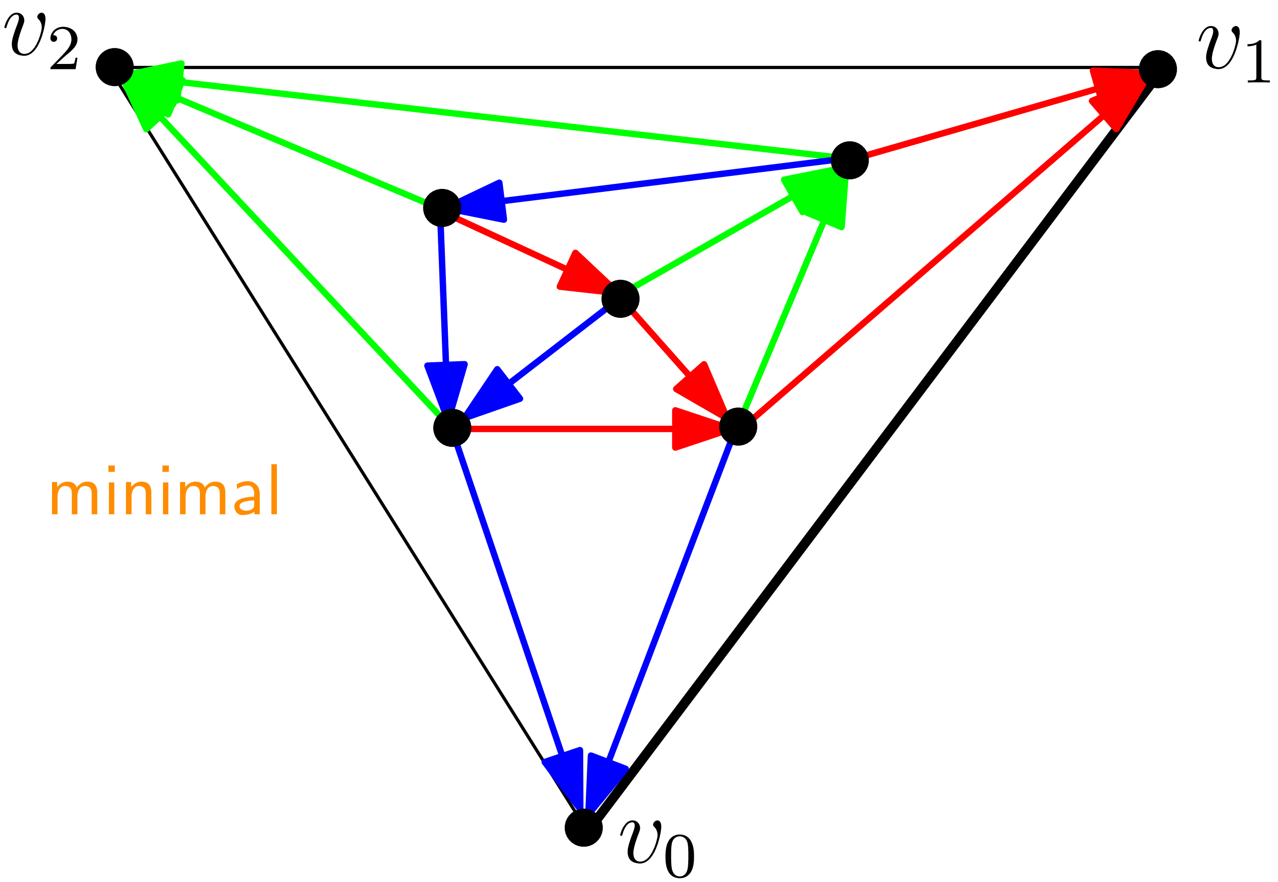


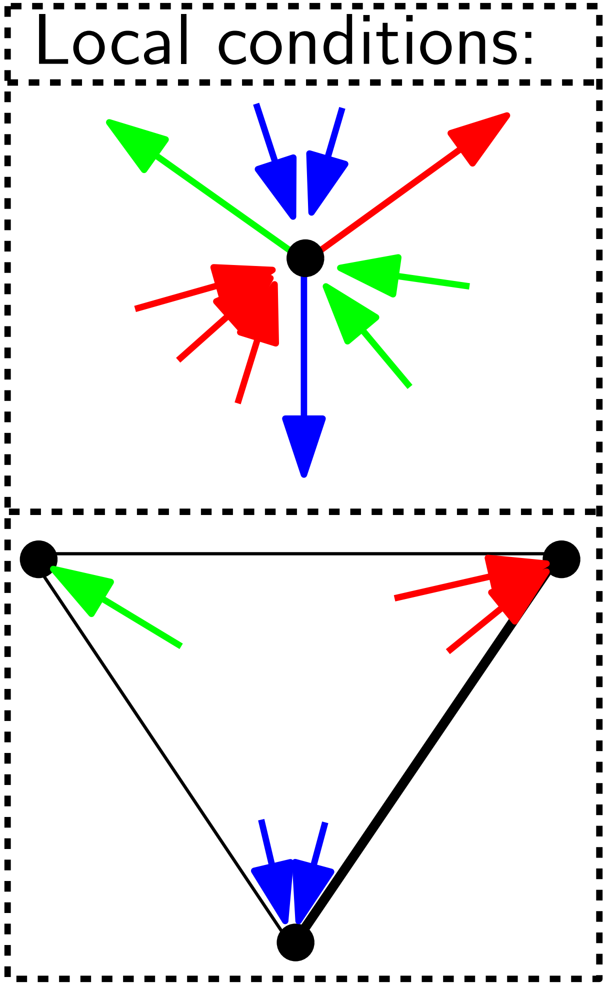


Theo: Any triangulation admits a Schnyder wood **[Schnyder'89]**

- A Schnyder wood with no cw circuit is called **minimal**

Schnyder woods

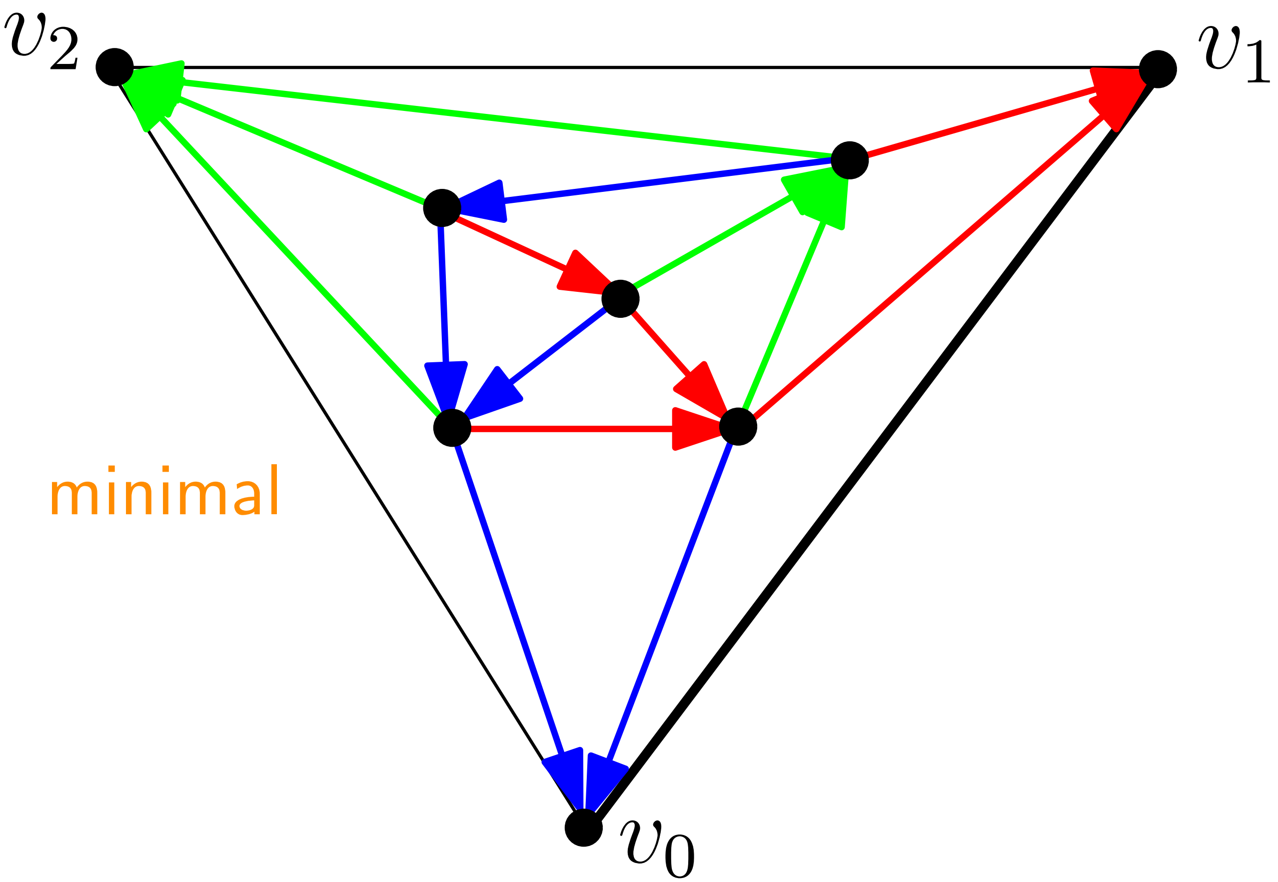


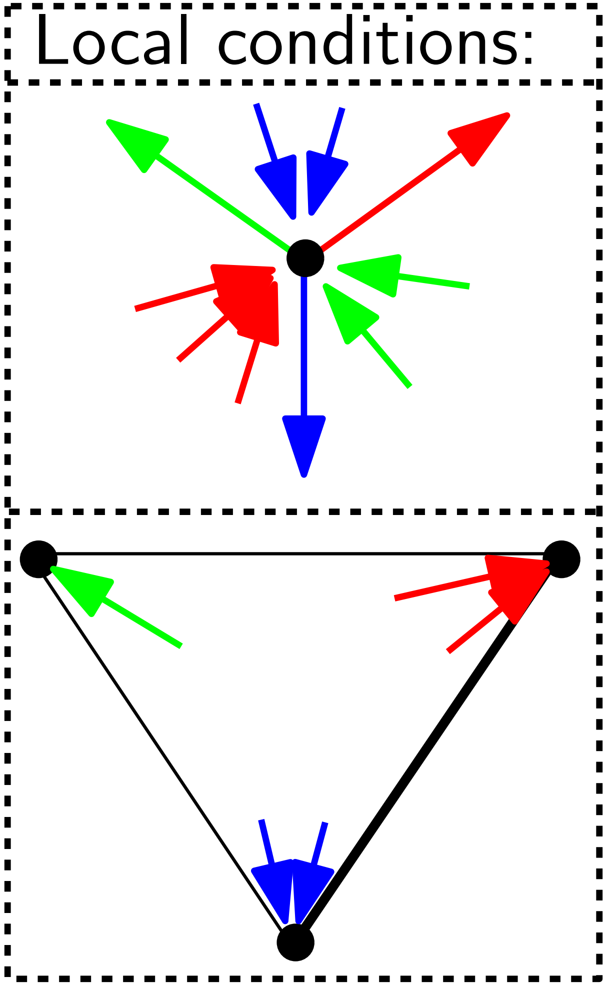


Theo: Any triangulation admits a Schnyder wood **[Schnyder'89]**

- A Schnyder wood with no cw circuit is called **minimal**

Schnyder woods





Theo: Any triangulation admits a Schnyder wood **[Schnyder'89]**

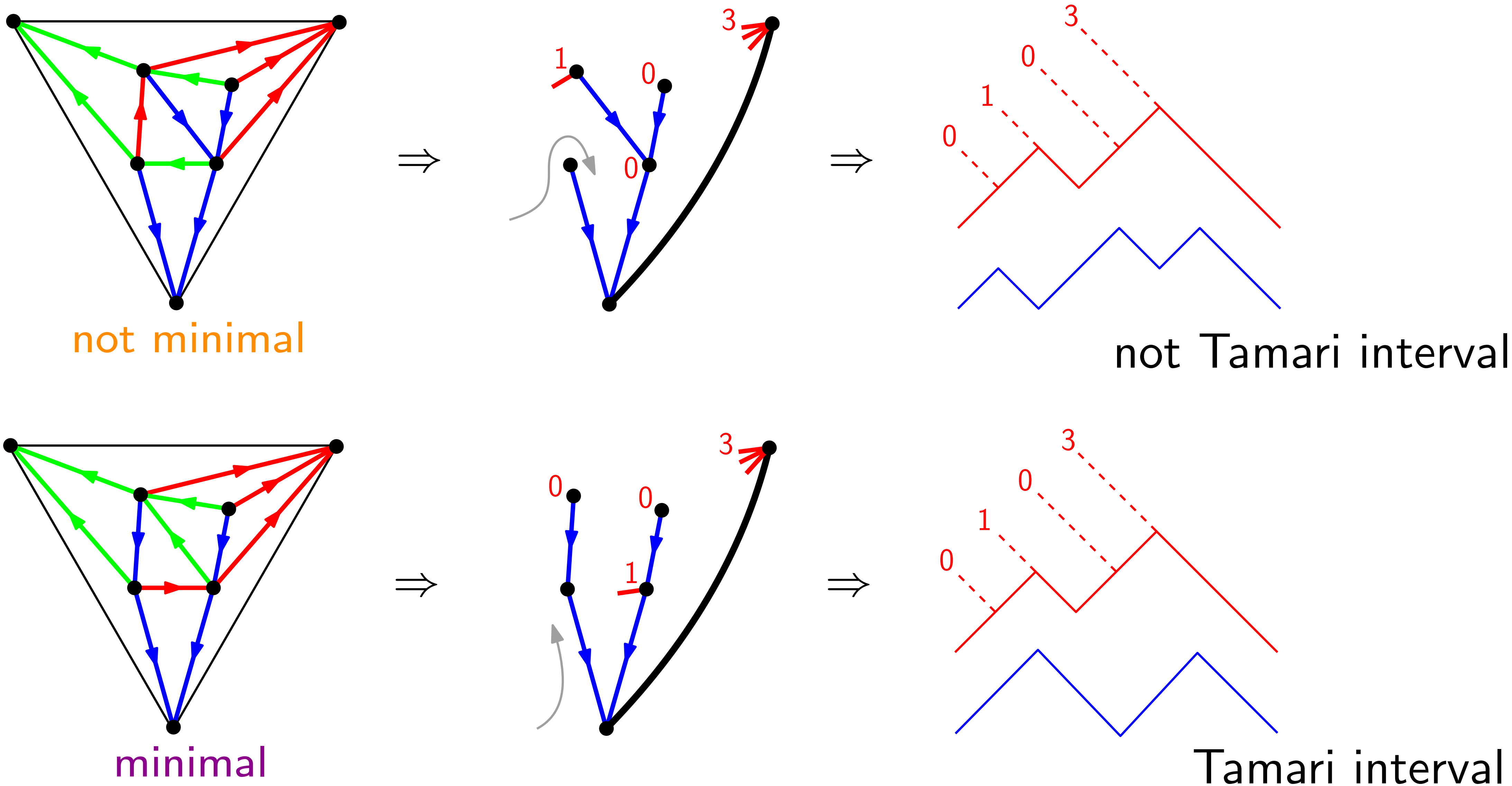
- A Schnyder wood with no cw circuit is called **minimal**

Theo: Any triangulation has a unique minimal Schnyder wood
(cf set of Schnyder woods on fixed triangulation is a distributive lattice)
[Ossona de Mendez'94, Brehm'03, Felsner'03]

The Bernardi-Bonichon bijection [Bernardi, Bonichon'09]

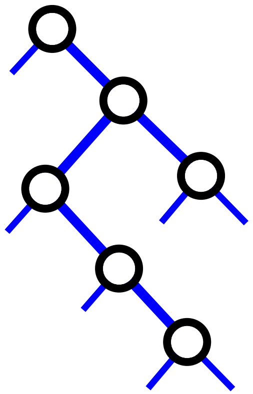
Schnyder woods on $n+3$ vertices $\updownarrow$ non-intersecting pairs of Dyck paths of lengths $2n$

minimal $\updownarrow$ Tamari interval

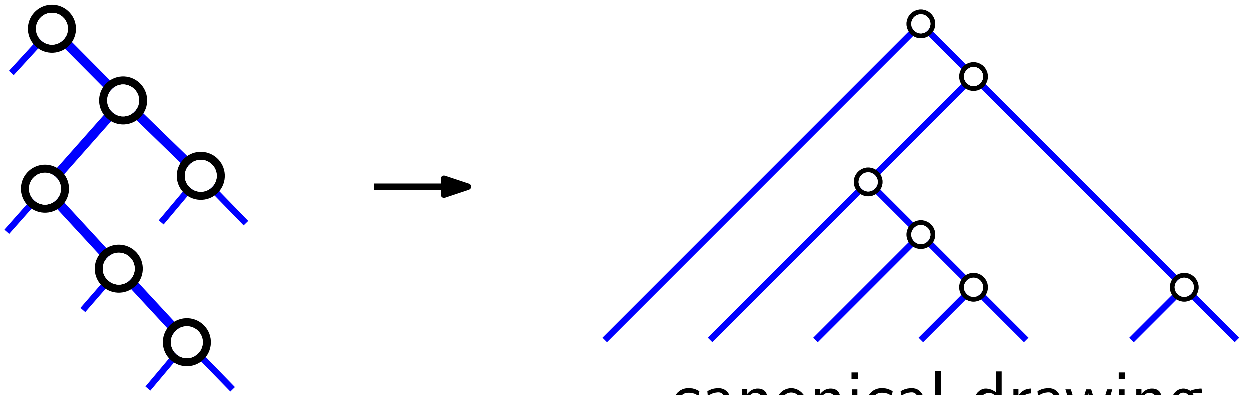


Direct bijection from Tamari intervals to blossoming trees

Representations of binary trees

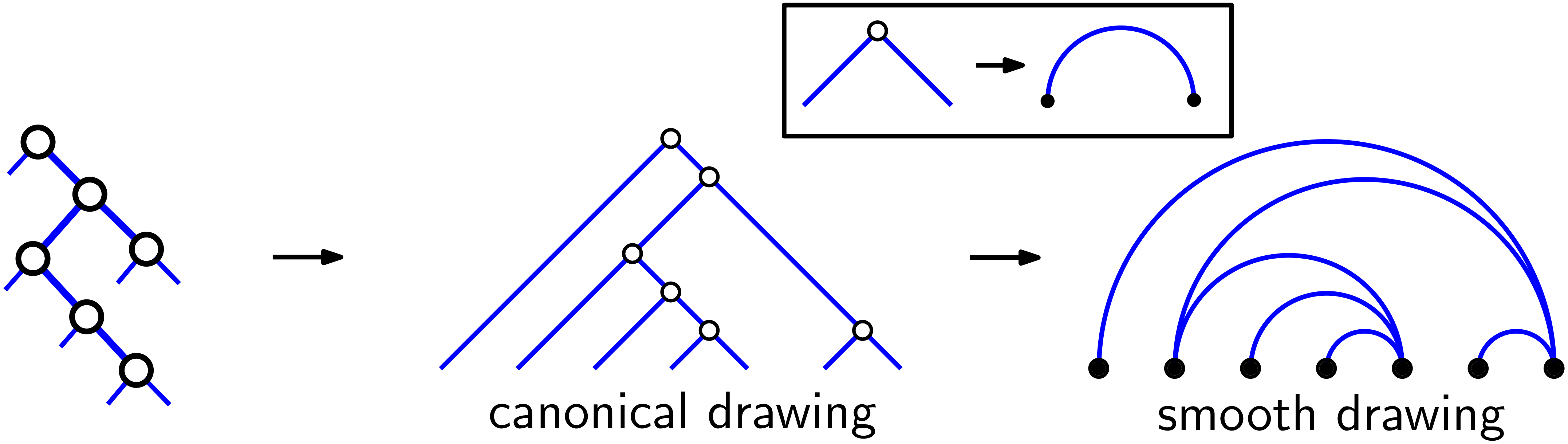


Representations of binary trees

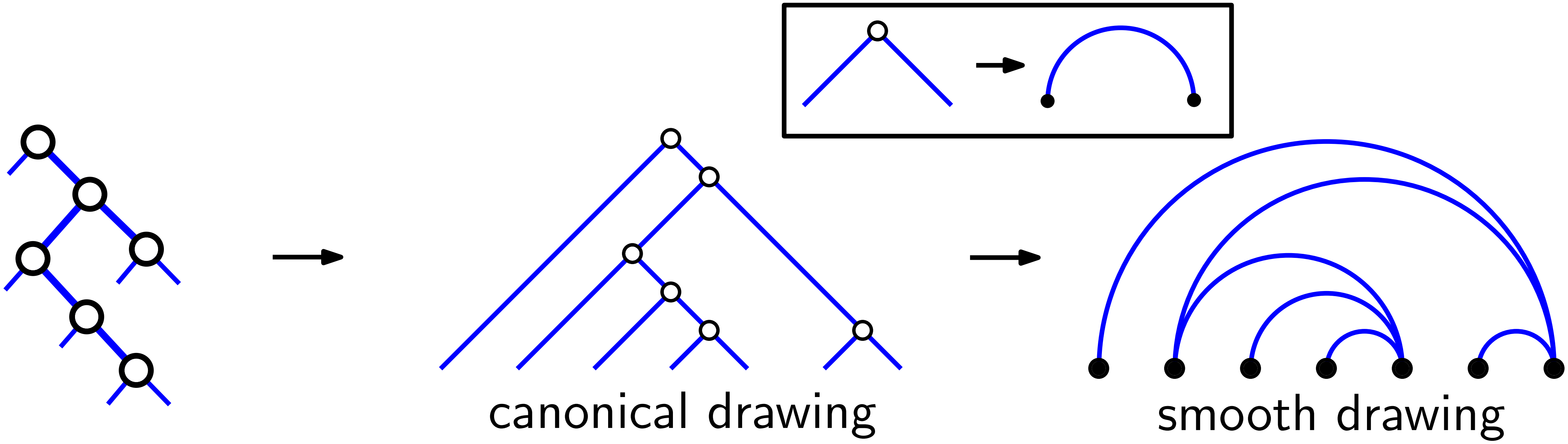


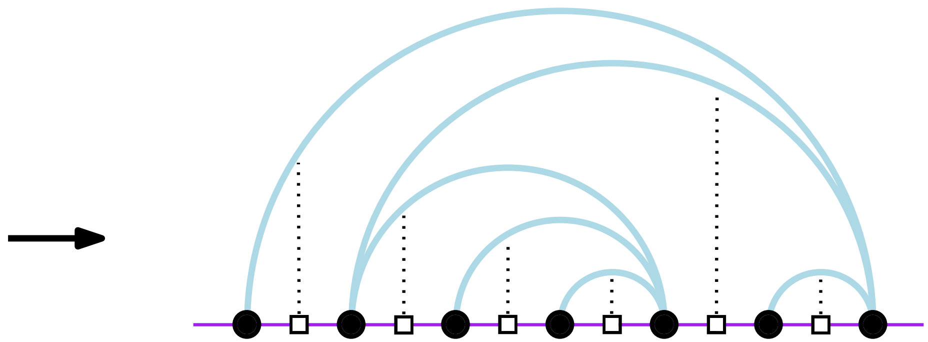
canonical drawing

Representations of binary trees

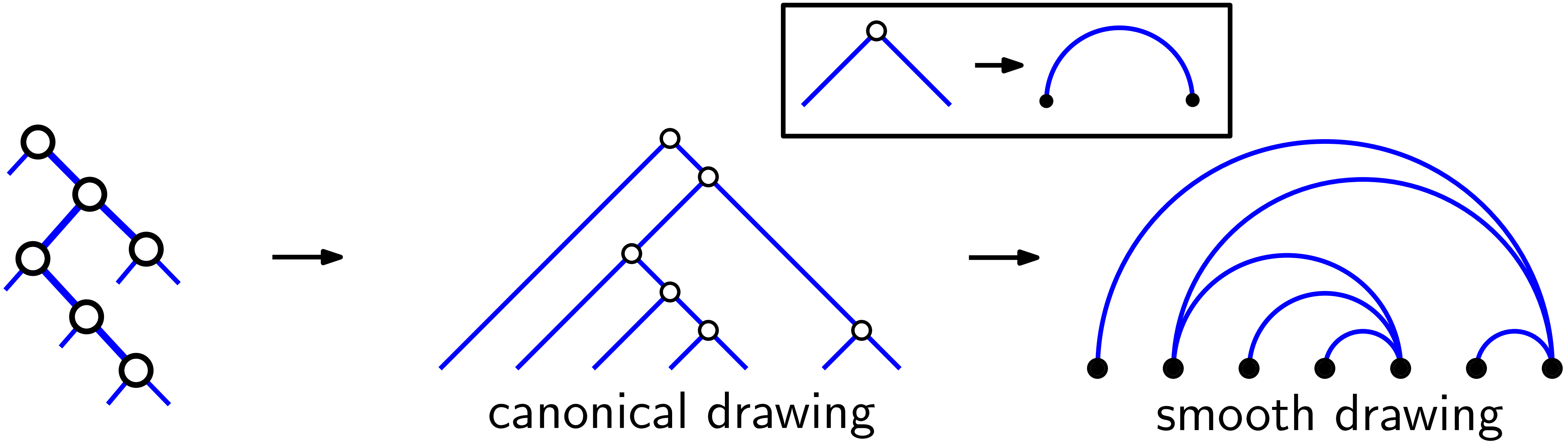


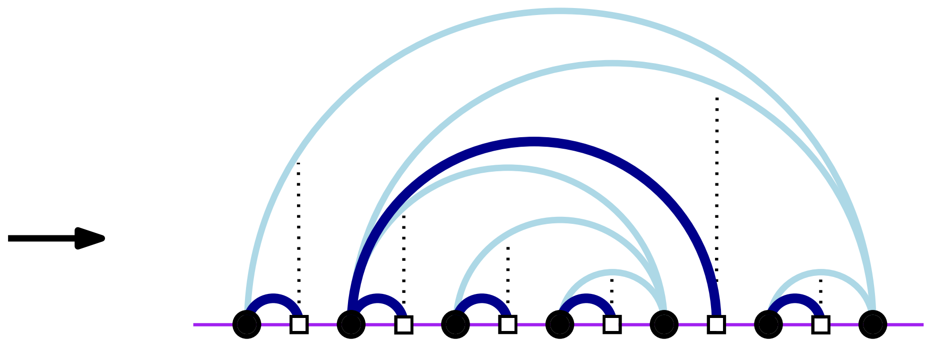
Representations of binary trees



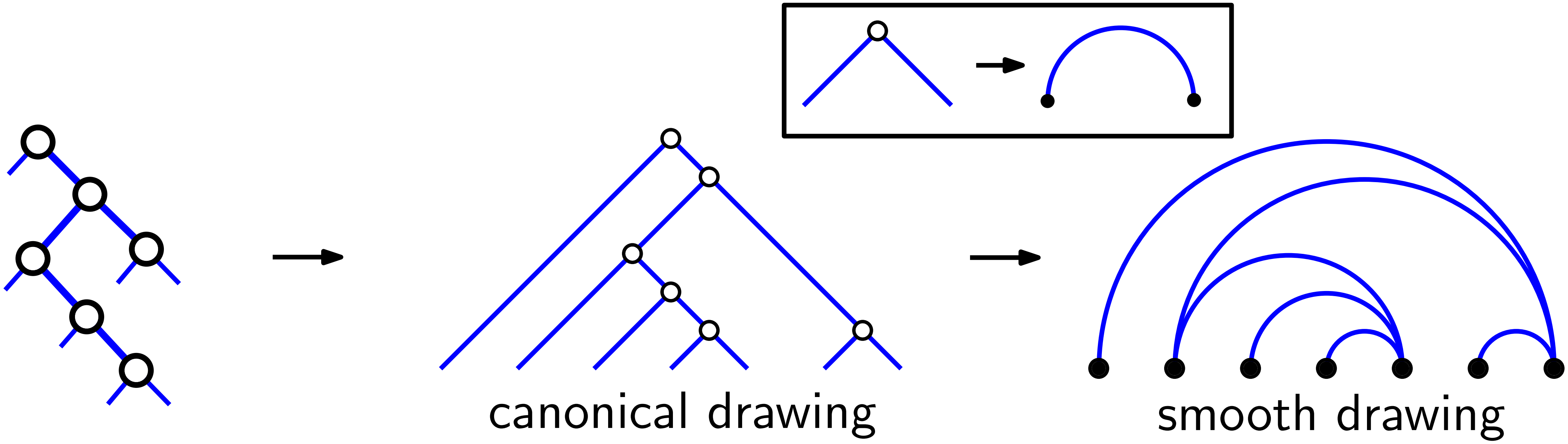


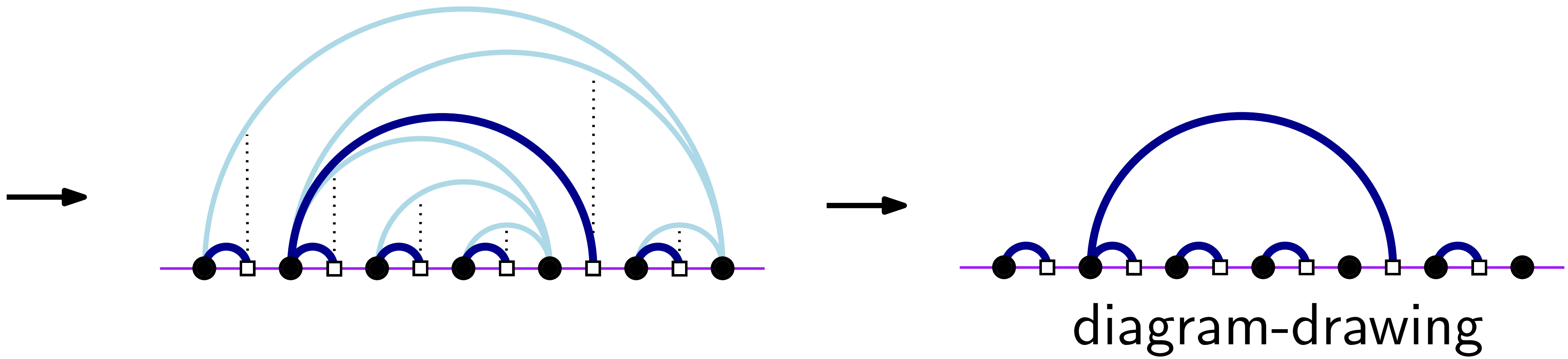
Representations of binary trees



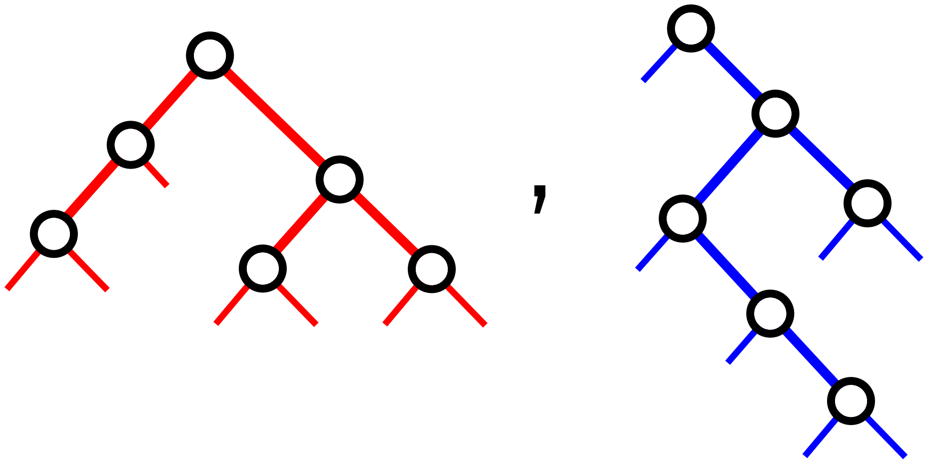


Representations of binary trees

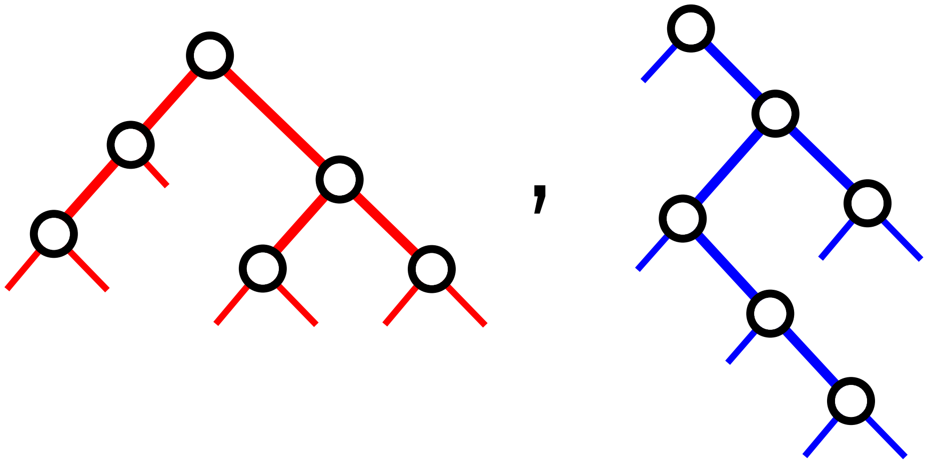


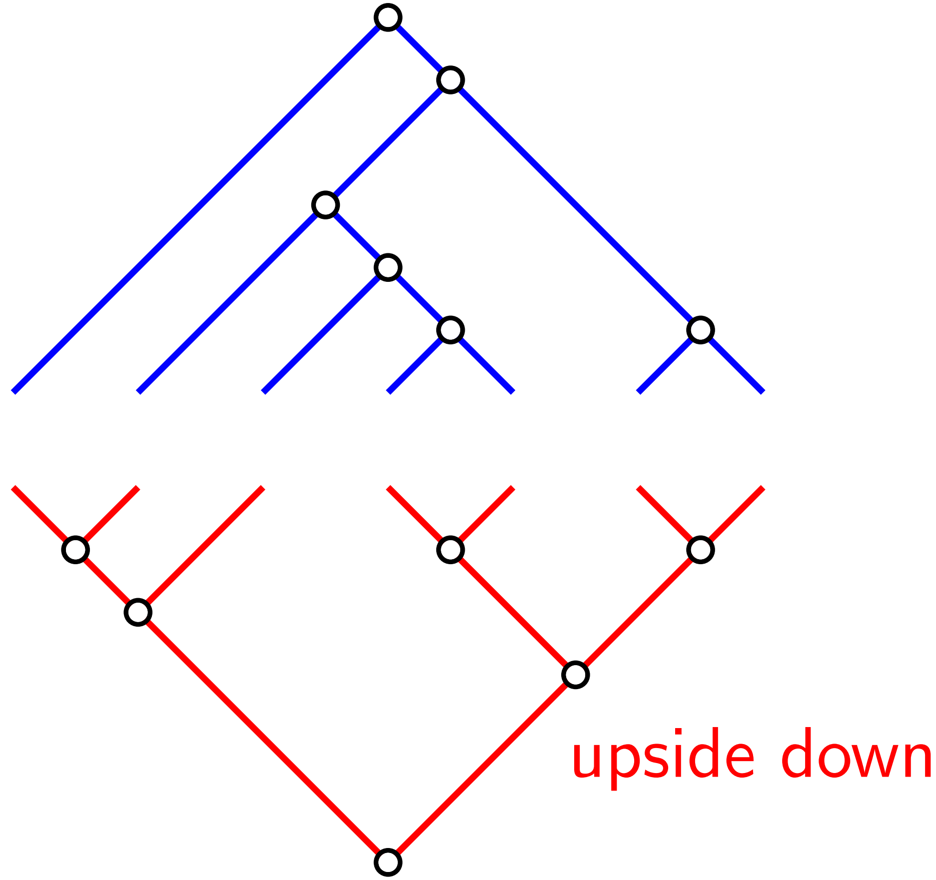


Representations of pairs of binary trees



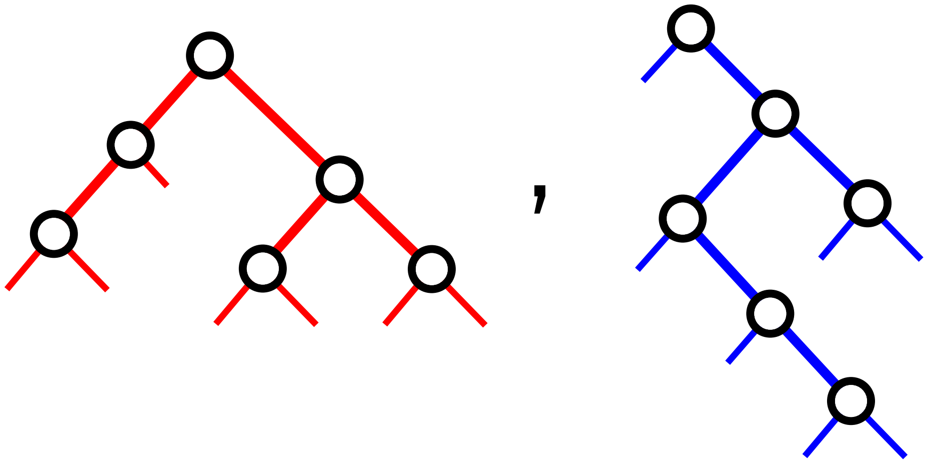
Representations of pairs of binary trees

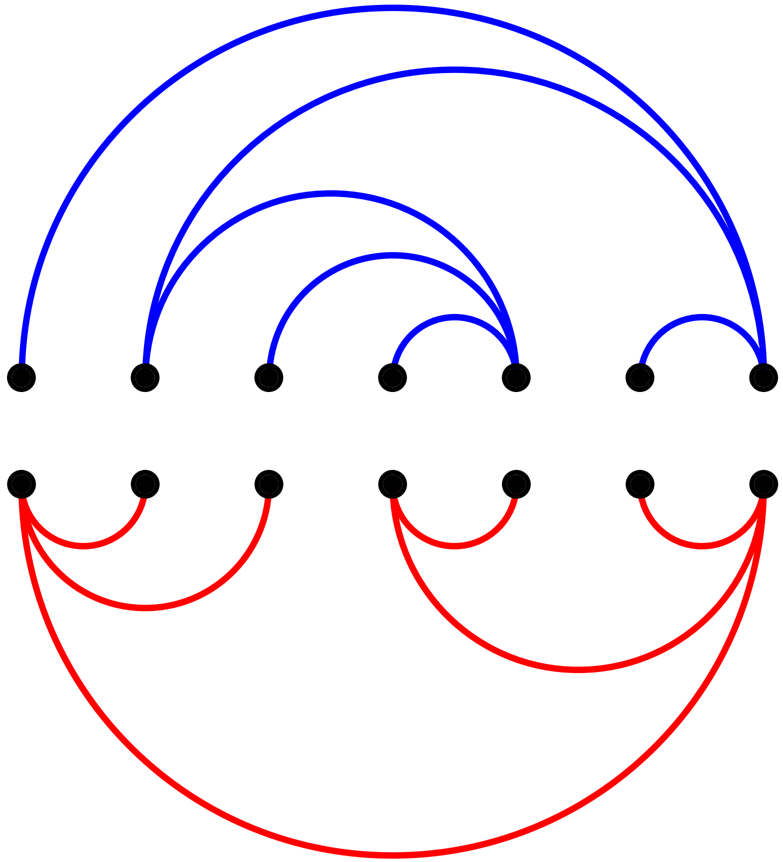




canonical drawings

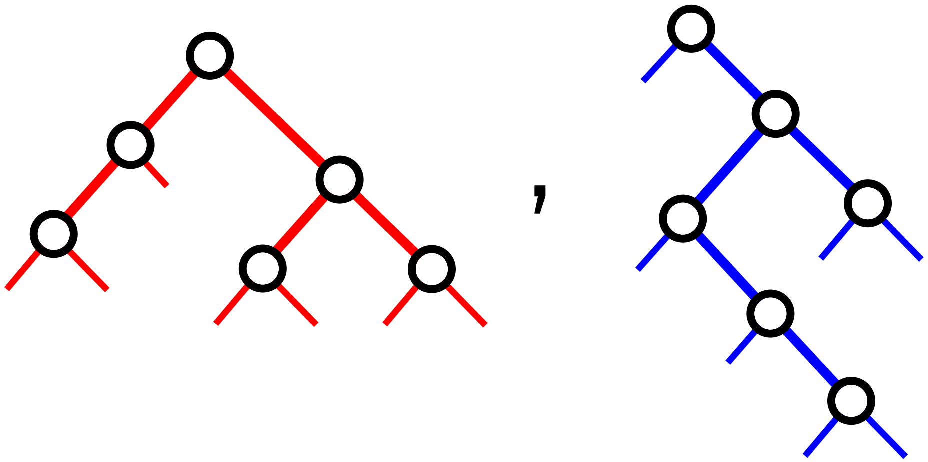
Representations of pairs of binary trees





smooth drawings

Representations of pairs of binary trees



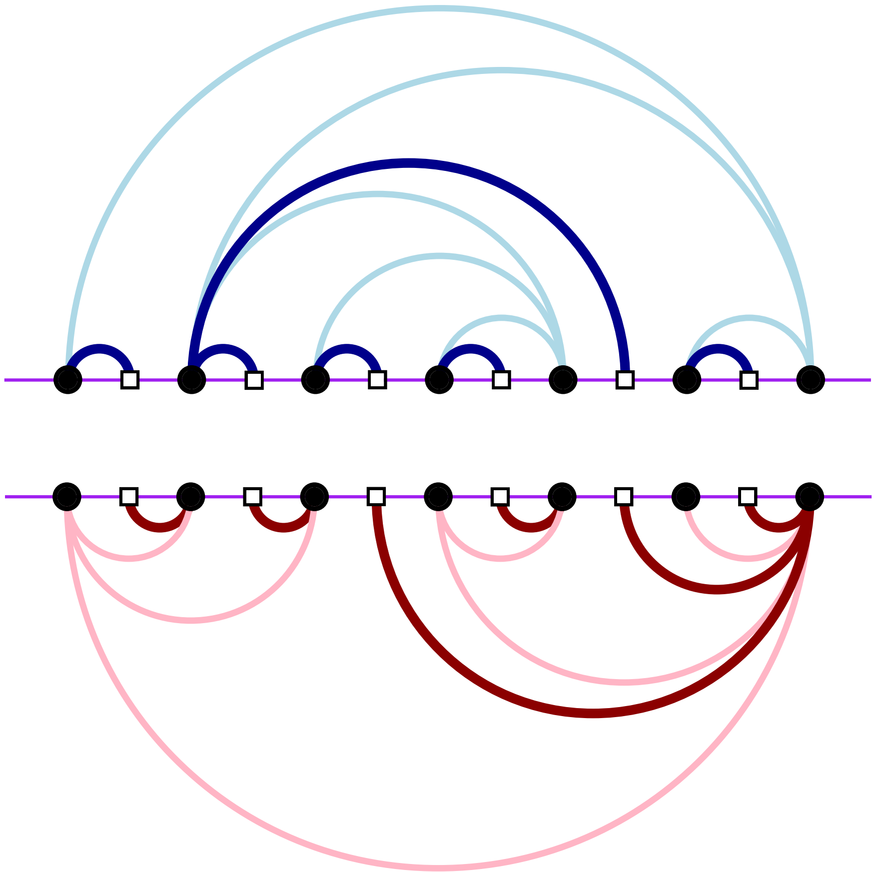
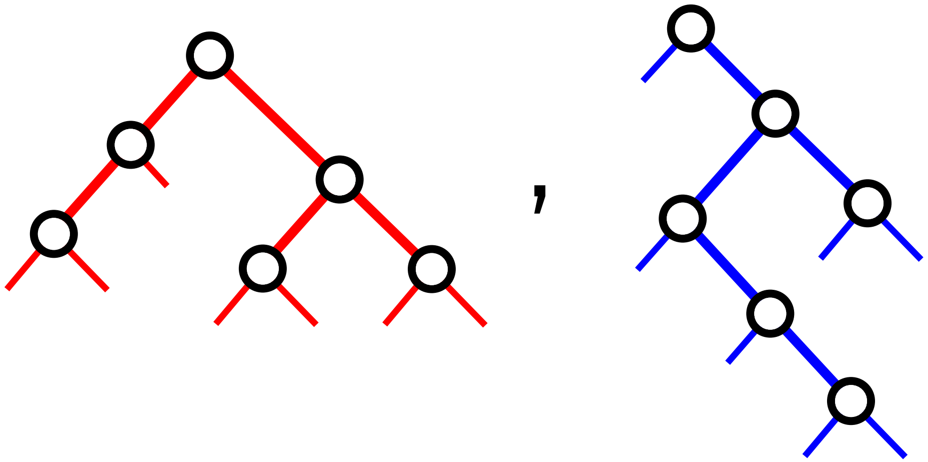


diagram drawings

Representations of pairs of binary trees



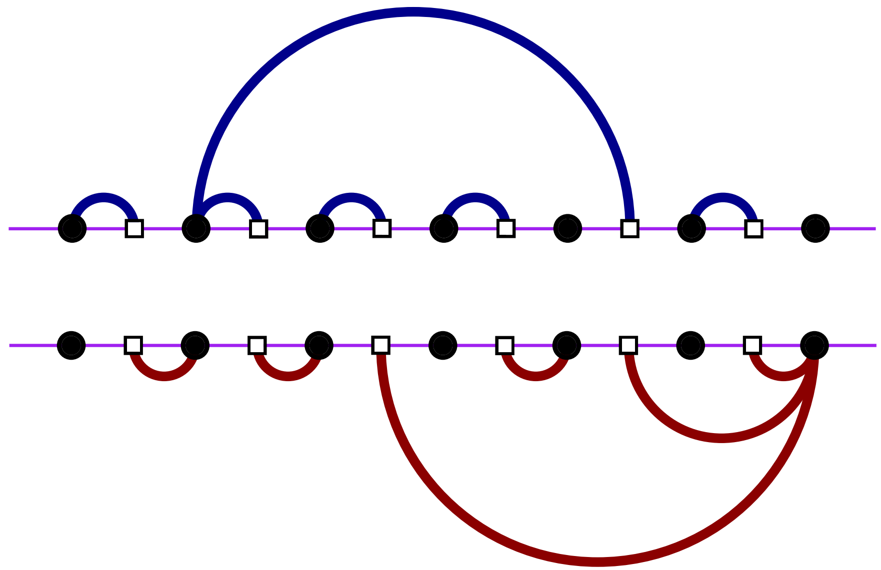
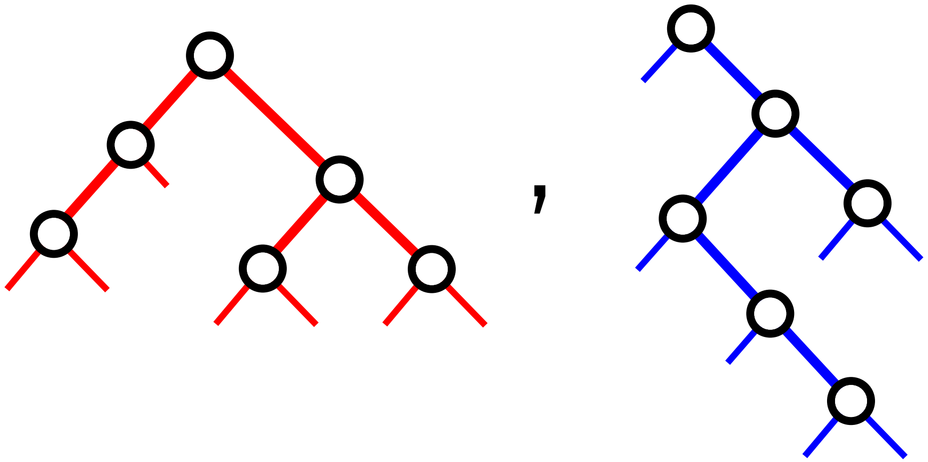
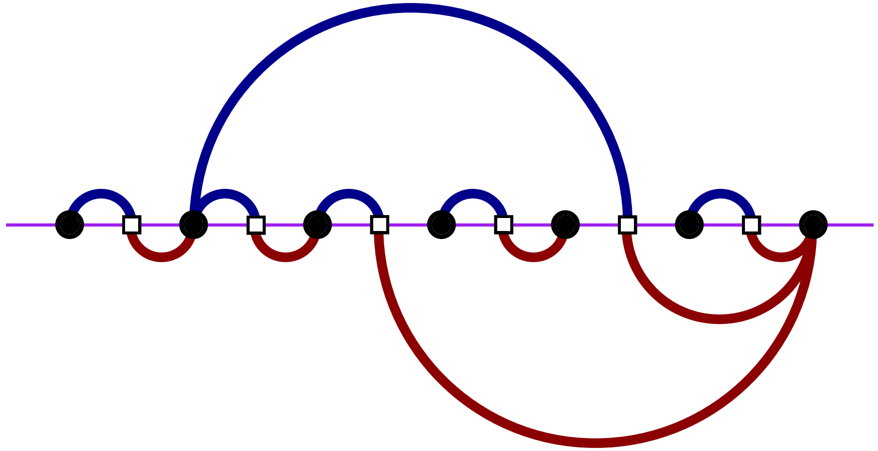


diagram drawings

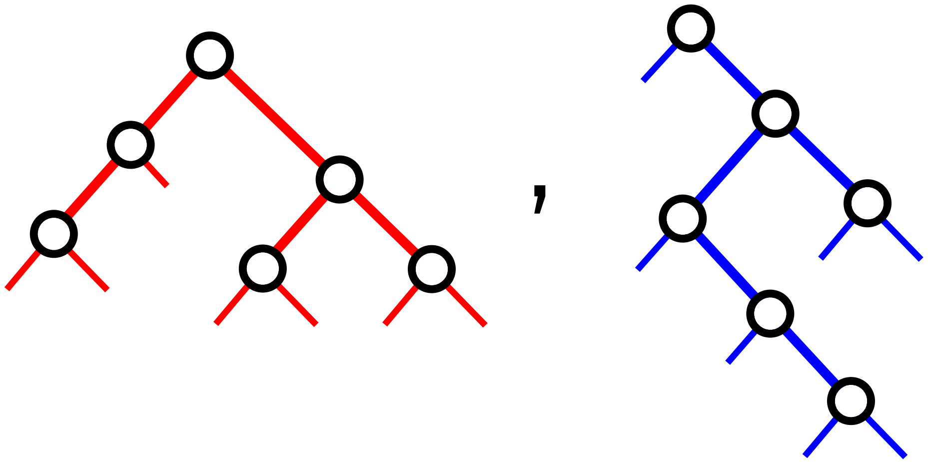
Representations of pairs of binary trees

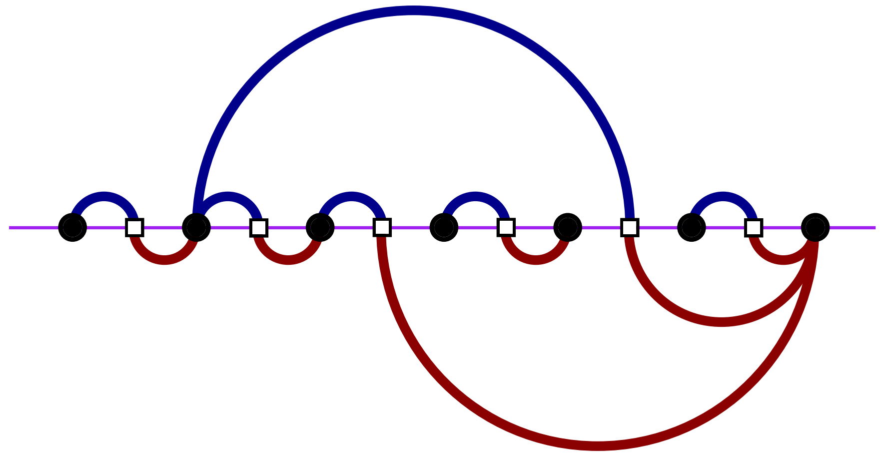




meandering diagram

Representations of pairs of binary trees



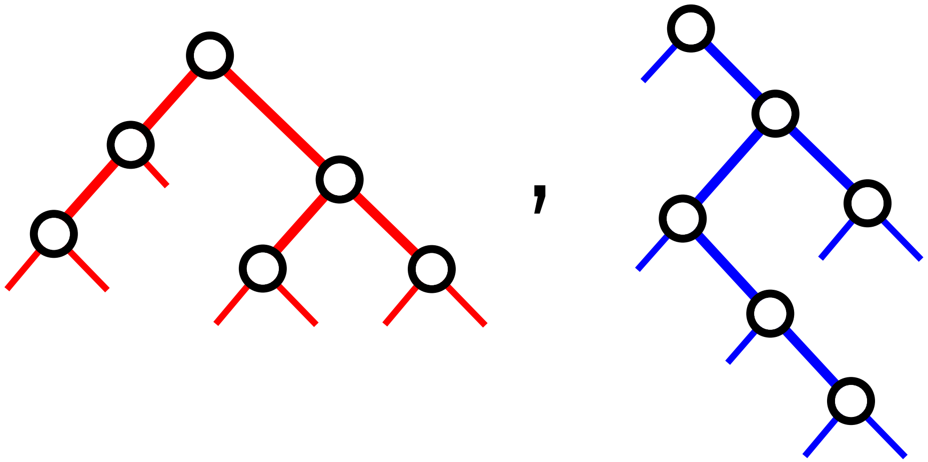


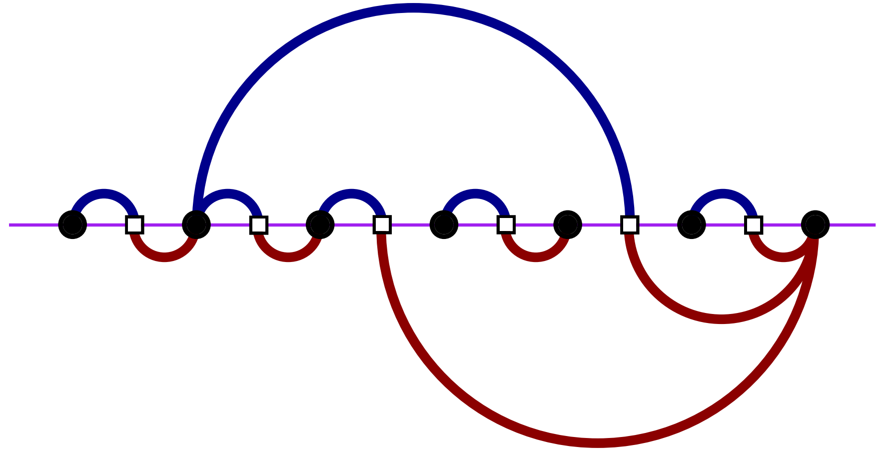
meandering diagram

meandering diagram M has underlying graph $G_M = (V, E)$

where $V \leftrightarrow \{\text{black points}\}$ and $E \leftrightarrow \{\text{white points}\}$

Representations of pairs of binary trees





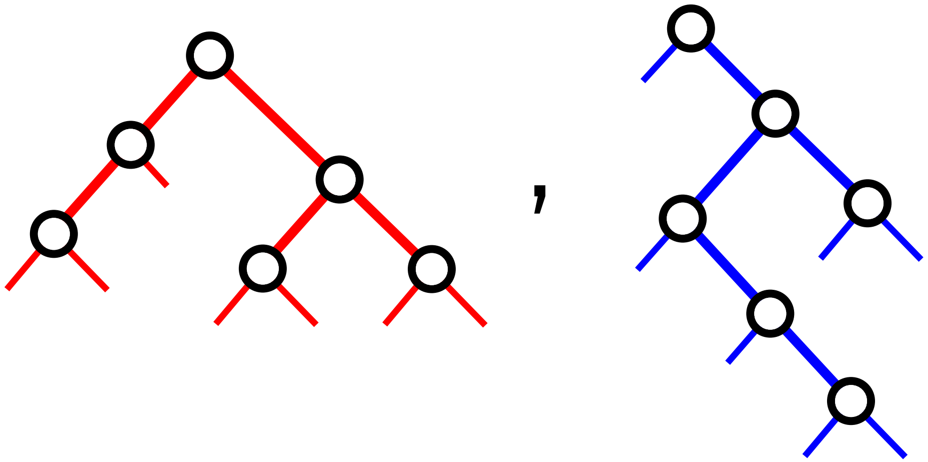
meandering diagram

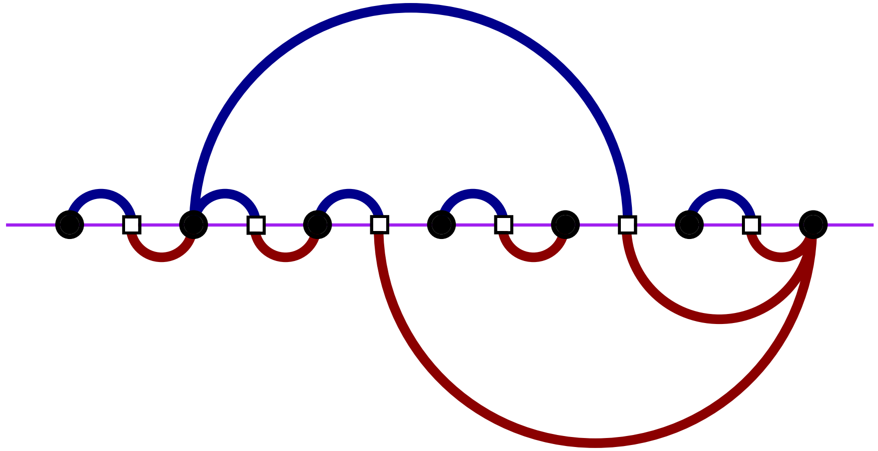
meandering diagram M has underlying graph $G_M = (V, E)$

where $V \leftrightarrow \{\text{black points}\}$ and $E \leftrightarrow \{\text{white points}\}$

(one more vertices than edges)

Representations of pairs of binary trees





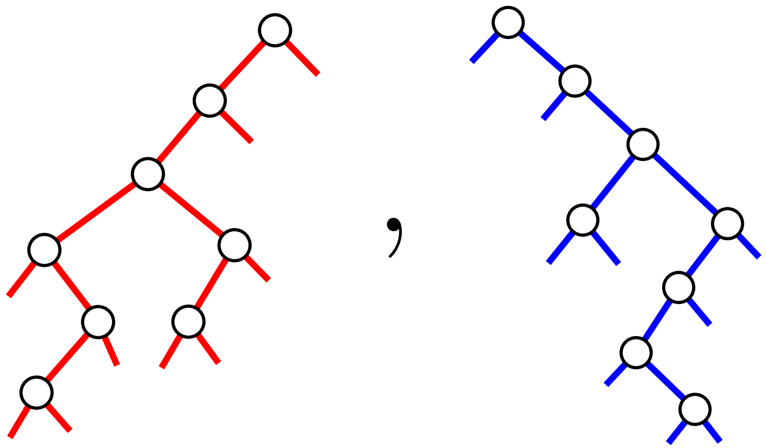
meandering diagram

meandering diagram M has underlying graph $G_M = (V, E)$
where $V \leftrightarrow \{\text{black points}\}$ and $E \leftrightarrow \{\text{white points}\}$
(one more vertices than edges)

Def: A meandering tree is a meandering diagram M such that G_M is a tree

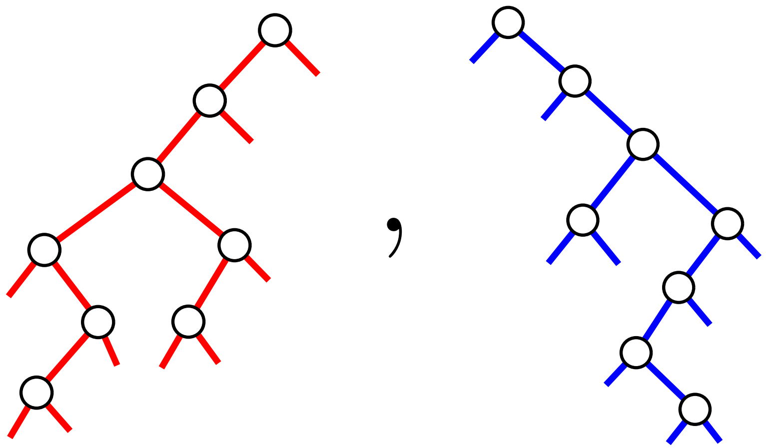
Tamari intervals correspond to meandering trees

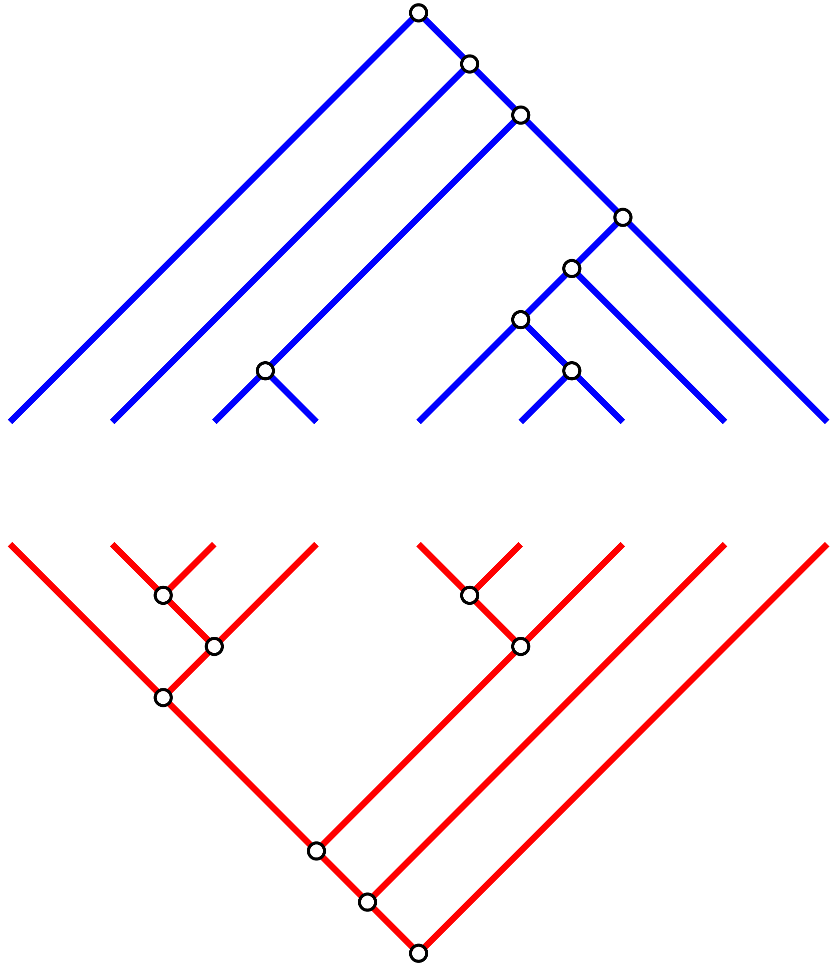
Theo: Let M = meandering diagram of (T, T').
Then M is a meandering tree iff $T \leq T'$



Tamari intervals correspond to meandering trees

Theo: Let M = meandering diagram of (T, T').
Then M is a meandering tree iff $T \leq T'$

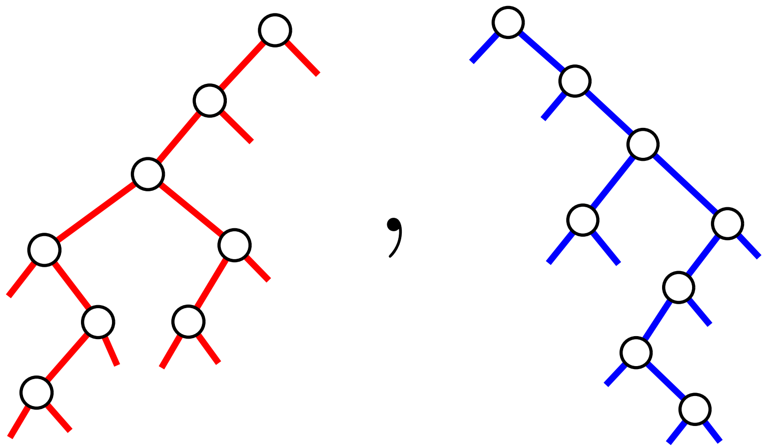


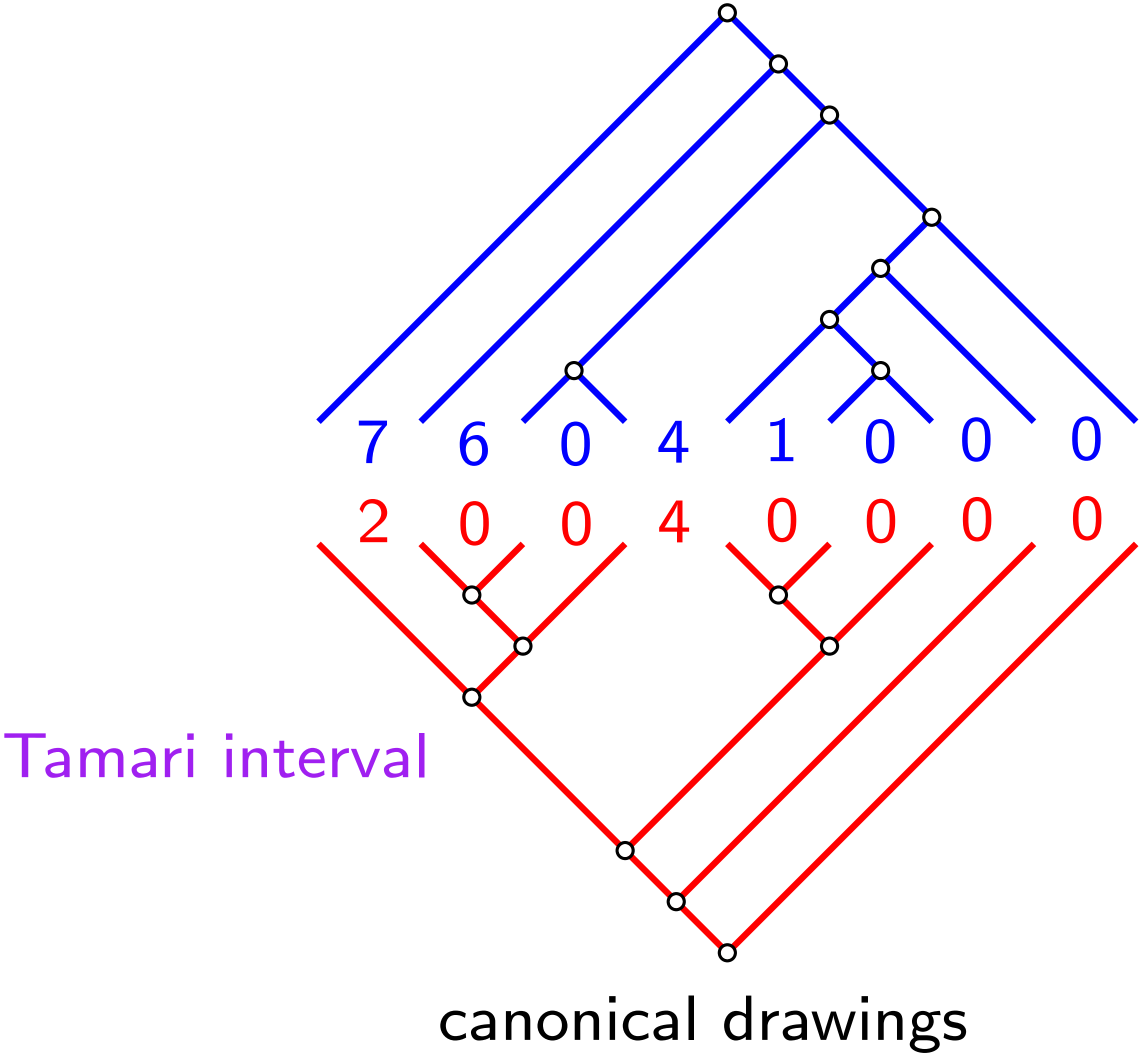


canonical drawings

Tamari intervals correspond to meandering trees

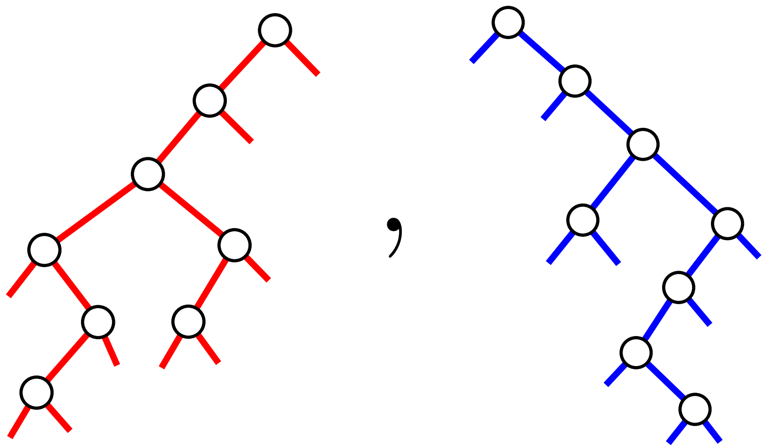
Theo: Let M = meandering diagram of (T, T').
Then M is a meandering tree iff $T \leq T'$

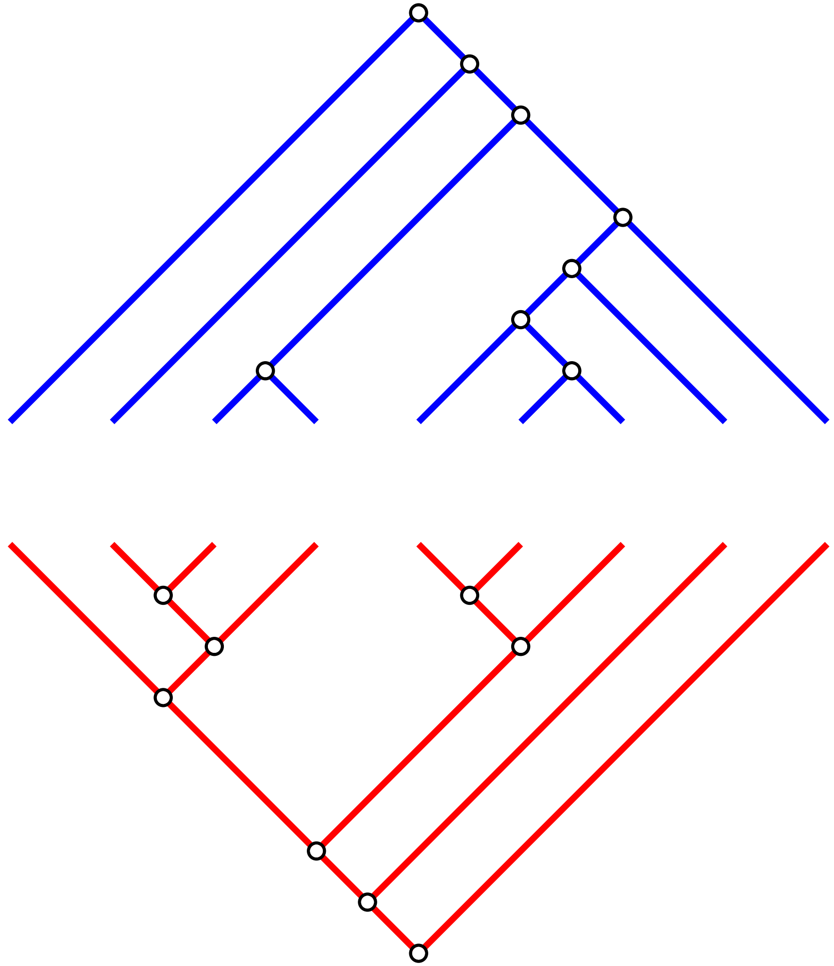




Tamari intervals correspond to meandering trees

Theo: Let $M =$ meandering diagram of (T, T').
Then M is a meandering tree iff $T \leq T'$

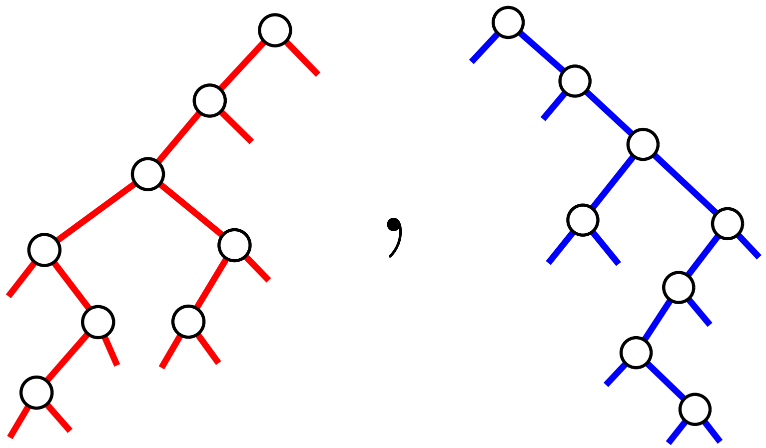


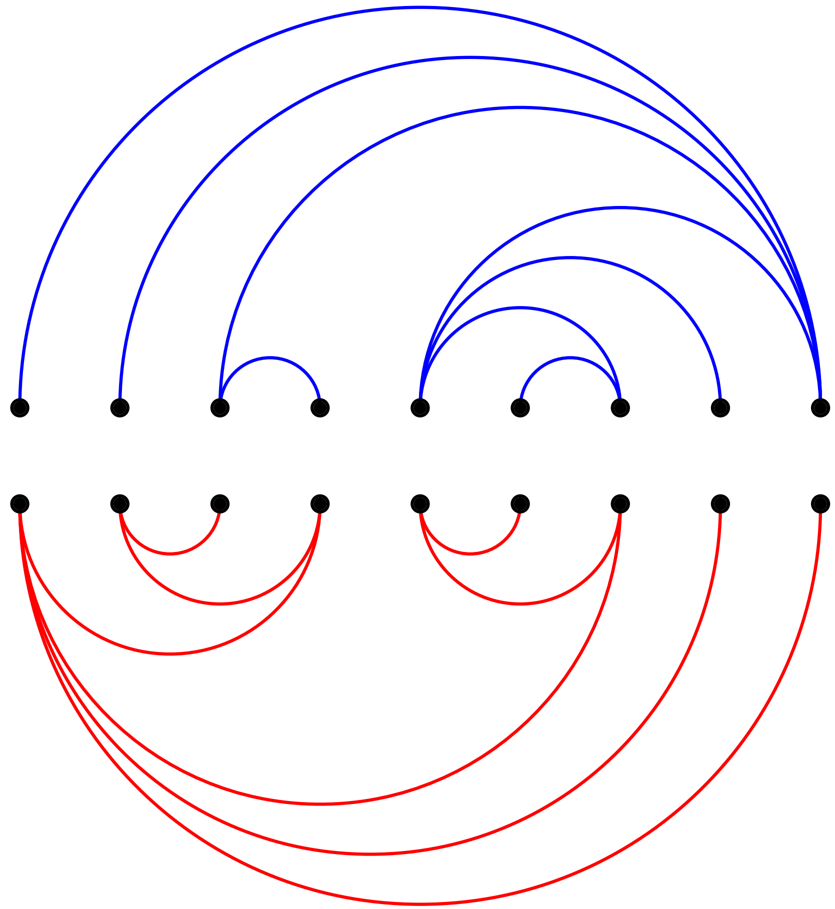


canonical drawings

Tamari intervals correspond to meandering trees

Theo: Let $M =$ meandering diagram of (T, T').
Then M is a meandering tree iff $T \leq T'$

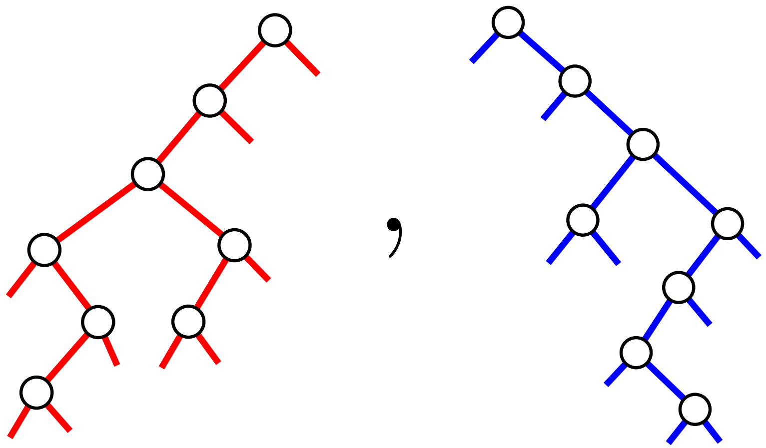




smooth drawings

Tamari intervals correspond to meandering trees

Theo: Let M = meandering diagram of (T, T').
Then M is a meandering tree iff $T \leq T'$



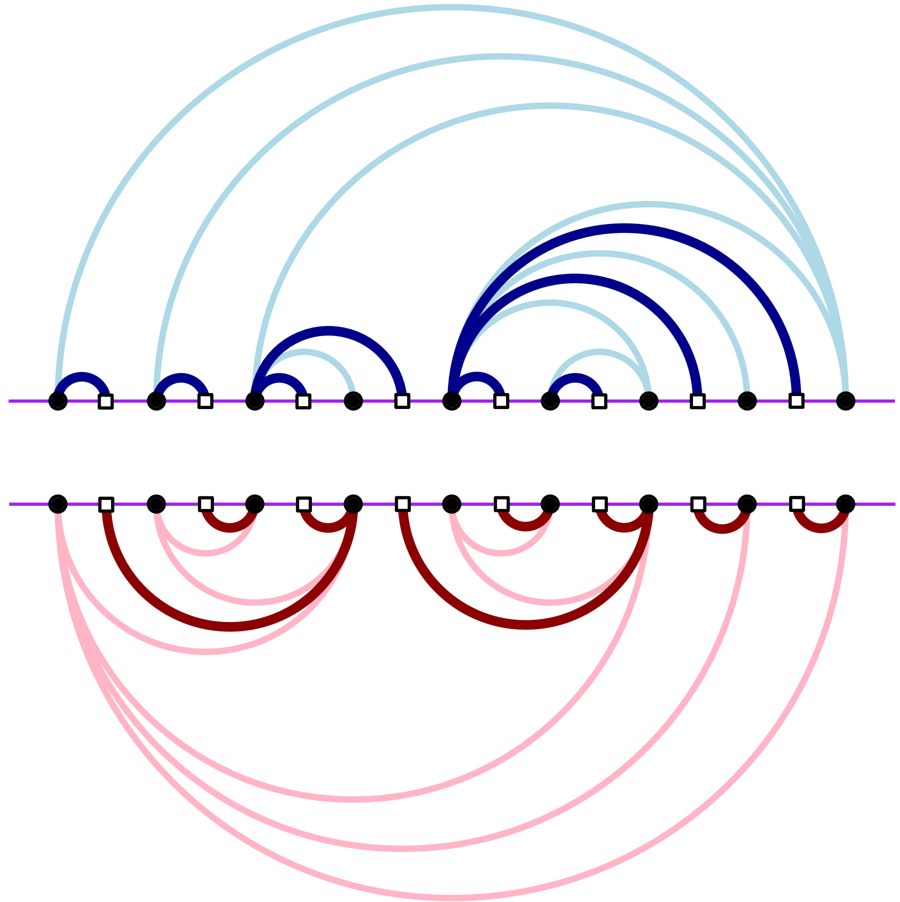
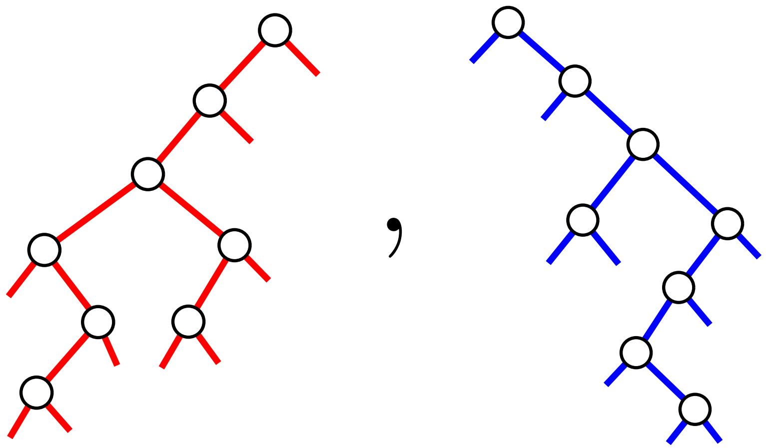
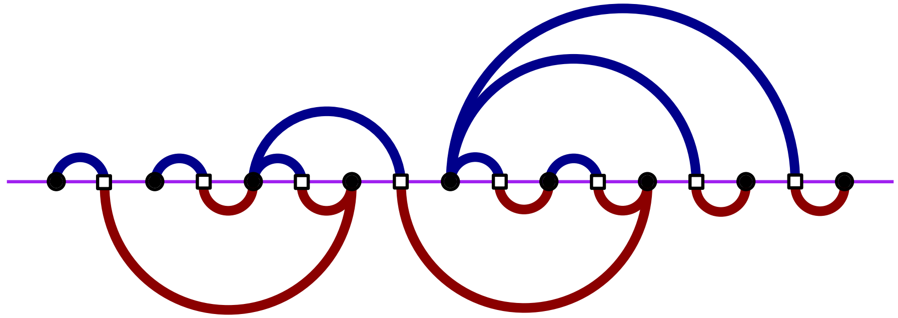


diagram drawings

Tamari intervals correspond to meandering trees

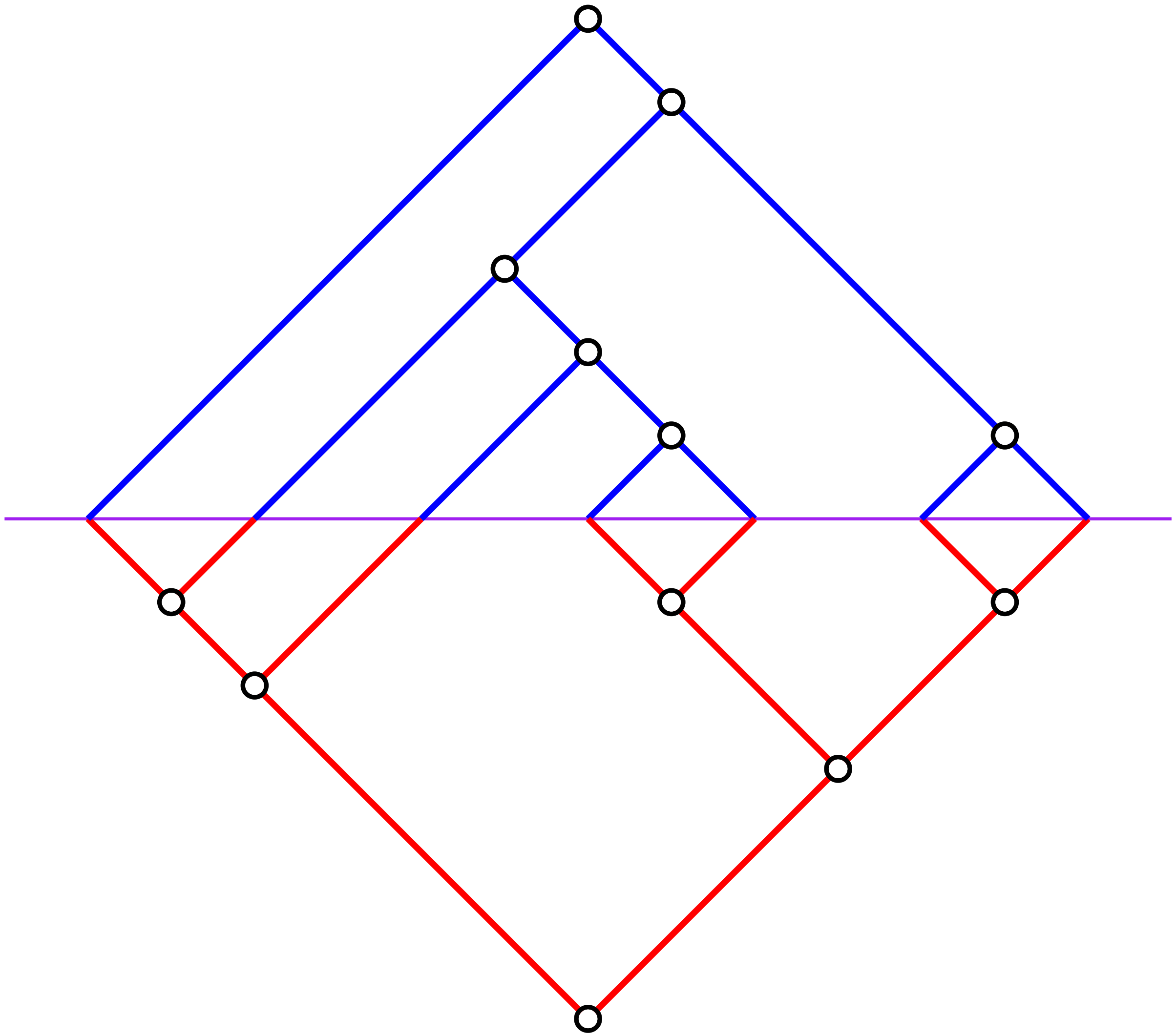
Theo: Let M = meandering diagram of (T, T').
Then M is a meandering tree iff $T \leq T'$



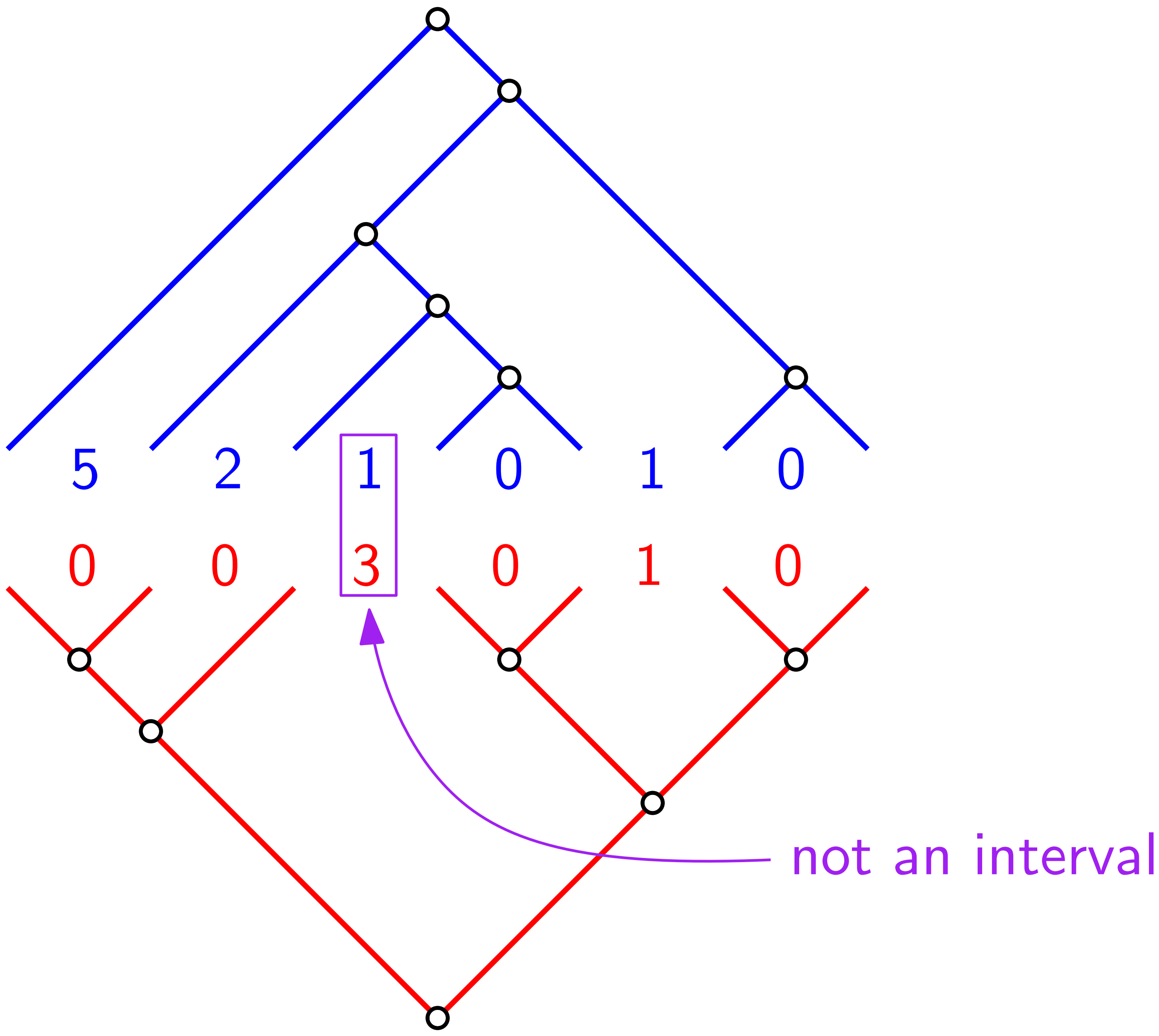


meandering tree

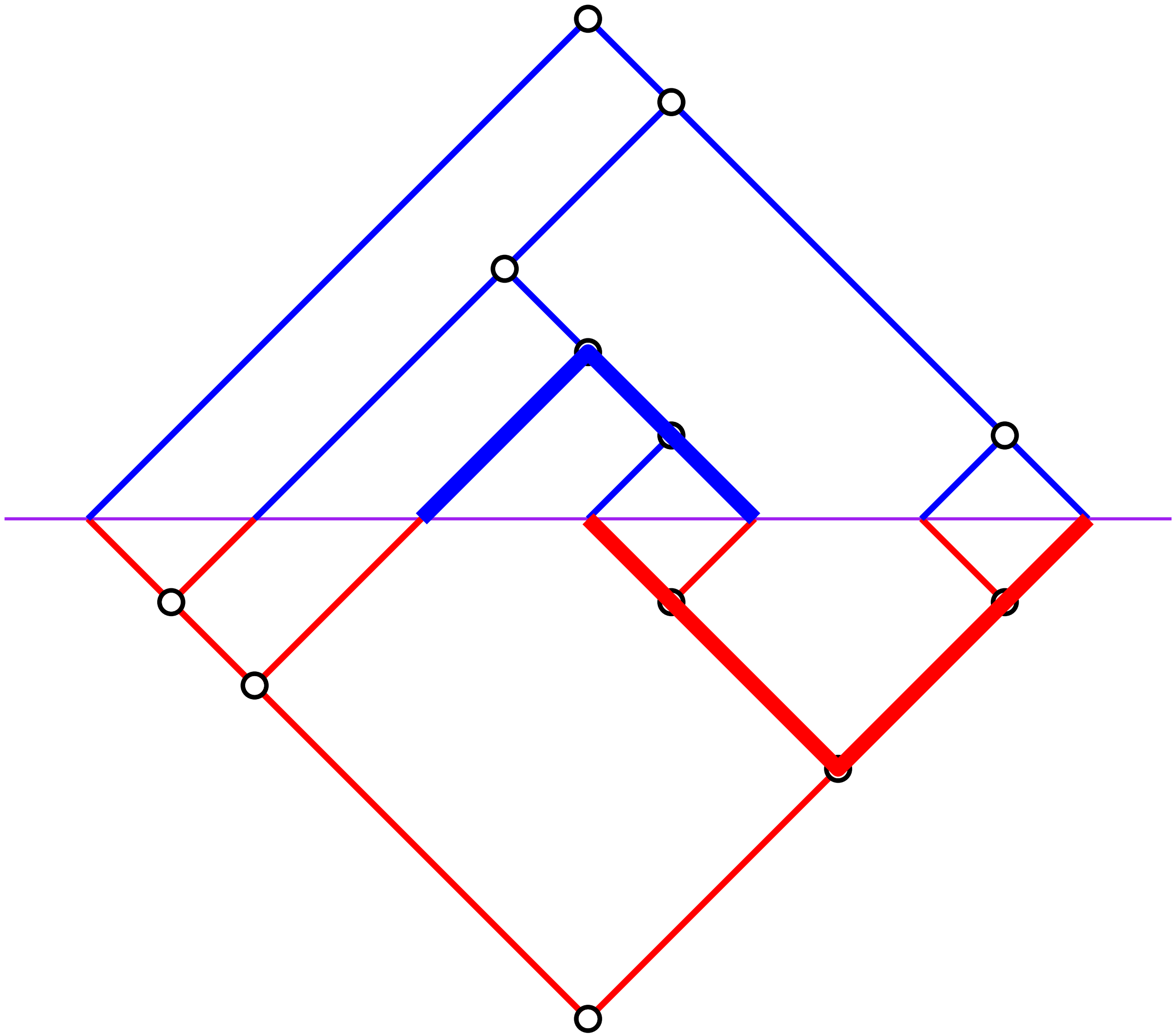
A pattern-avoiding explanation



A pattern-avoiding explanation



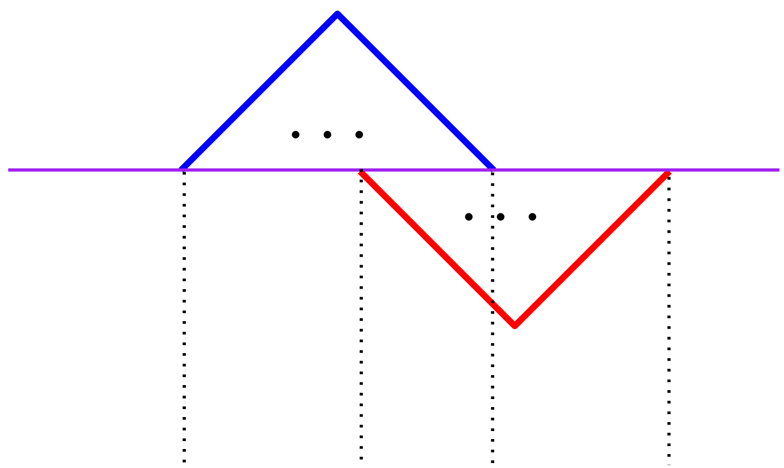
A pattern-avoiding explanation



not an interval

$\Updownarrow$

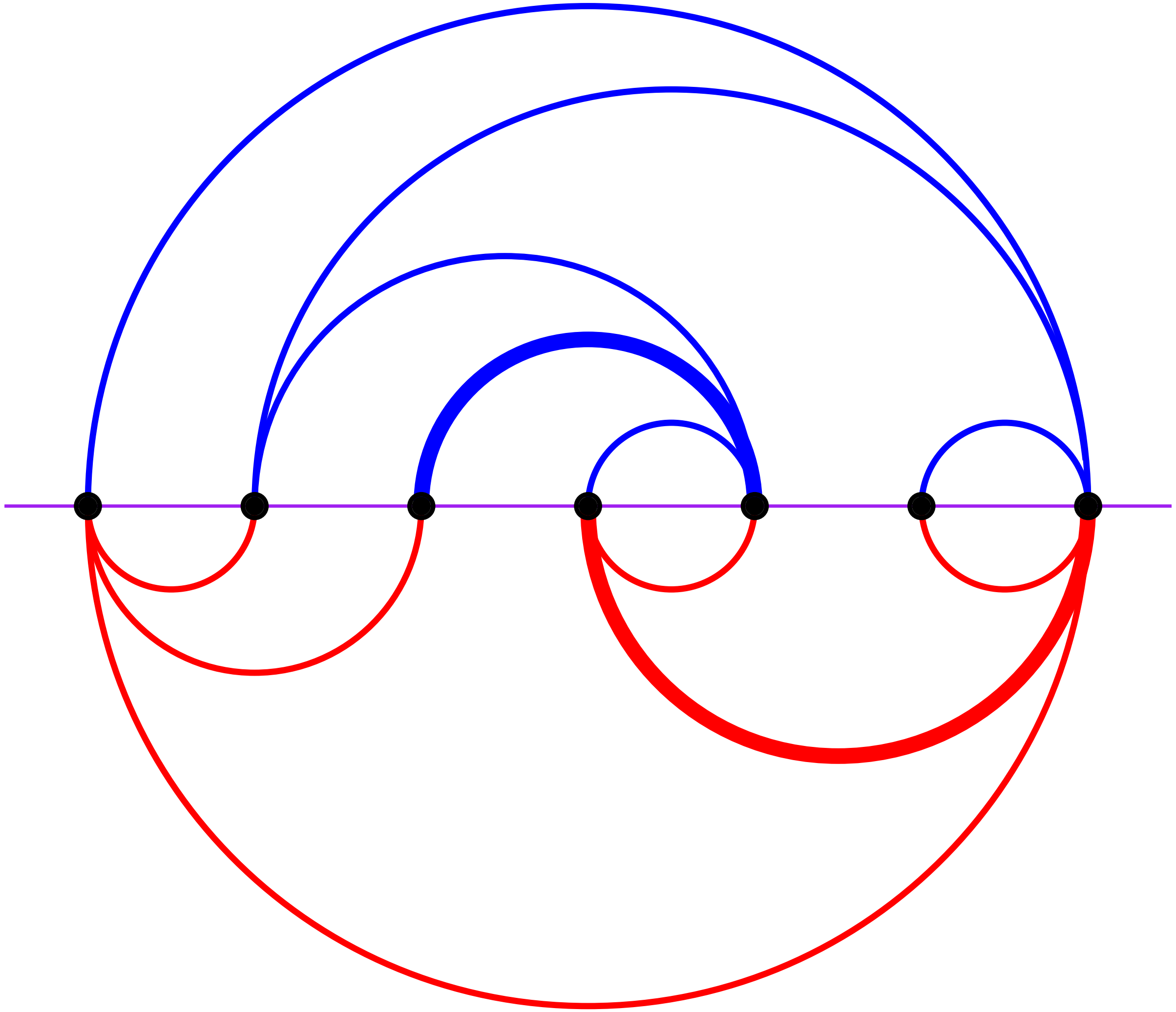
contains pattern



$a < b \leq c < d$

cf [Combe'19]

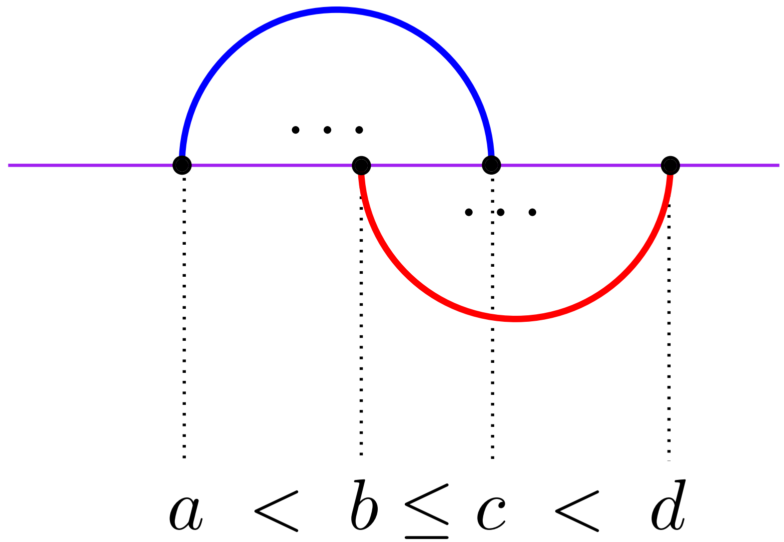
A pattern-avoiding explanation



not an interval

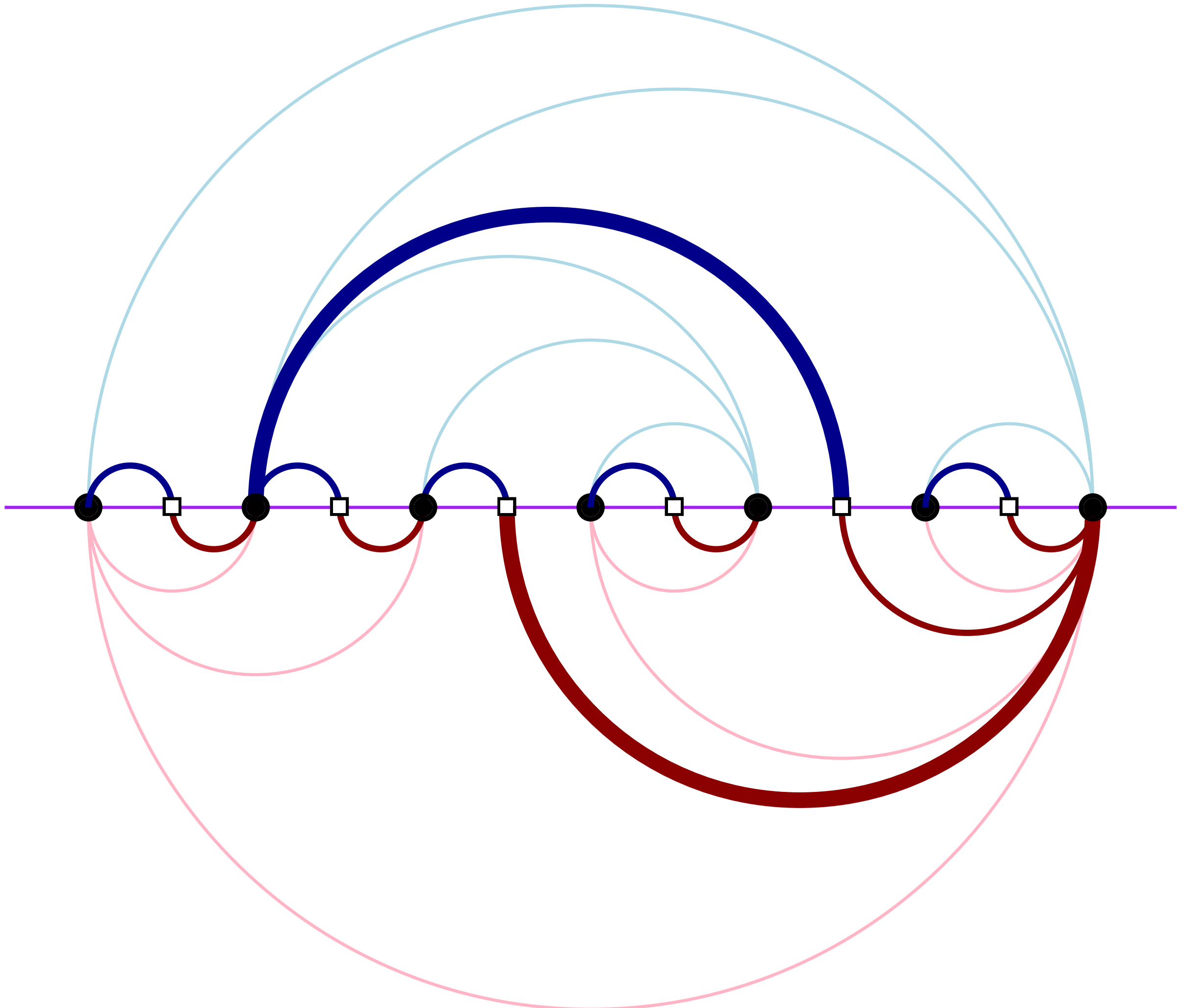
$\Updownarrow$

contains pattern



cf [Combe'19]

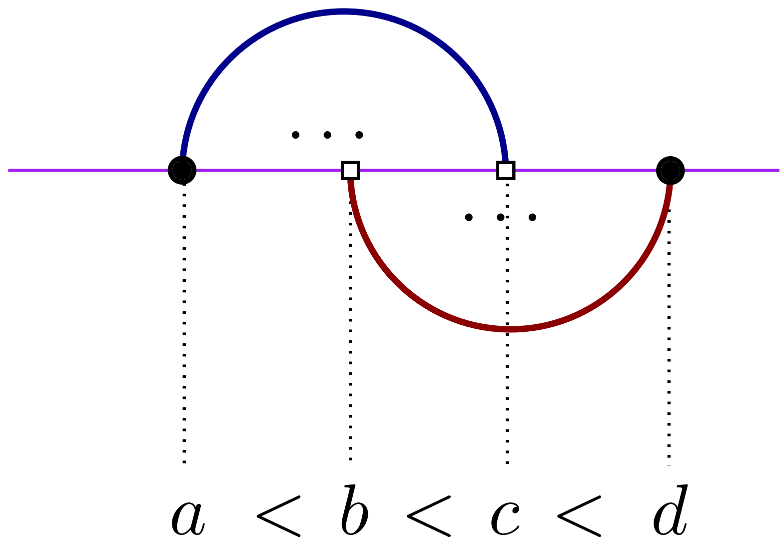
A pattern-avoiding explanation



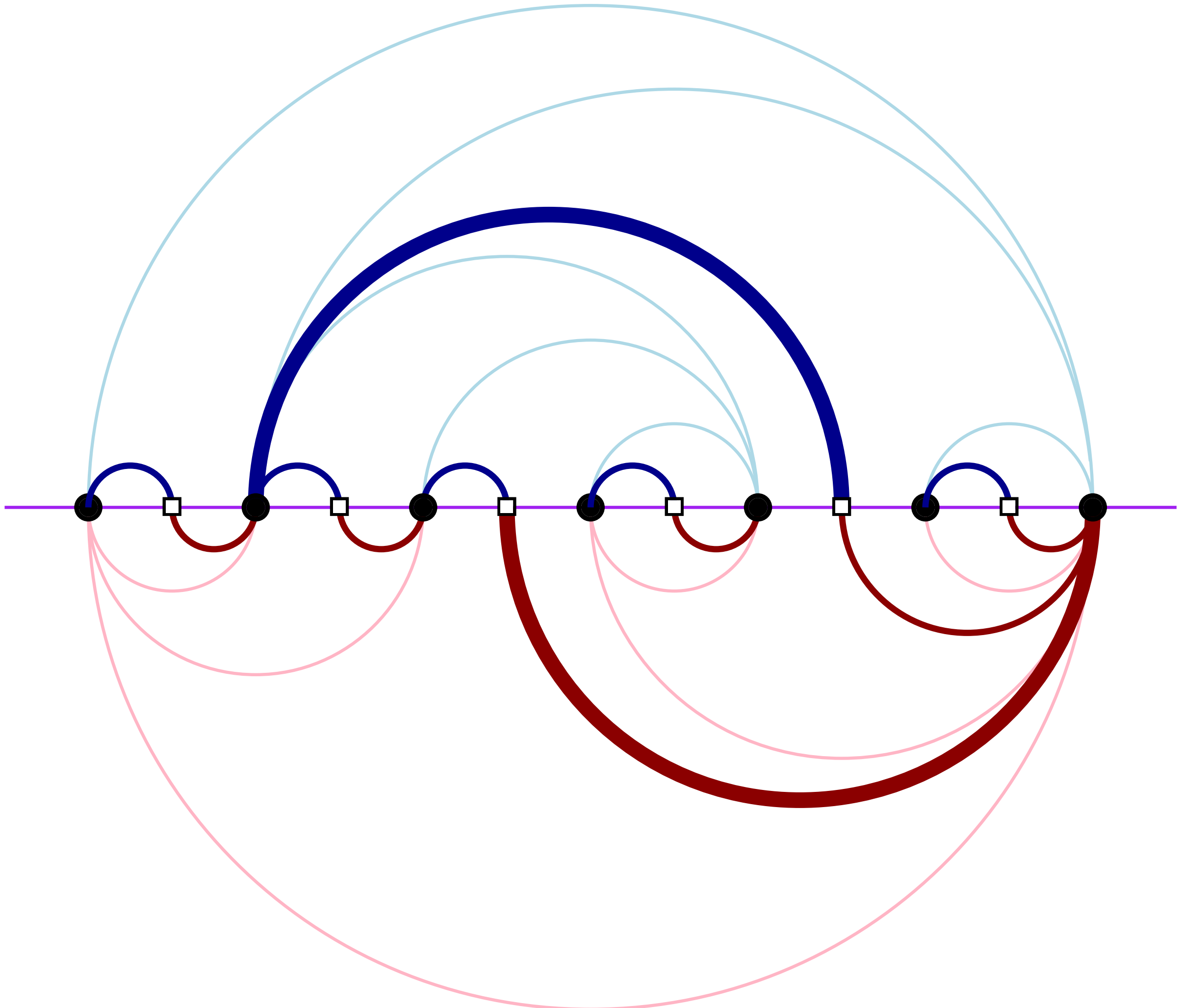
not an interval

⇕

contains pattern



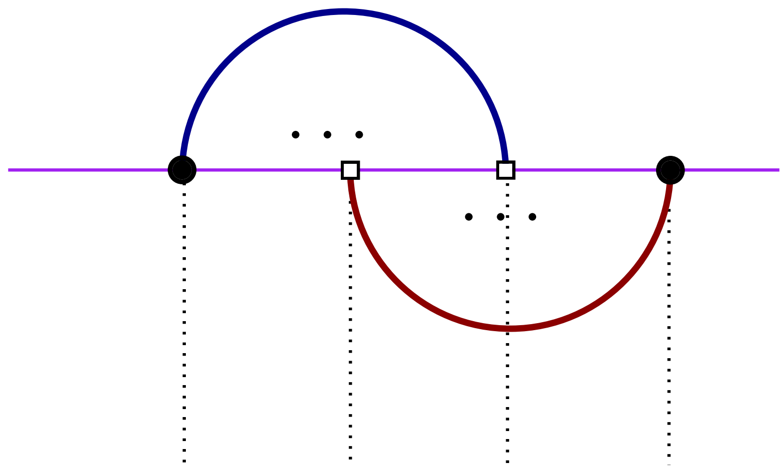
A pattern-avoiding explanation



not an interval

$\Updownarrow$

contains pattern

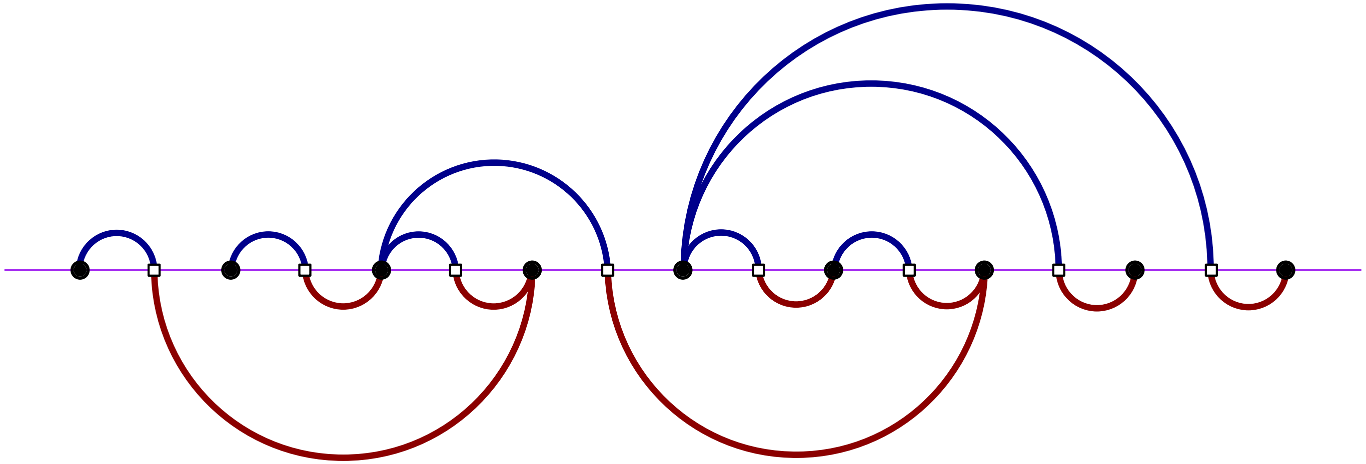


$a < b < c < d$

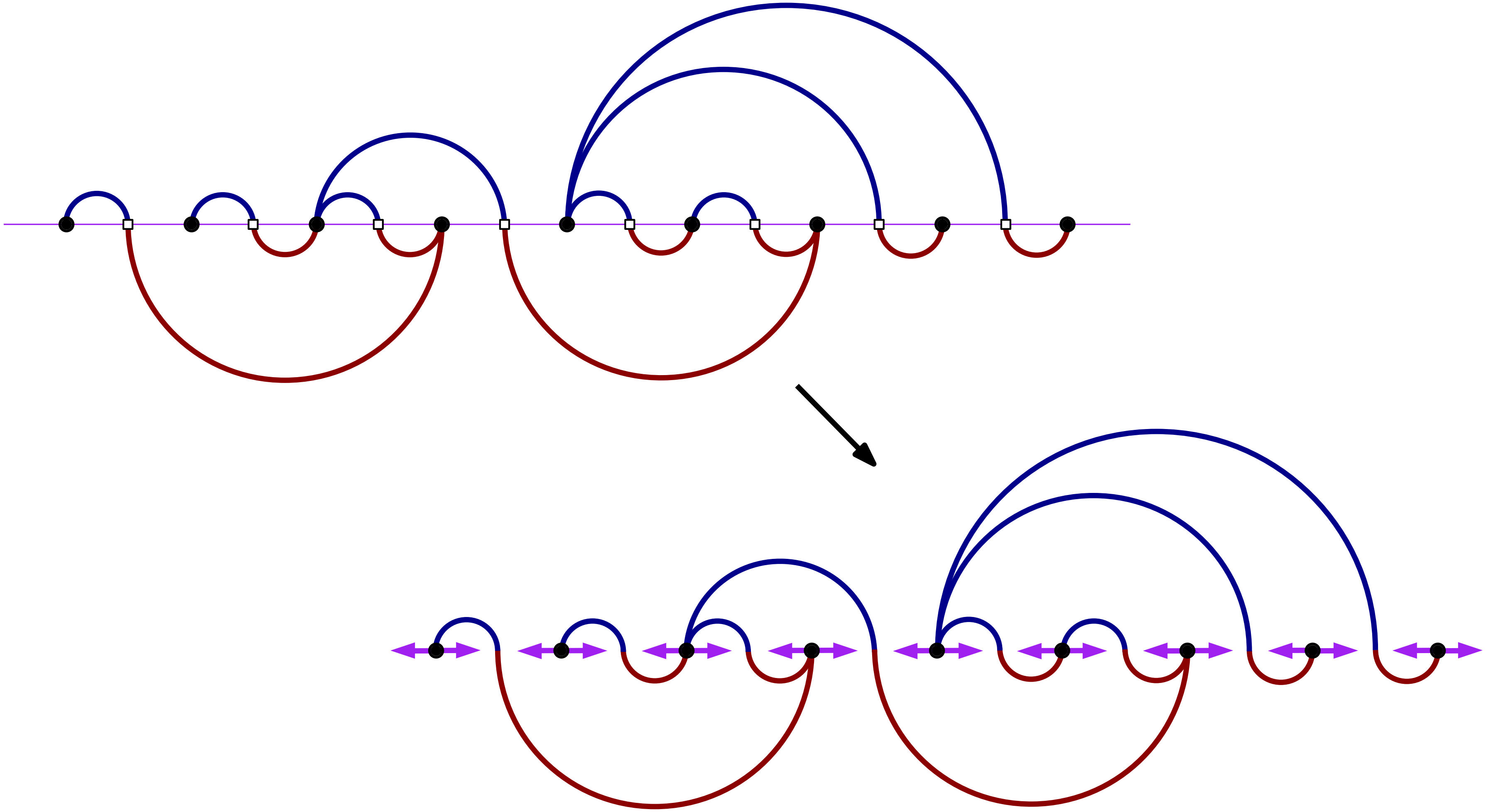
$\Updownarrow$

not a tree

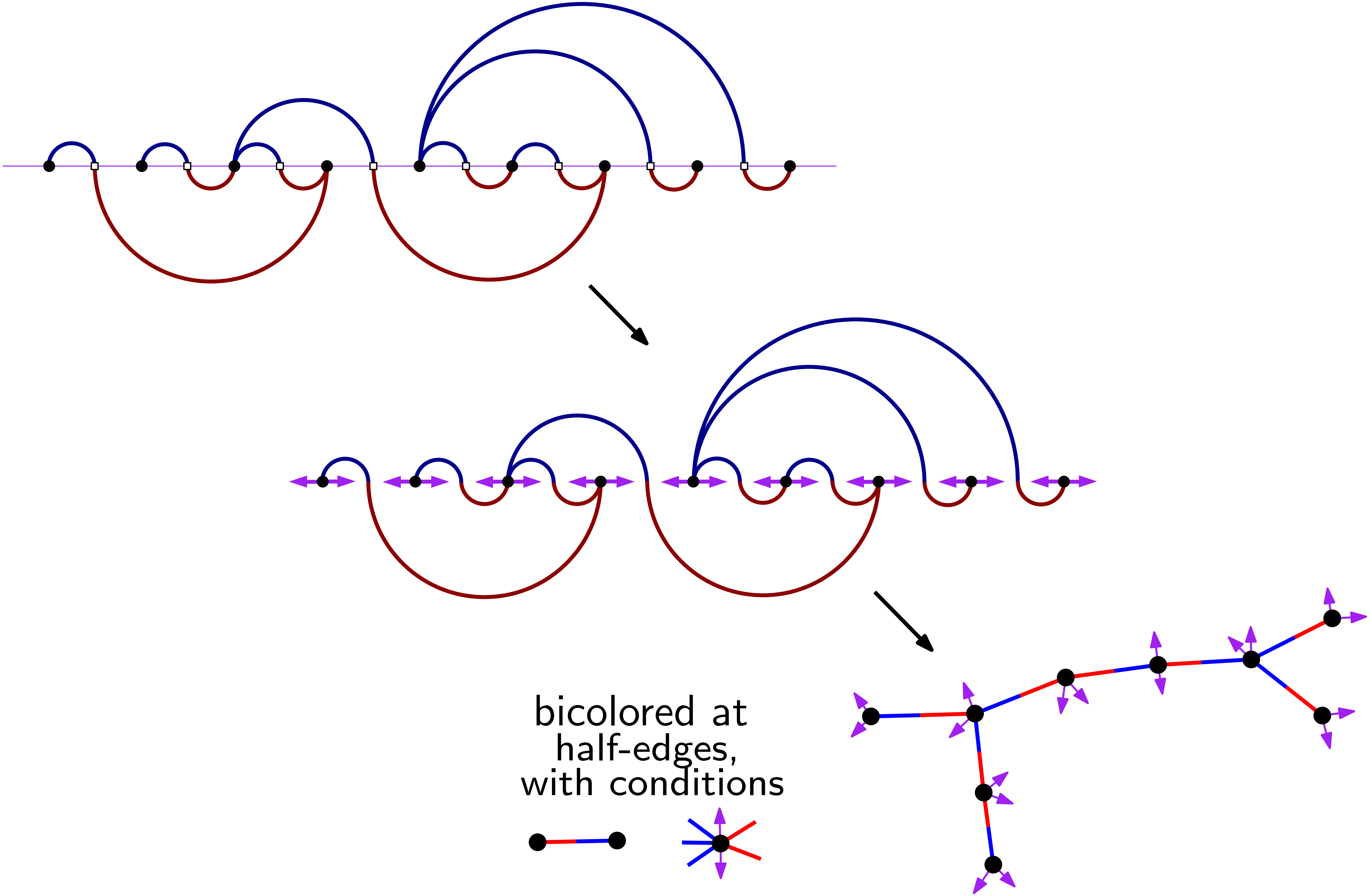
From meandering trees to blossoming trees



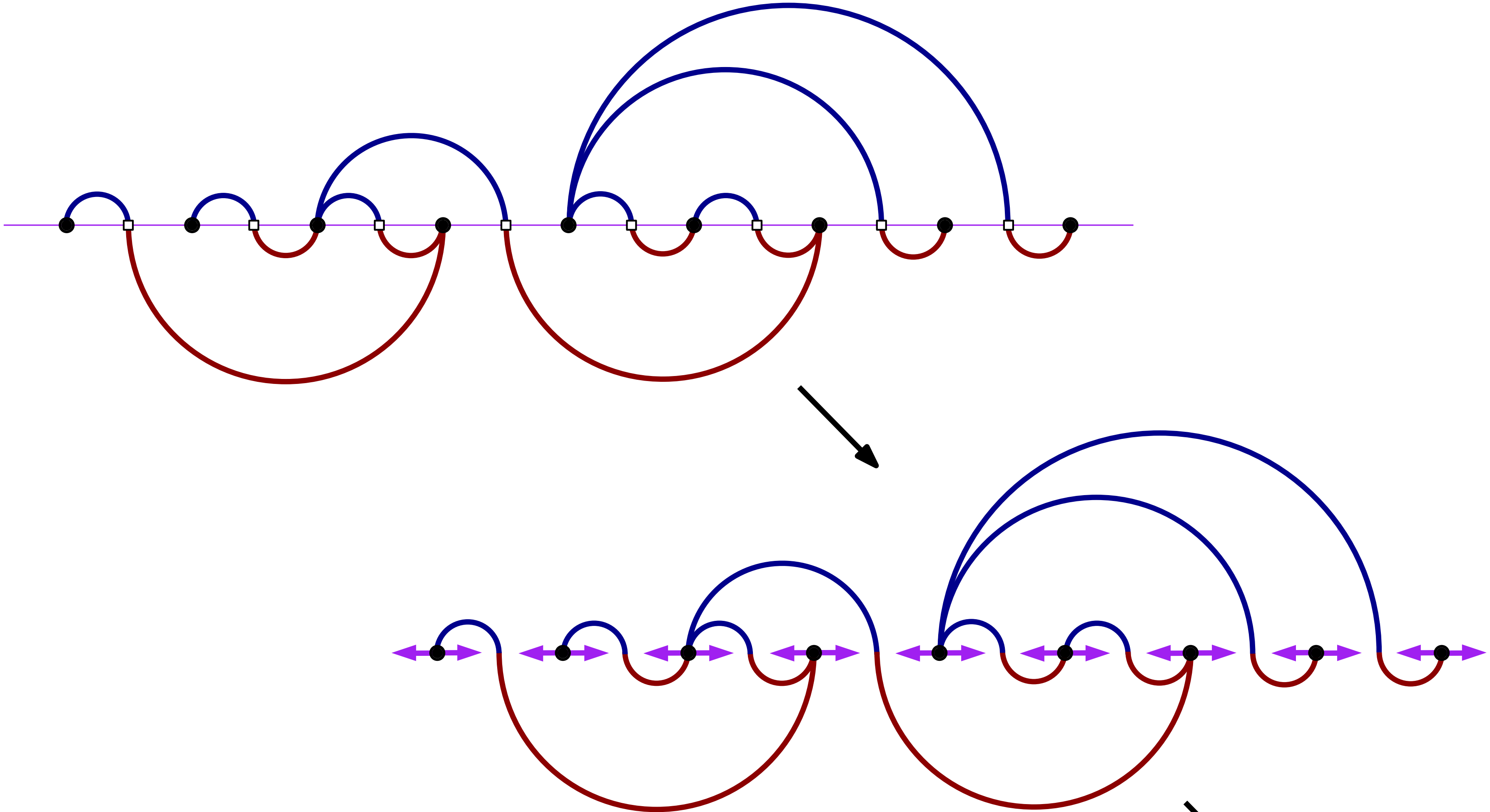
From meandering trees to blossoming trees

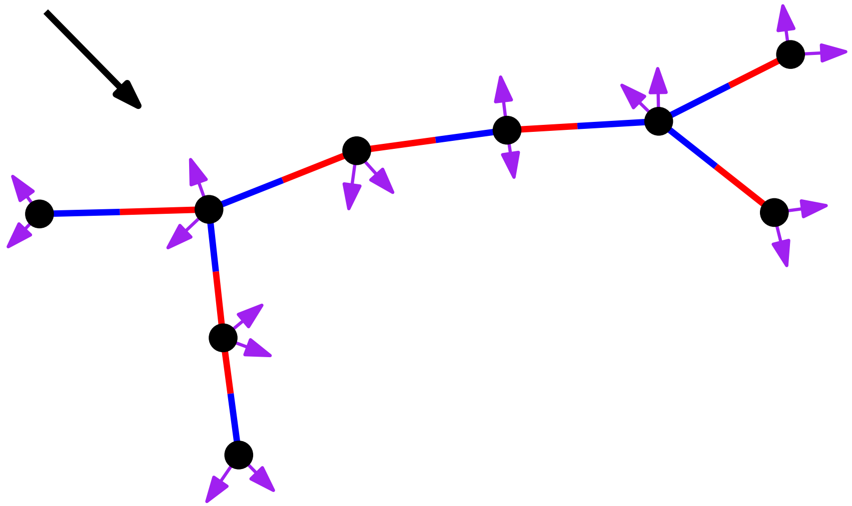


From meandering trees to blossoming trees



From meandering trees to blossoming trees





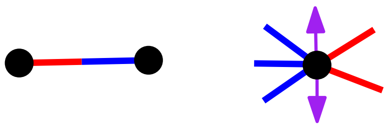
bicolored blossoming trees

$\Updownarrow$

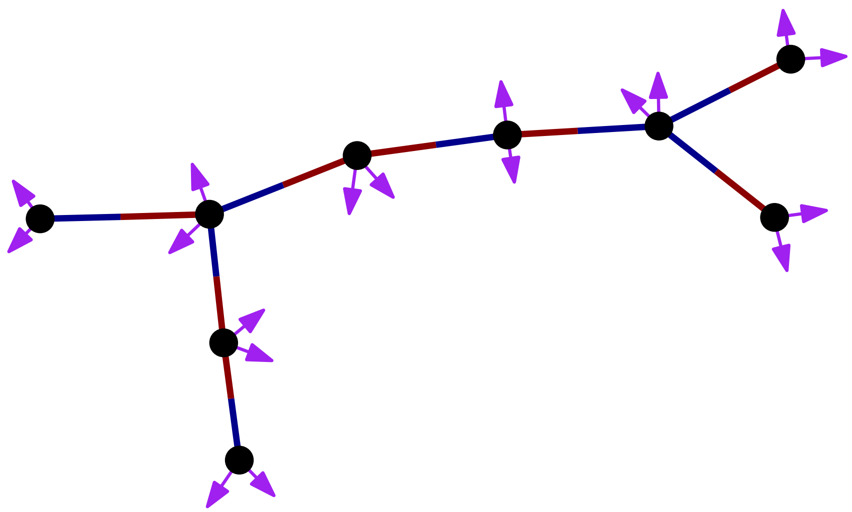
blossoming trees

with weight 1 or 2 whether half-turn symmetric or not

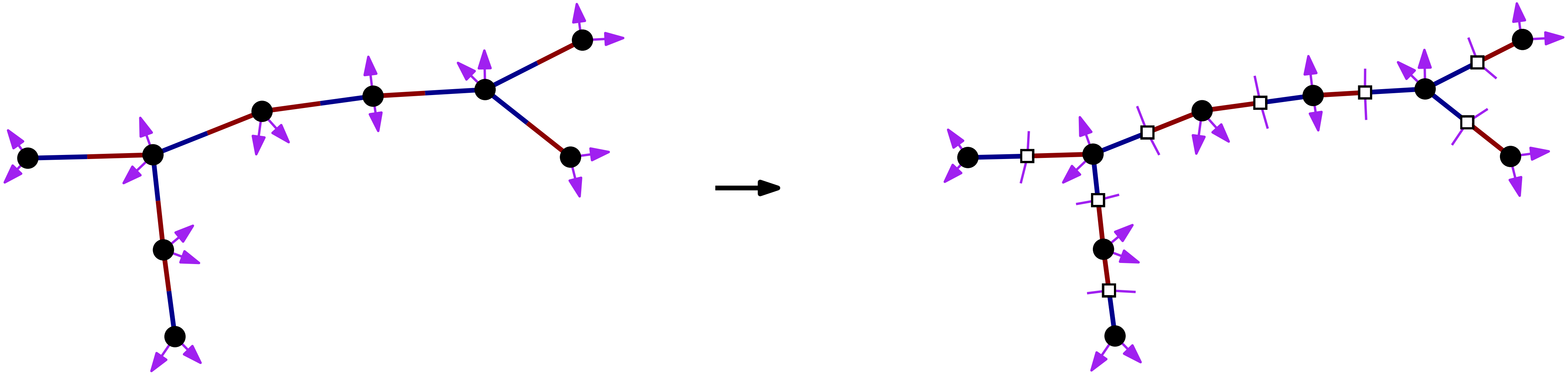
bicolored at half-edges, with conditions



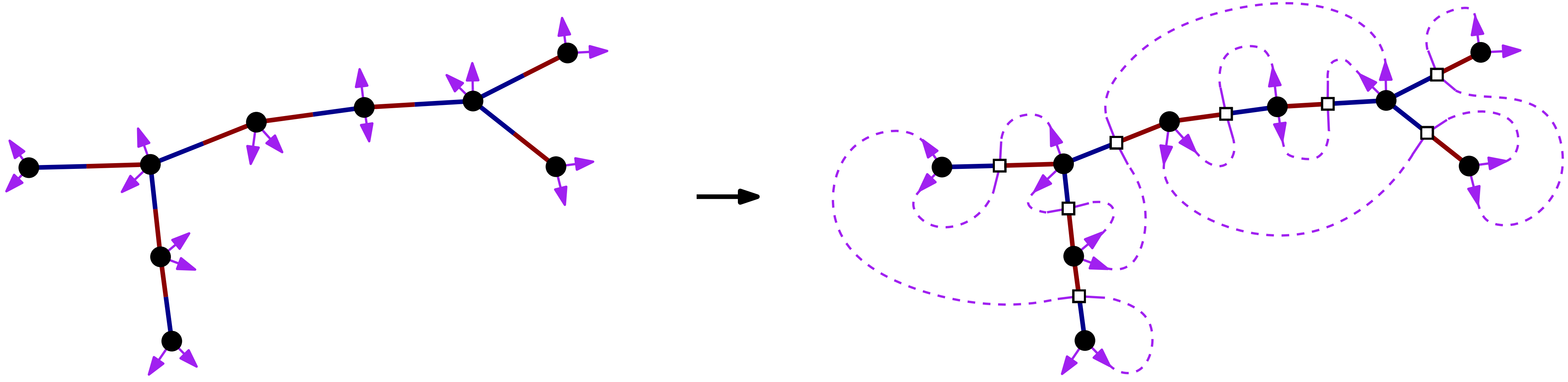
From blossoming trees to meandering trees



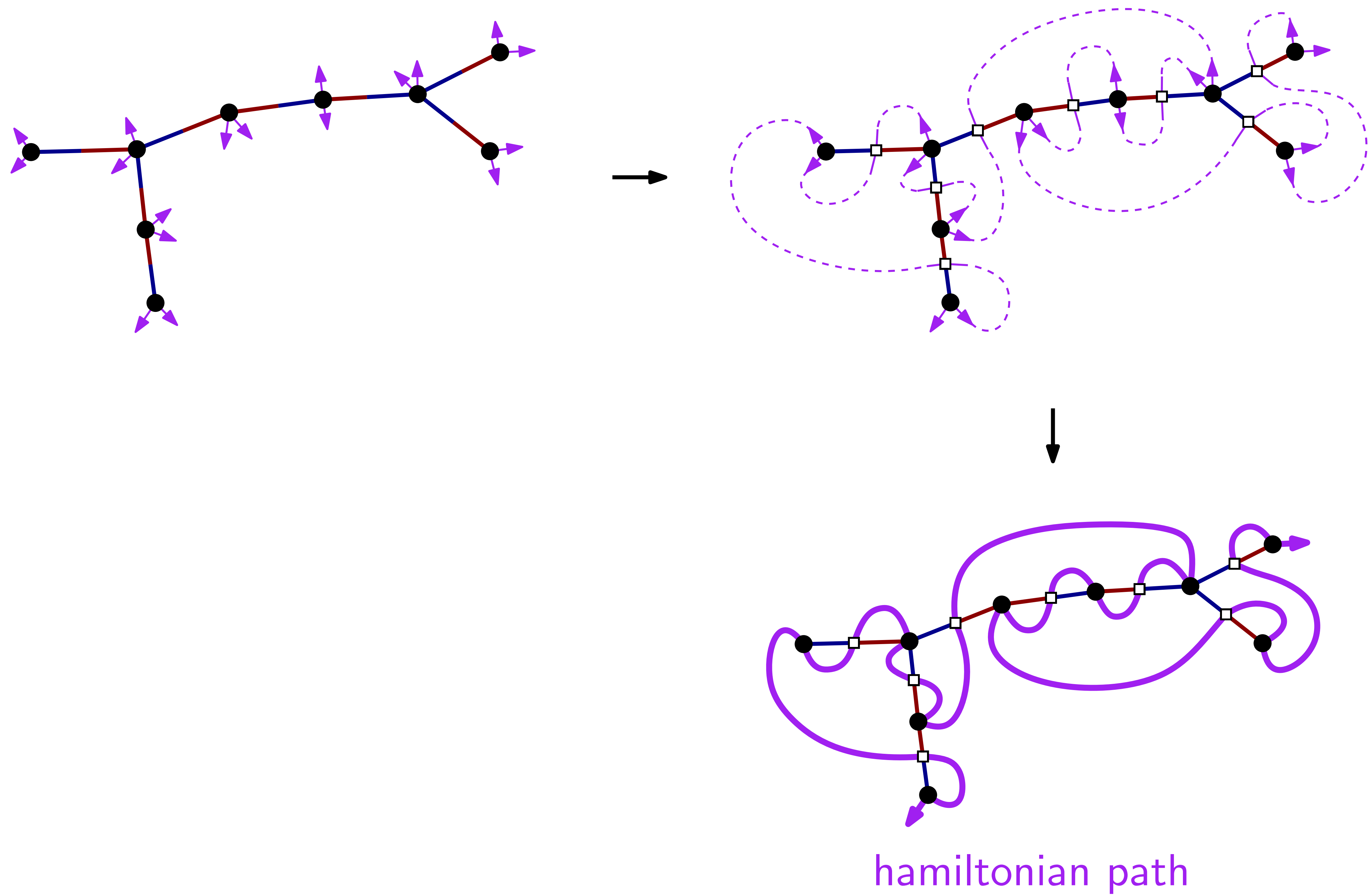
From blossoming trees to meandering trees



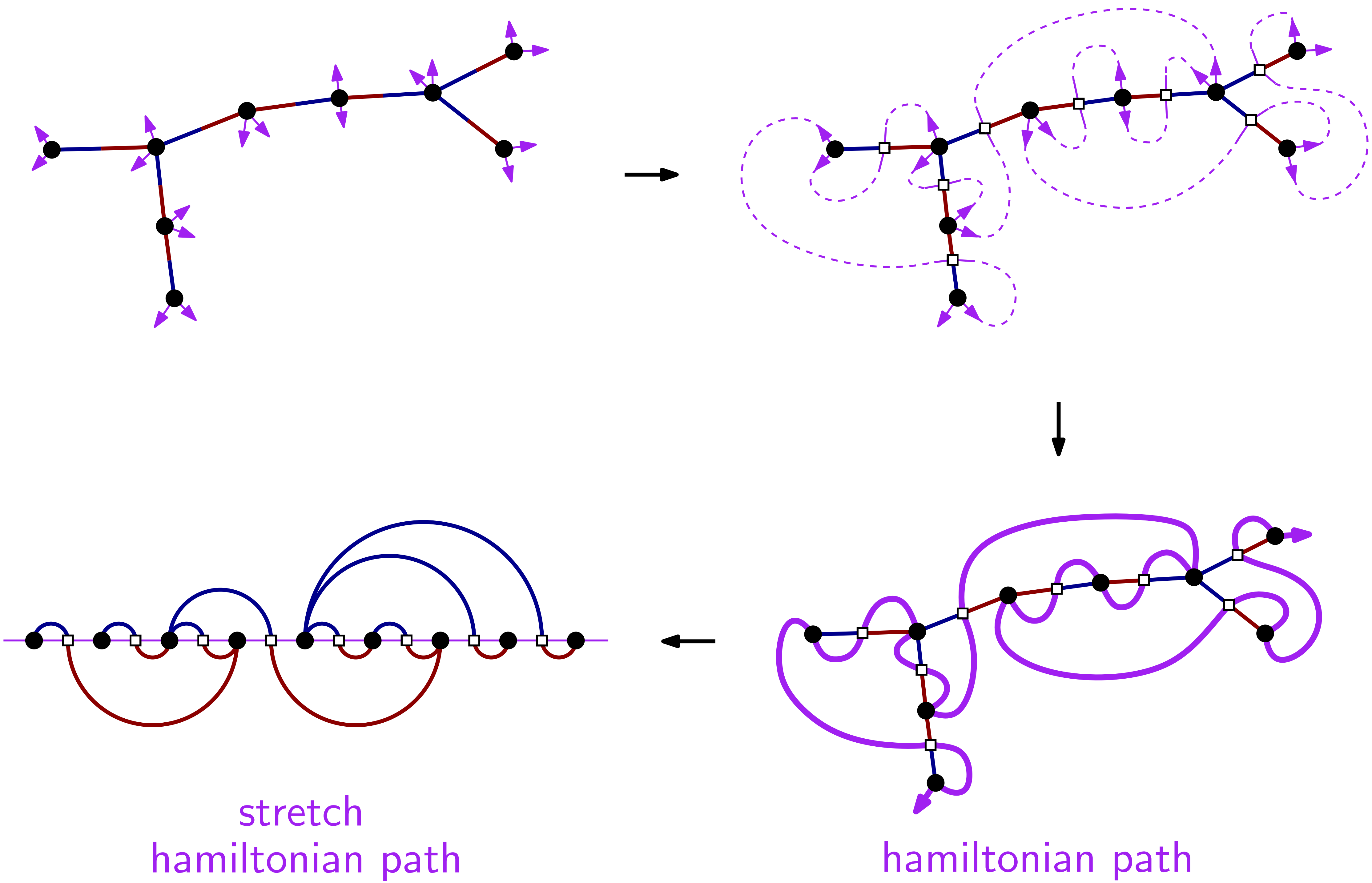
From blossoming trees to meandering trees



From blossoming trees to meandering trees



From blossoming trees to meandering trees



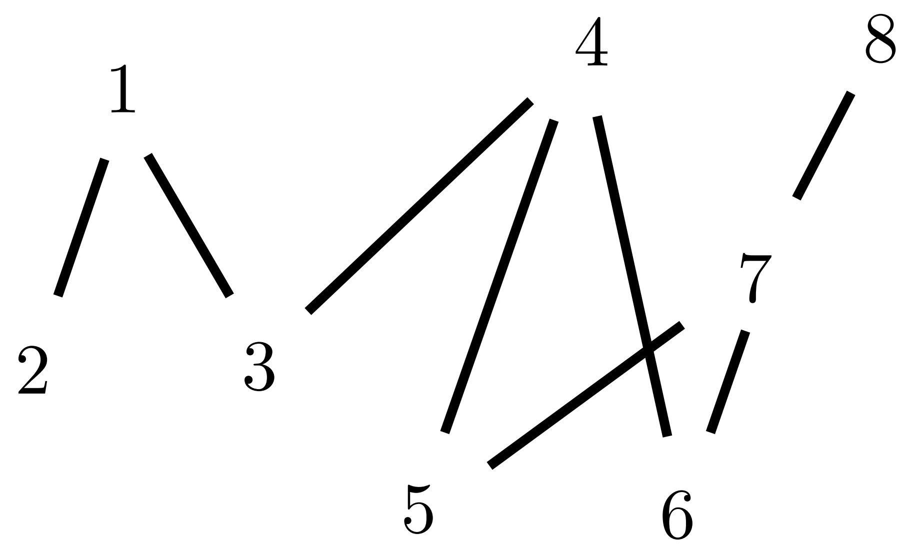
Link to interval-posets

Let P be a poset on $[n] = \{1, \ldots, n\}$

for $x \in [n]$, let $I_x := \{y \in [n],\ y \preceq_P x\}$

Def: P is an **interval-poset** if $\forall x \in [n]$, I_x is an interval of $[n]$

(of the form $[i..j]$ with $i \leq x \leq j$)



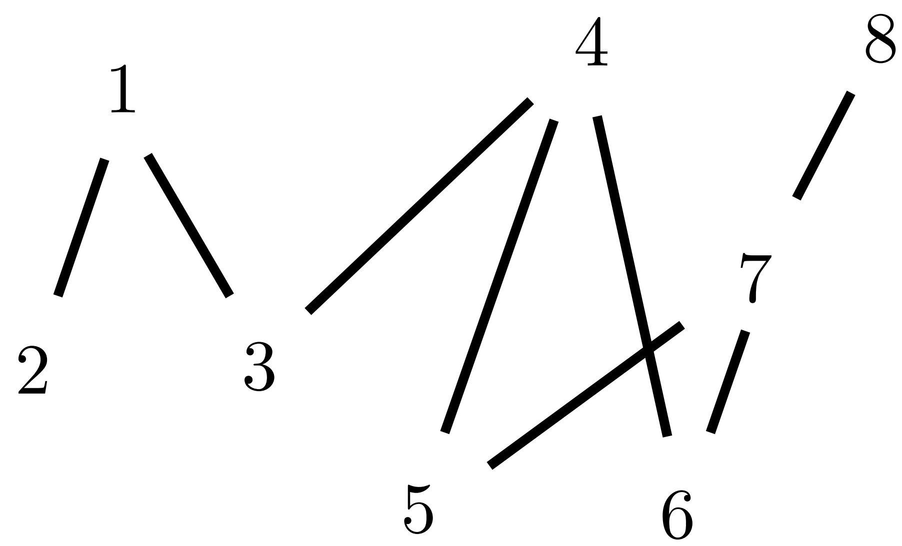
Link to interval-posets

[Chatel-Pons'13]

Let P be a poset on $[n] = \{1, \ldots, n\}$

for $x \in [n]$, let $I_x := \{y \in [n],\ y \preceq_P x\}$

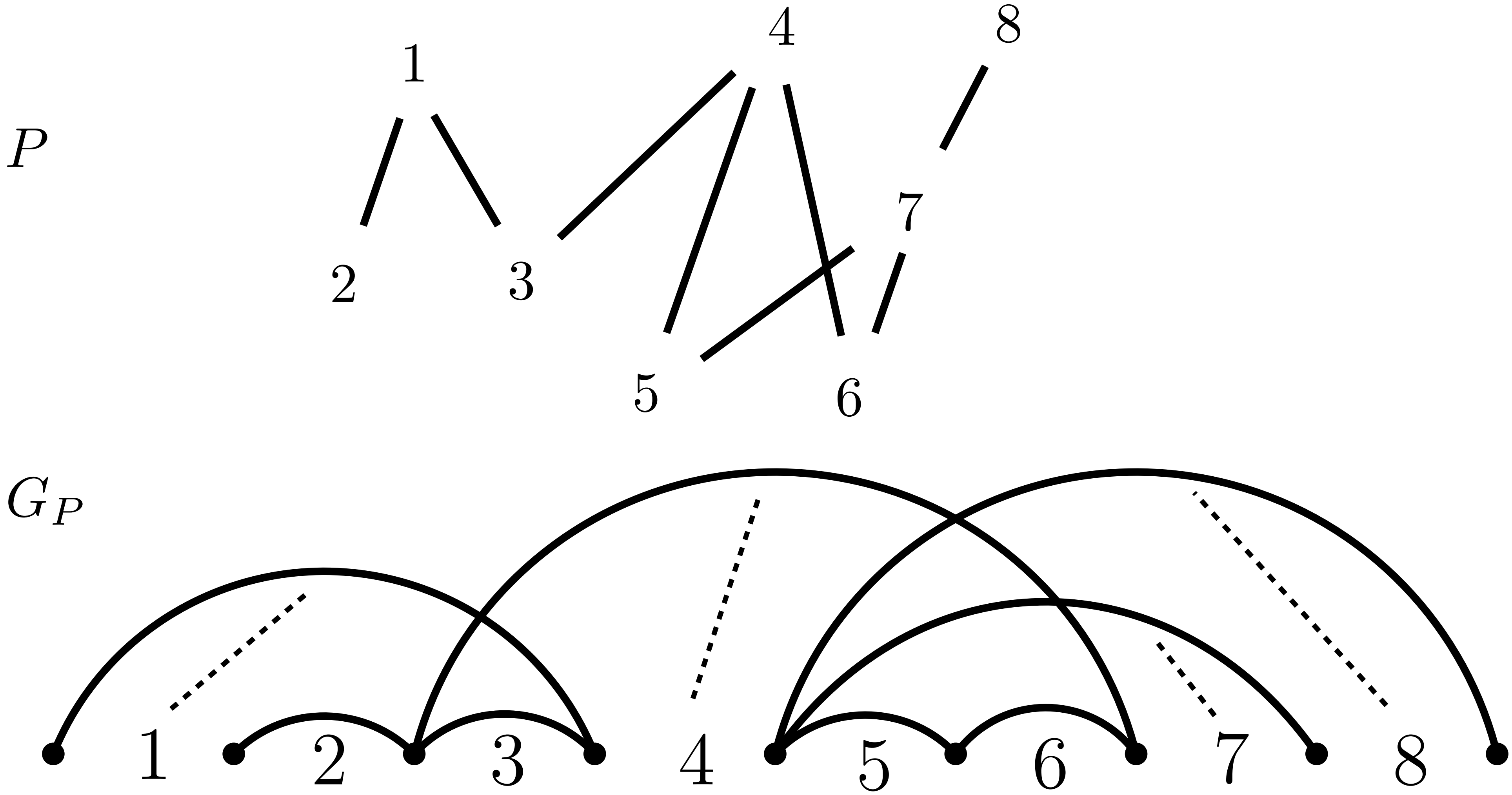
Def: P is an **interval-poset** if $\forall x \in [n]$, I_x is an interval of $[n]$
(of the form $[i..j]$ with $i \leq x \leq j$)



Tamari intervals ⟷ *bijection* ⟷ Interval-posets

Link to interval-posets

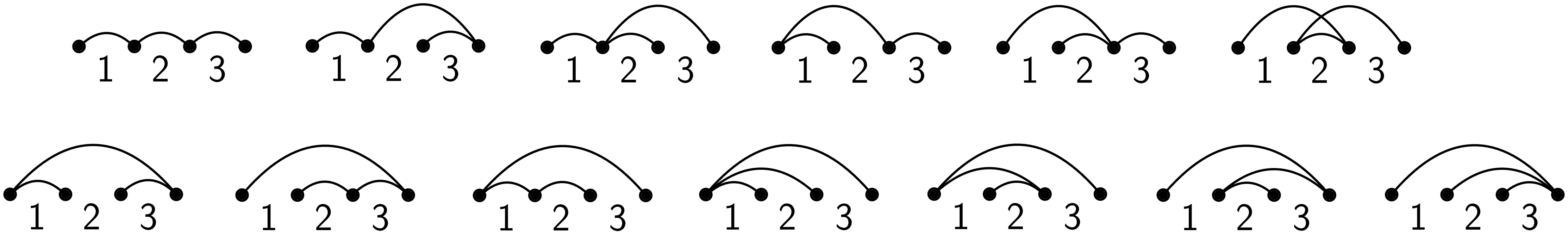
Tree-encoding (cf **[Rognerud'18]**)



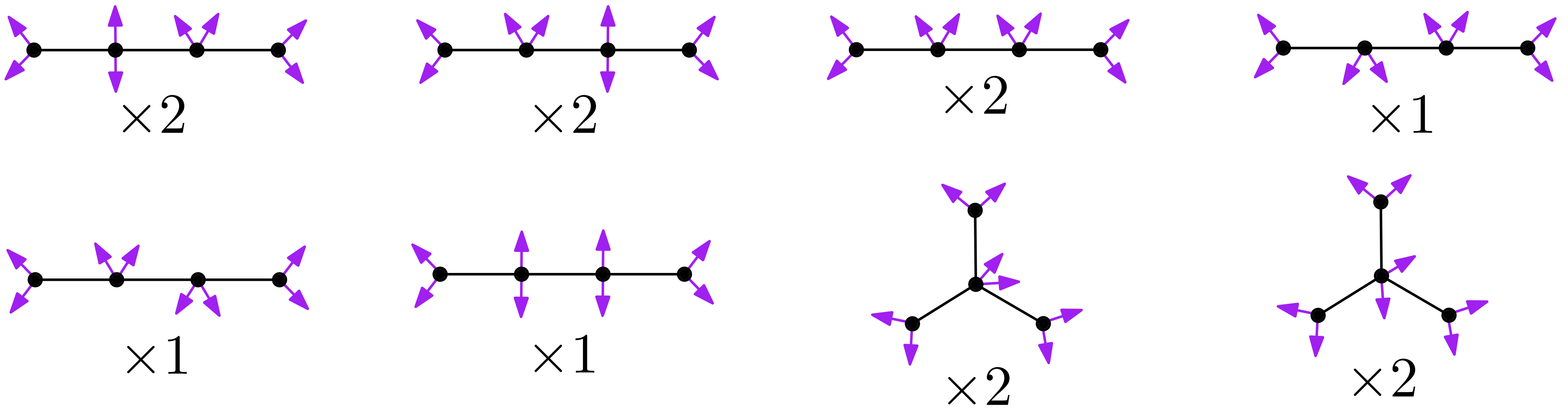
For each $x \in [1..n]$ place an arc above x that covers I_x

Link to interval-posets

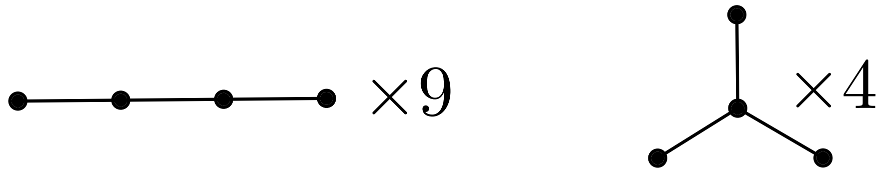
The 13 interval-poset trees of size 3



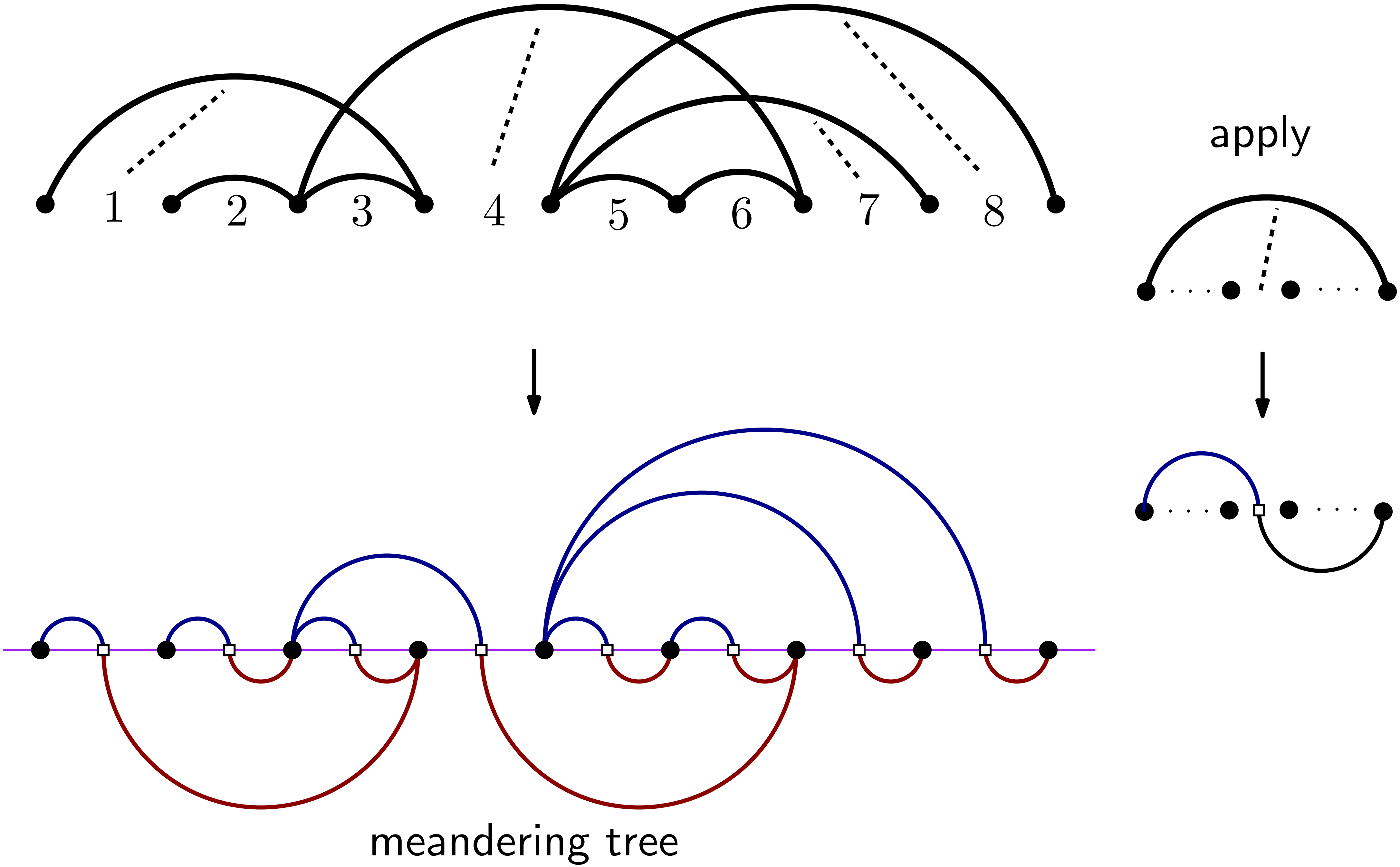
The 13 blossoming trees of size 3



In each case we have

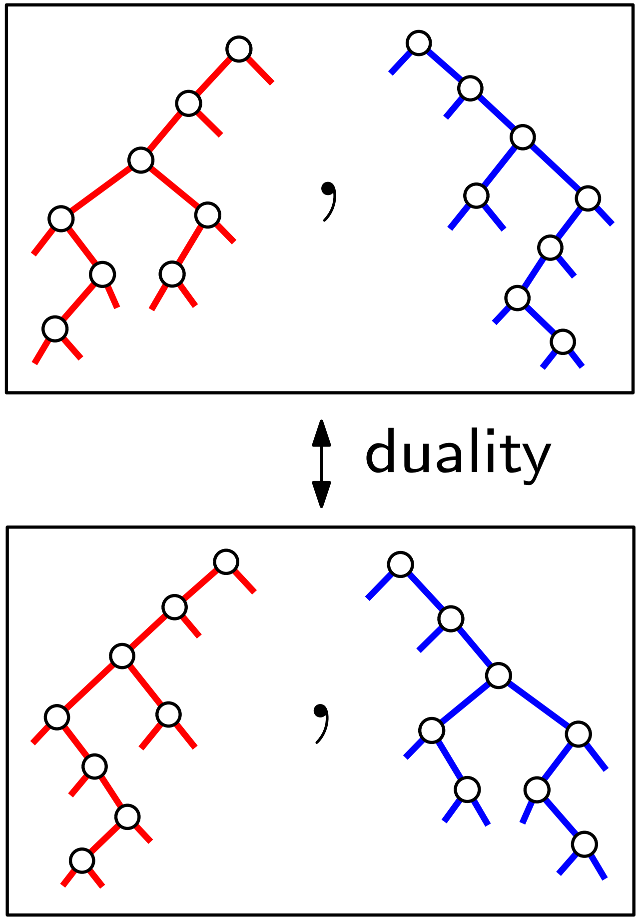


Link to interval-posets

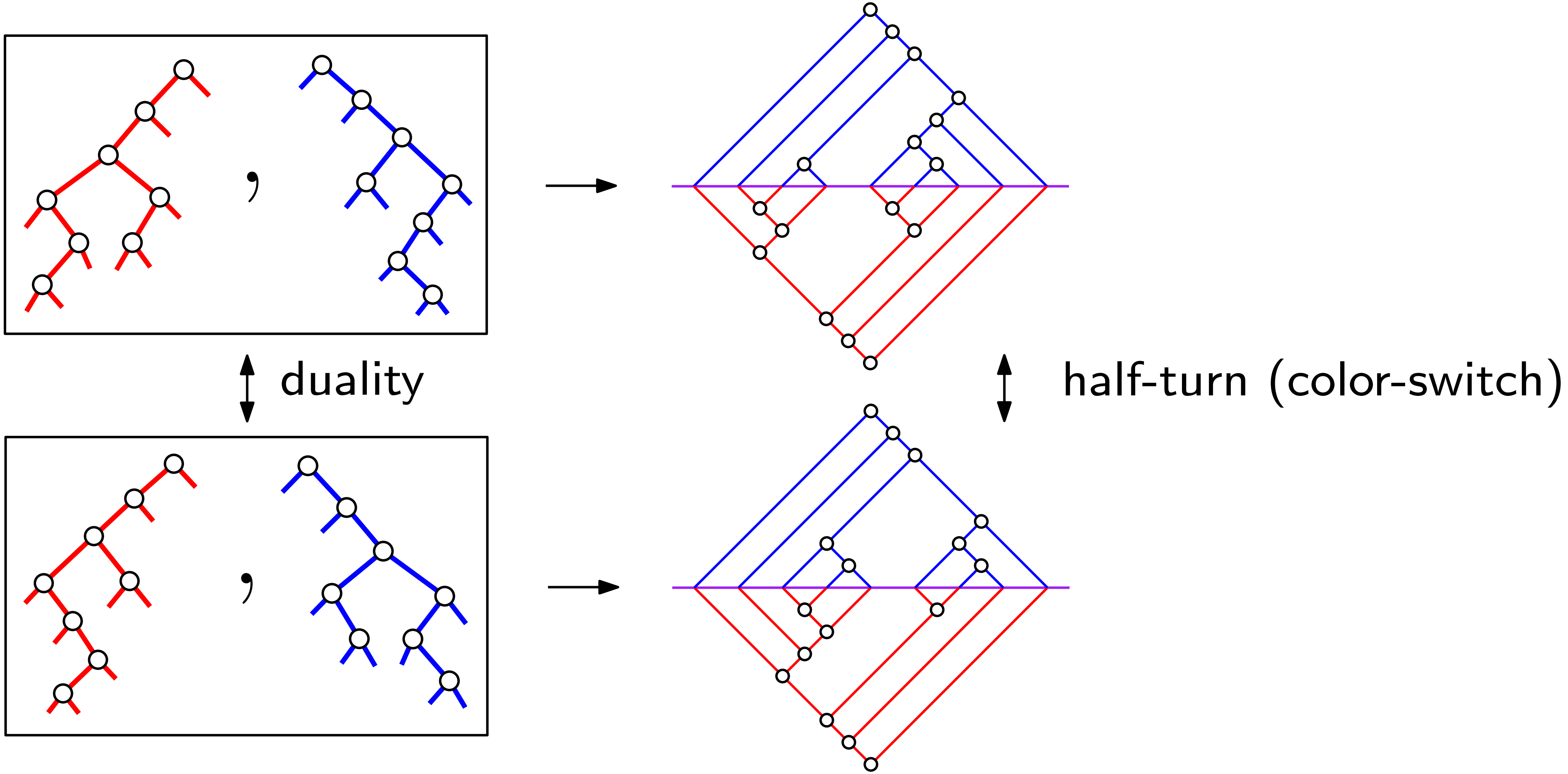


Properties of the bijection, specializations

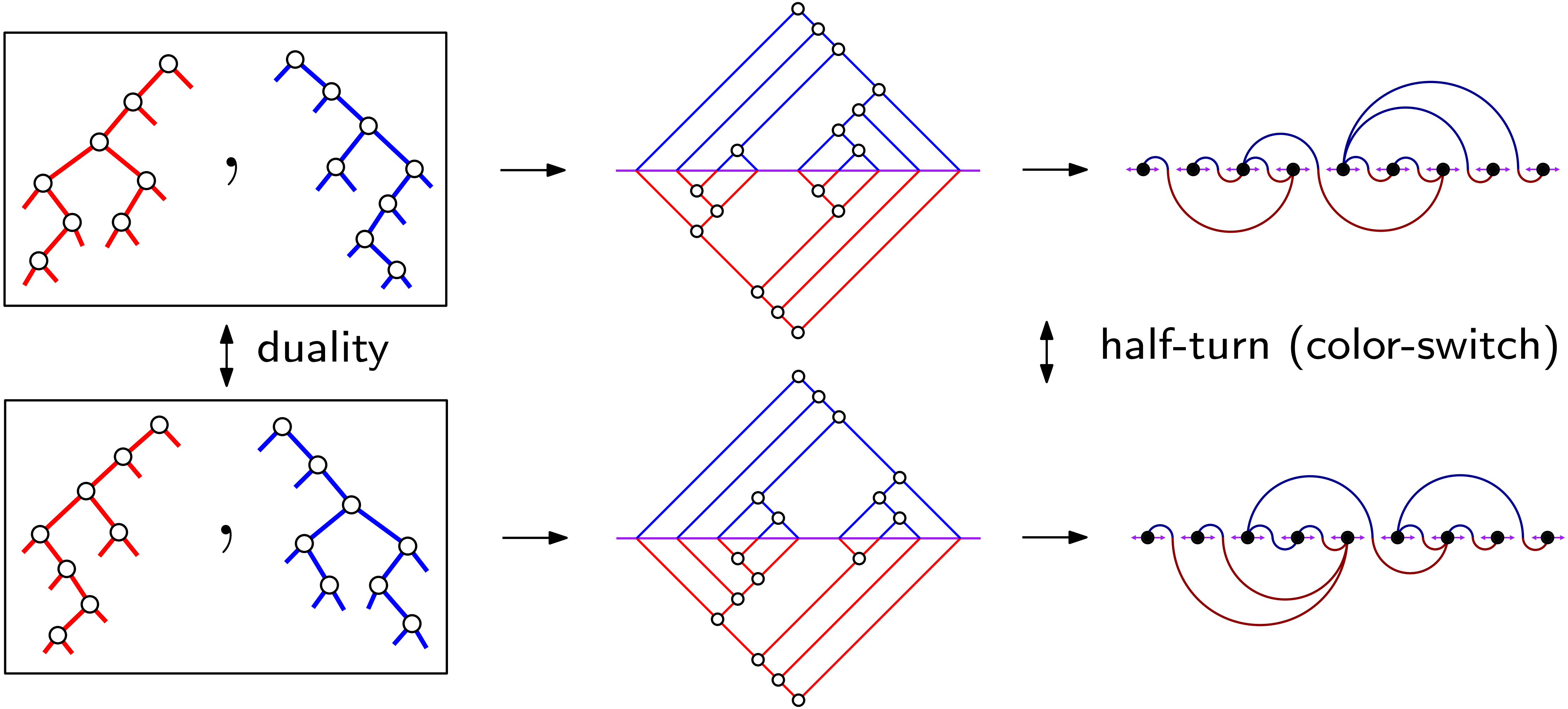
Commutation with duality of intervals



Commutation with duality of intervals

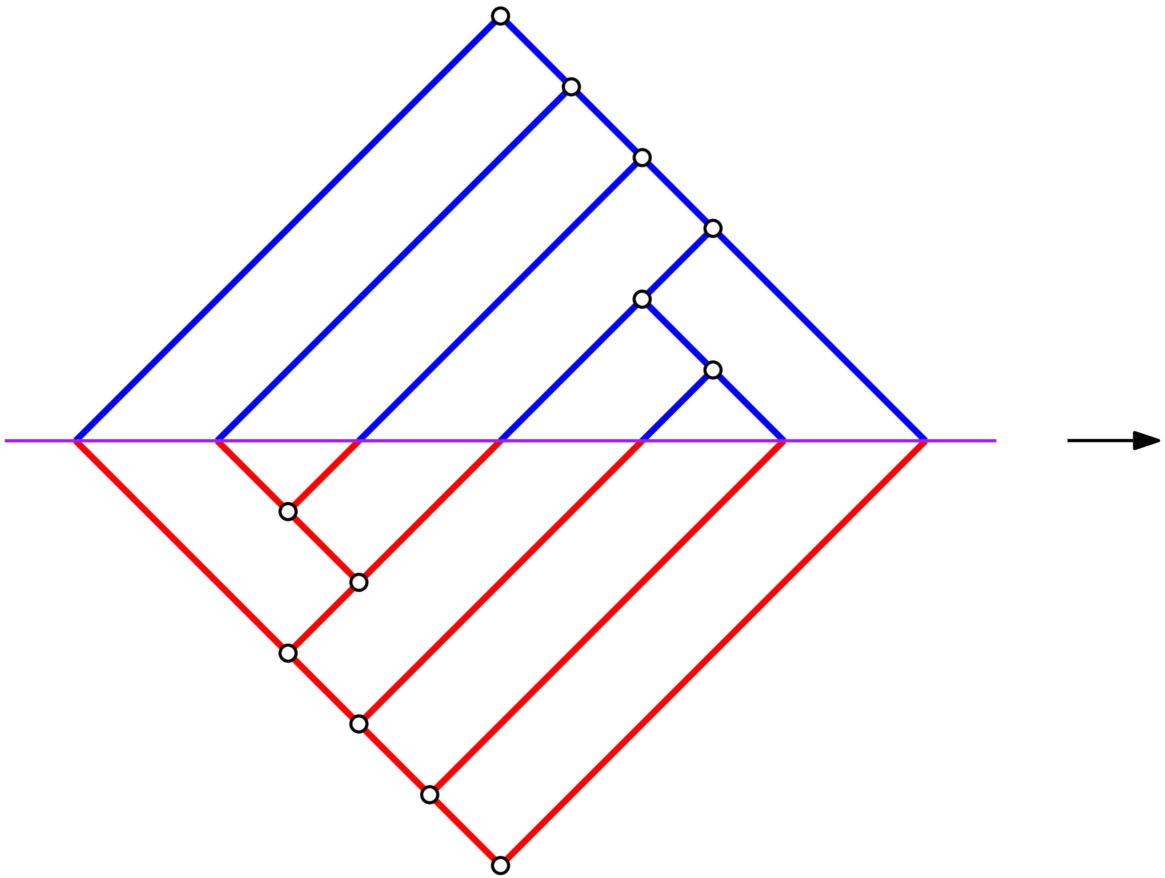


Commutation with duality of intervals



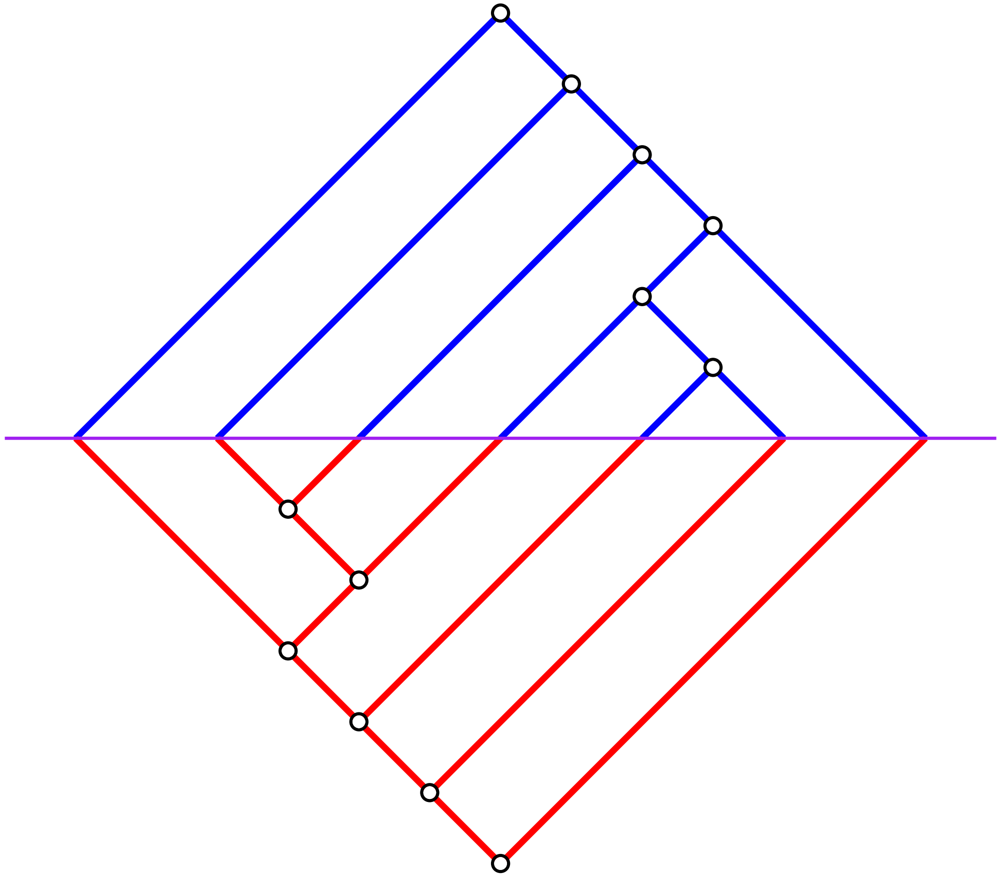
Commutation with duality of intervals

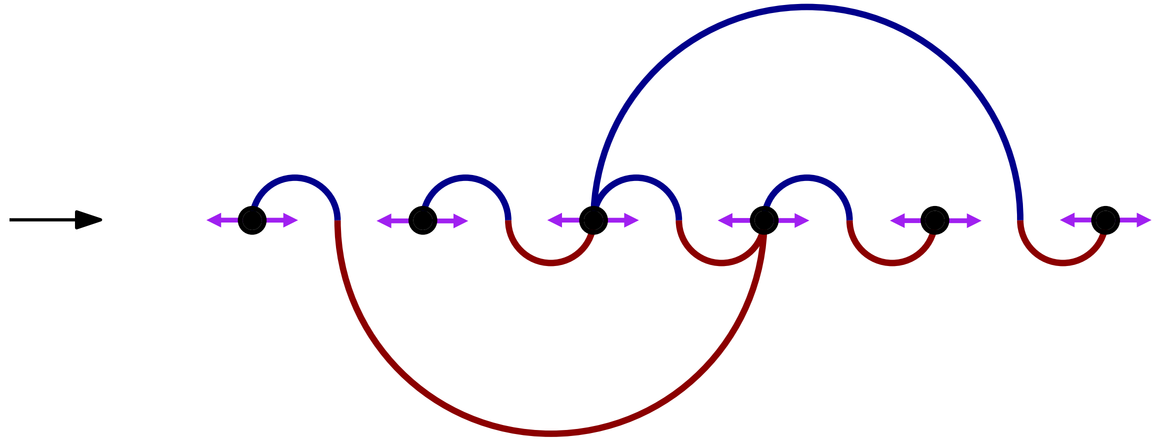
Corollary: self-dual Tamari intervals ⟷ blossoming trees with half-turn symmetry



Commutation with duality of intervals

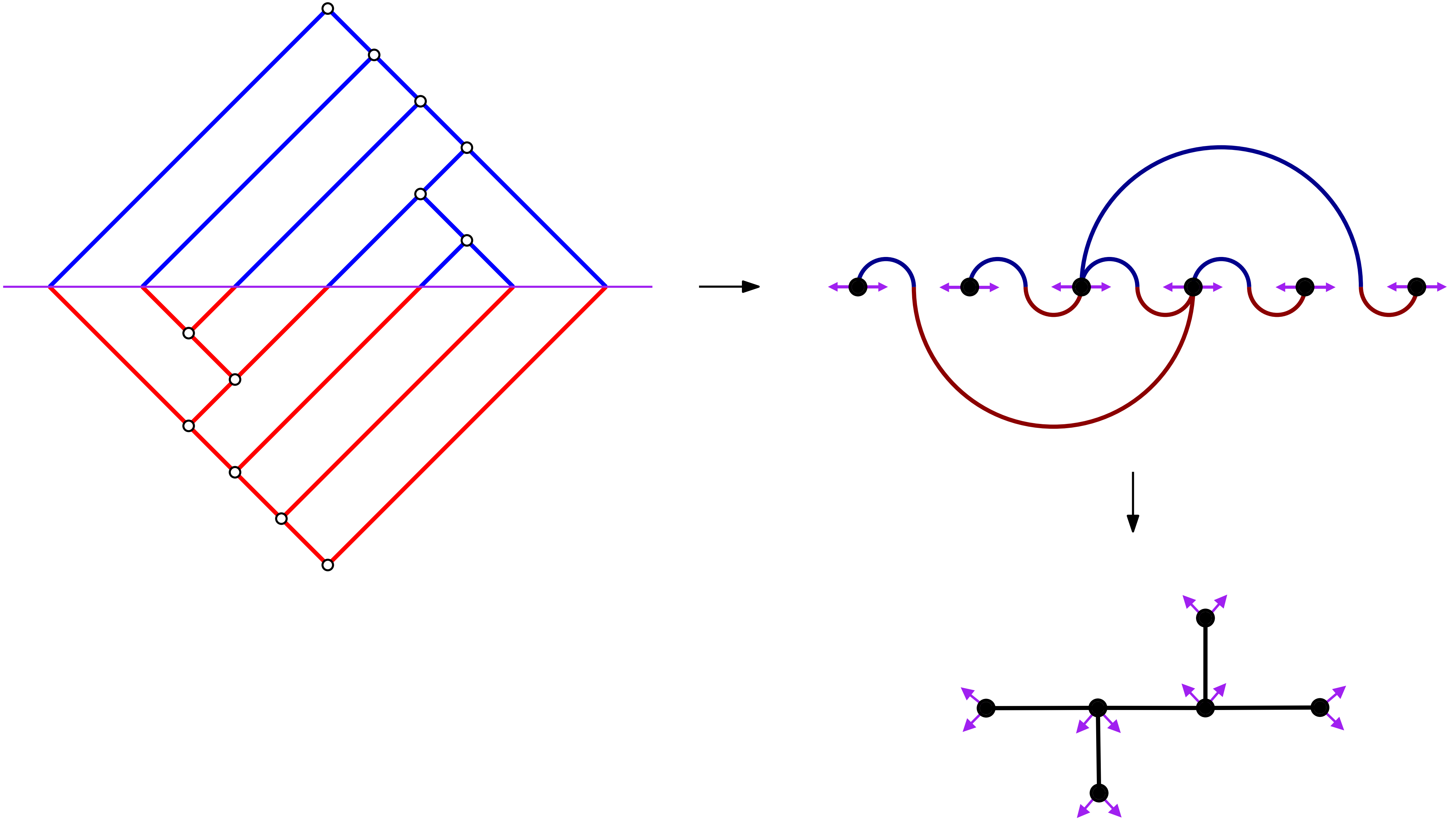
Corollary: self-dual Tamari intervals ⟷ blossoming trees with half-turn symmetry





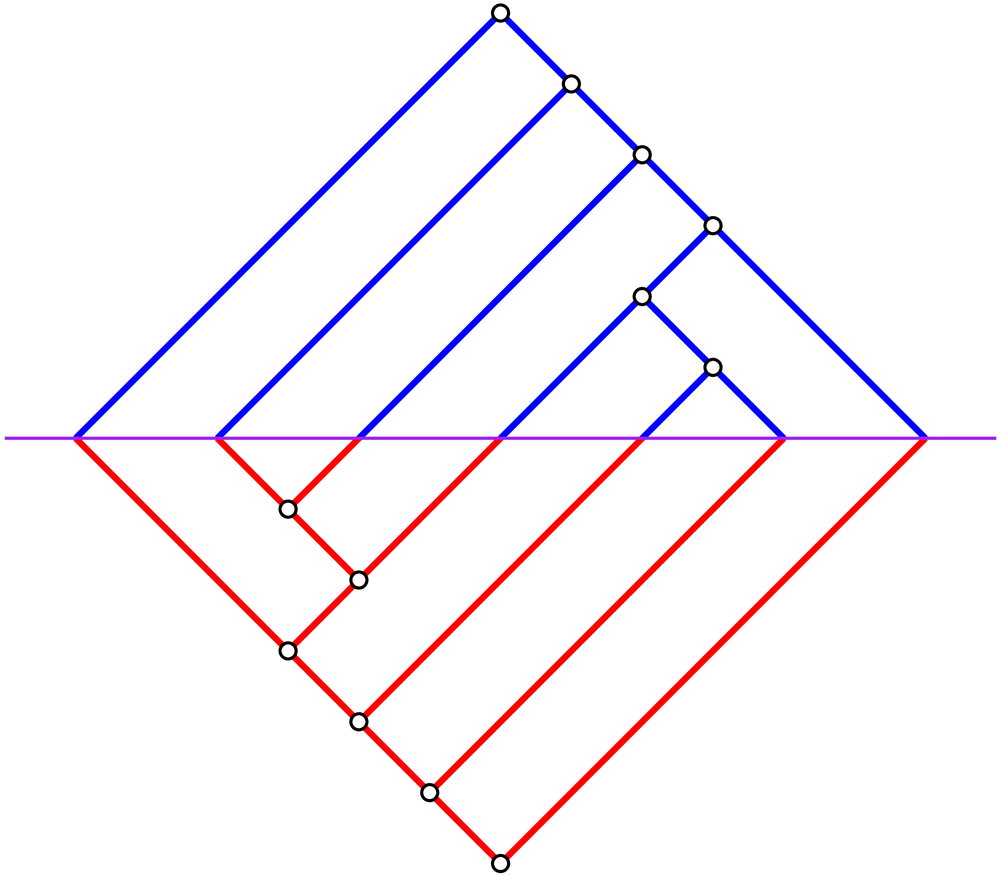
Commutation with duality of intervals

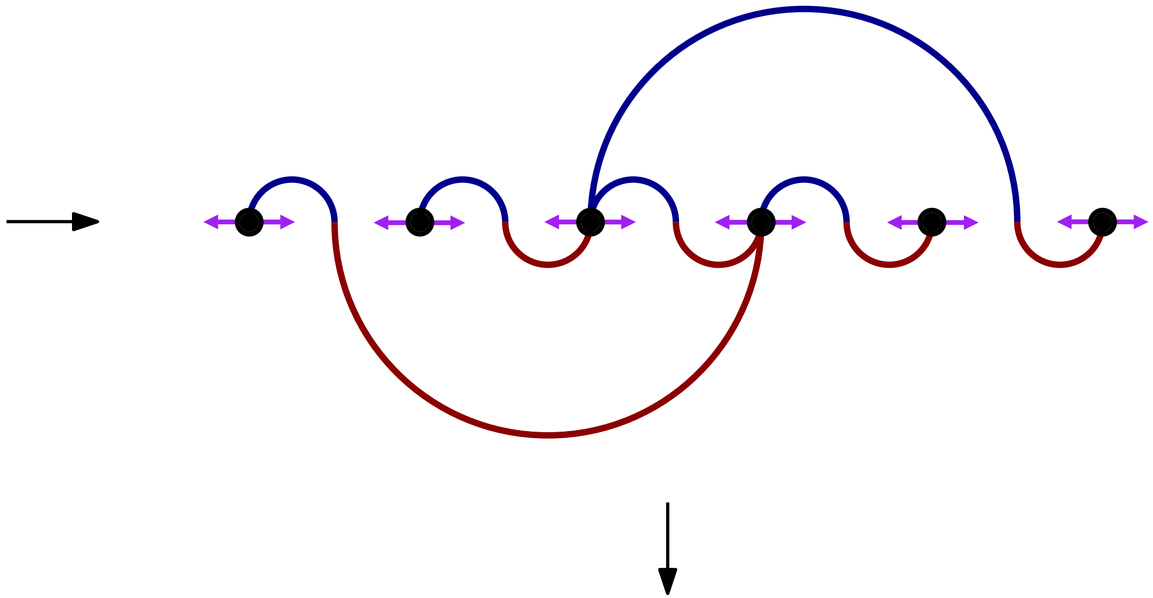
Corollary: self-dual Tamari intervals ⟷ blossoming trees with half-turn symmetry

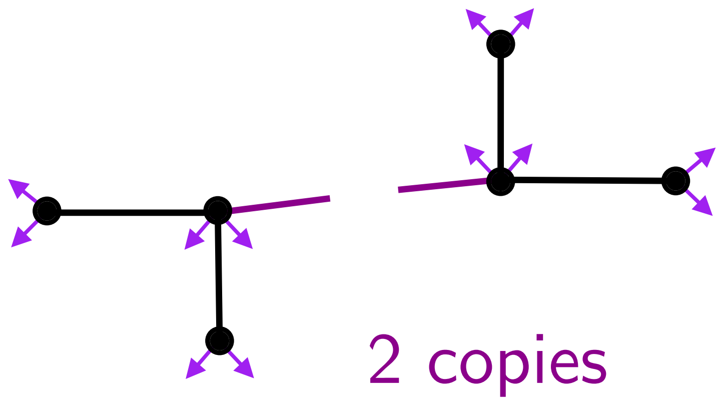


Commutation with duality of intervals

Corollary: self-dual Tamari intervals ⟷ blossoming trees with half-turn symmetry







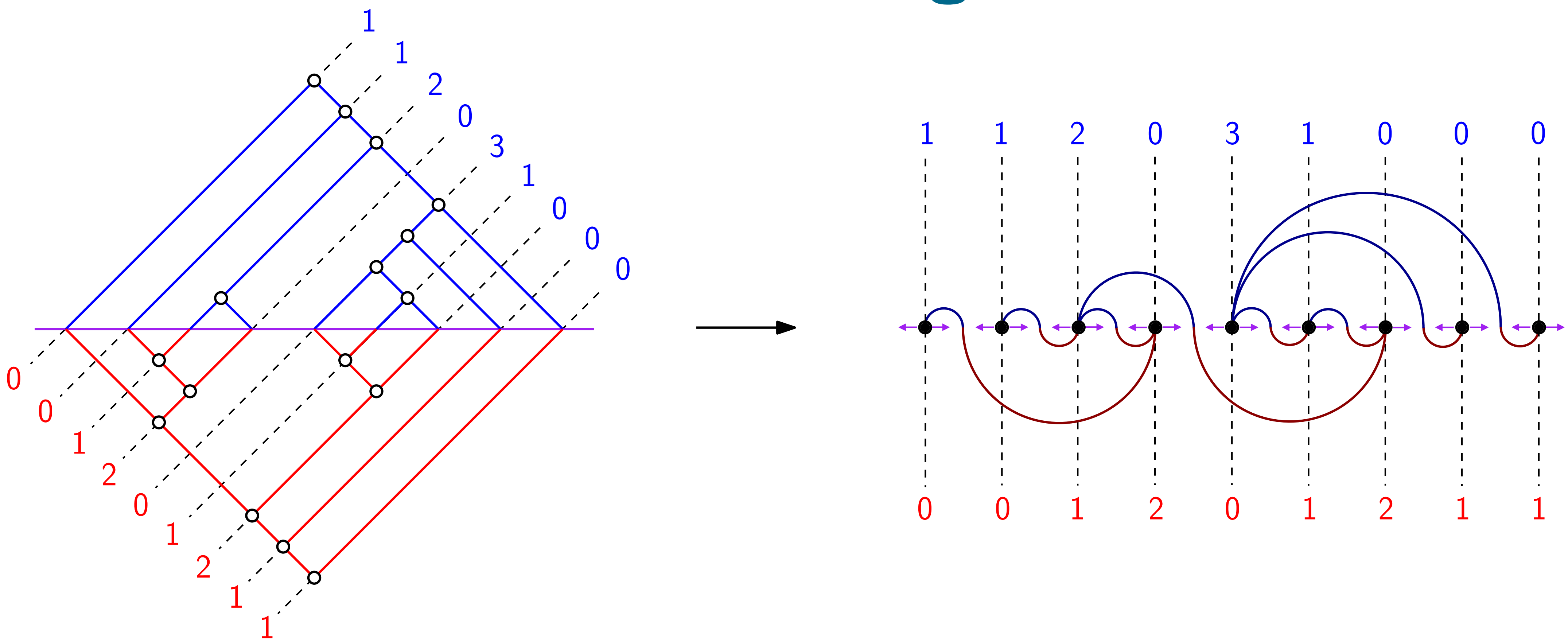
The number of self-dual intervals of size n is

$\frac{1}{r}\binom{4r}{r-1}$ if n is even, $n = 2r$

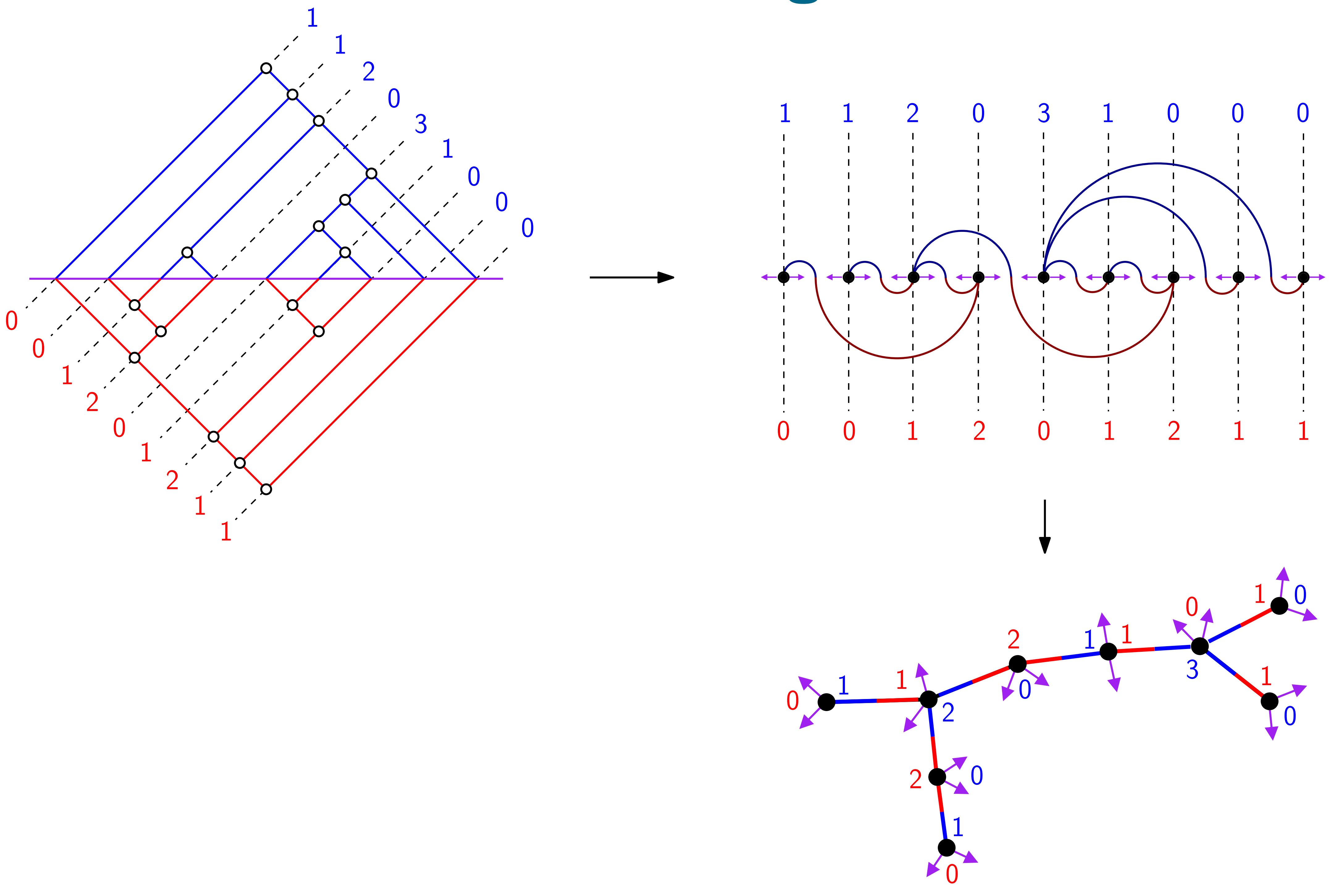
$\frac{1}{r}\binom{4r-2}{r-1}$ if n is odd, $n = 2r - 1$

1, 1, 3, 4, 15, 22, 91, 140, 612, 969

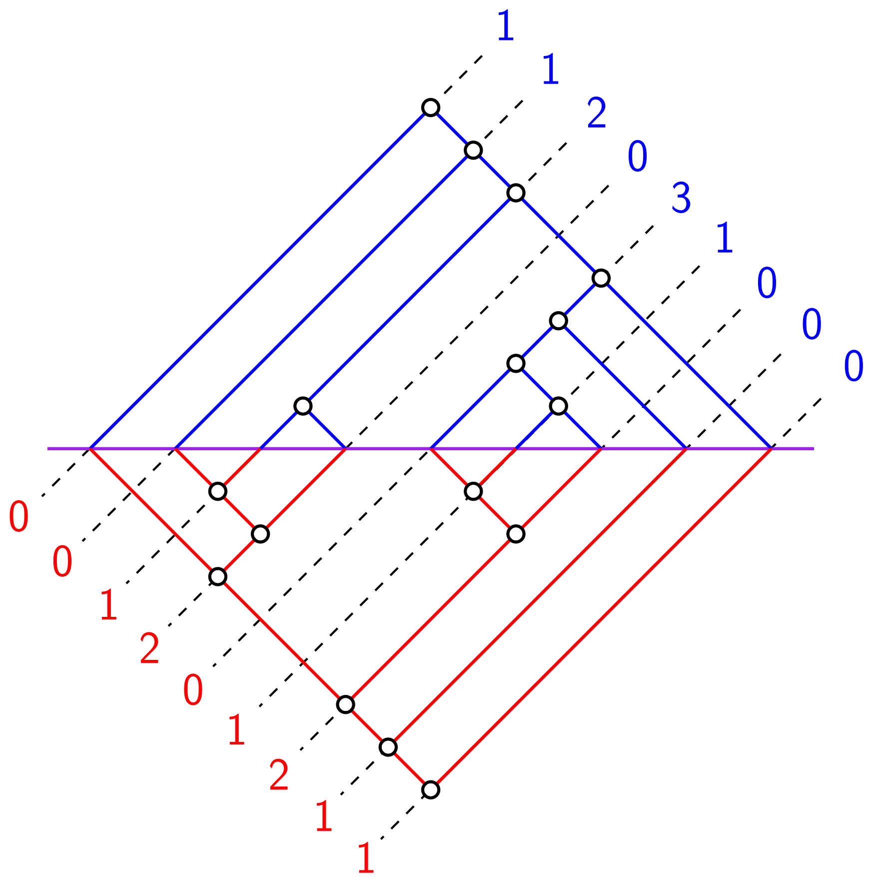
Control on some branch-lengths

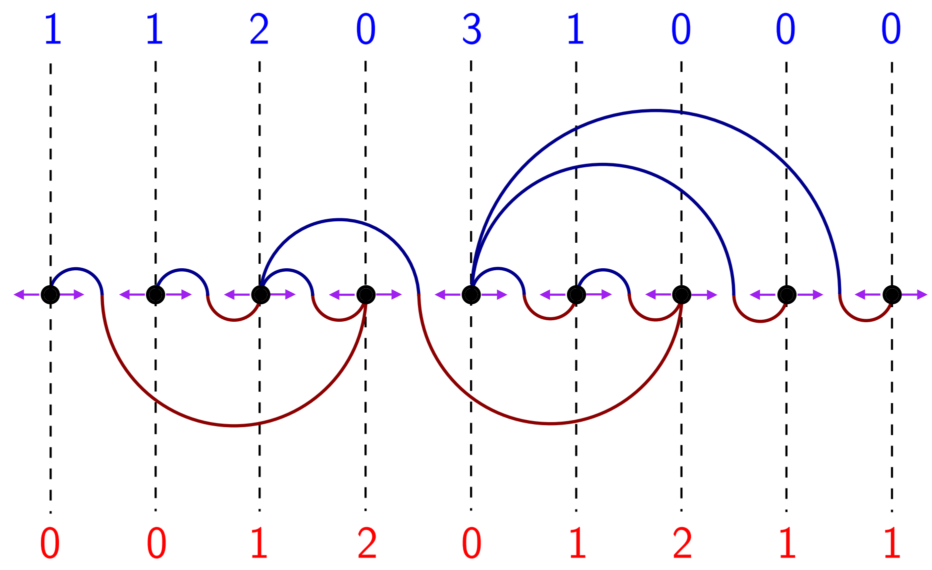


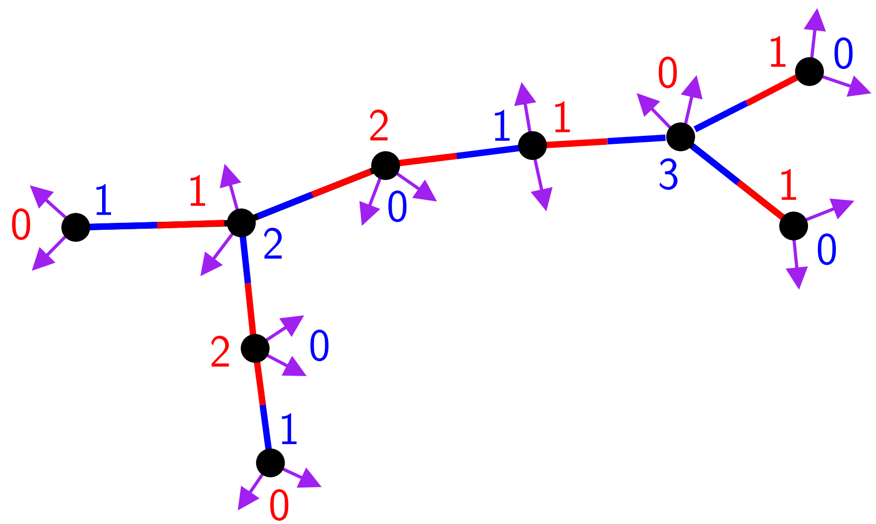
Control on some branch-lengths



Control on some branch-lengths





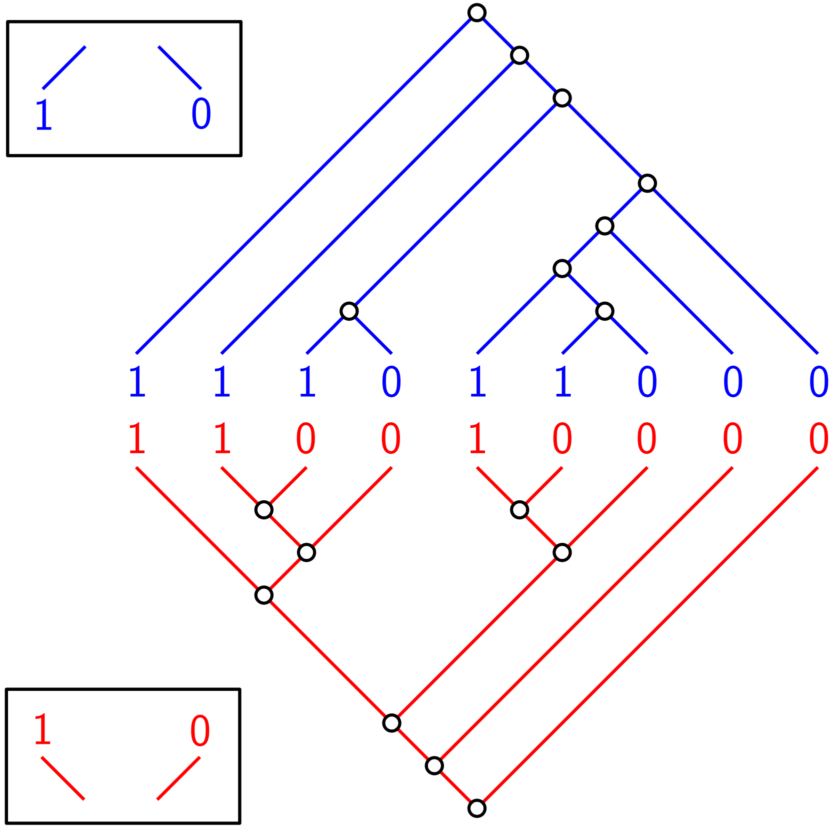


Rk: To have specialization to

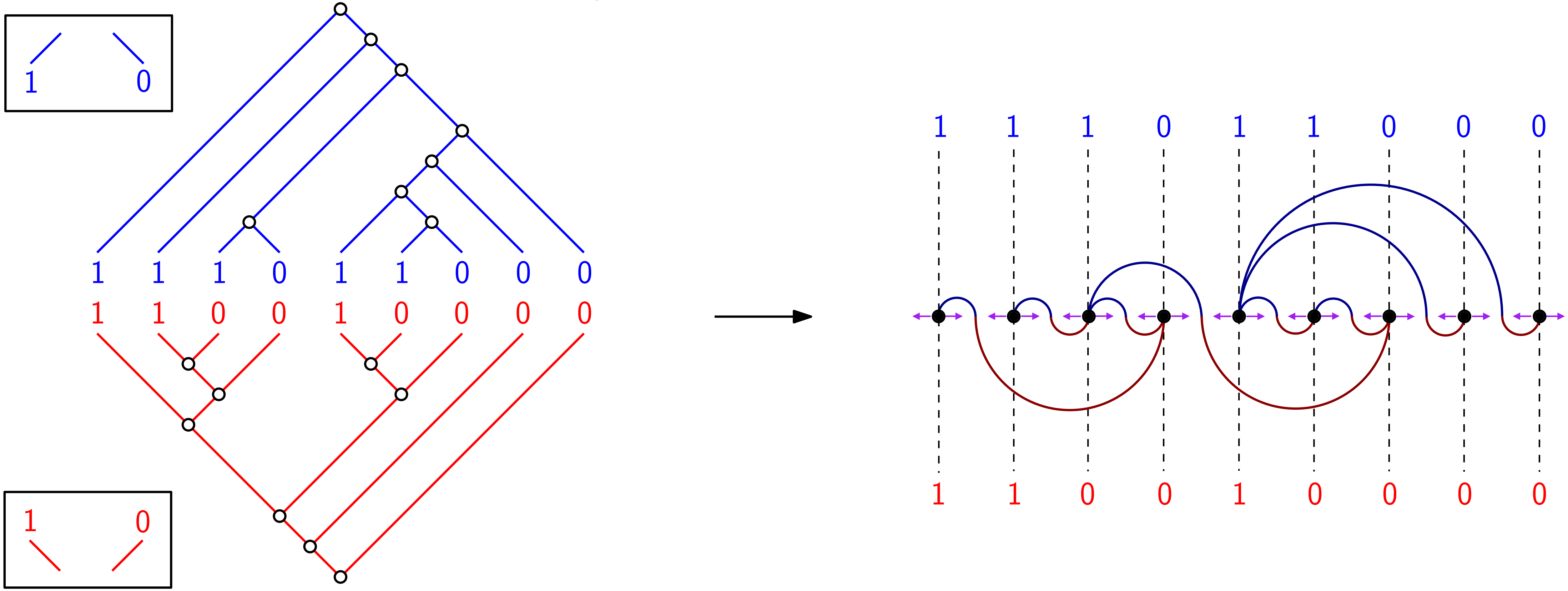
- labeled Tamari intervals
- m-Tamari intervals, cf [Pons'19]

would need to control lengths of branches of slope -1

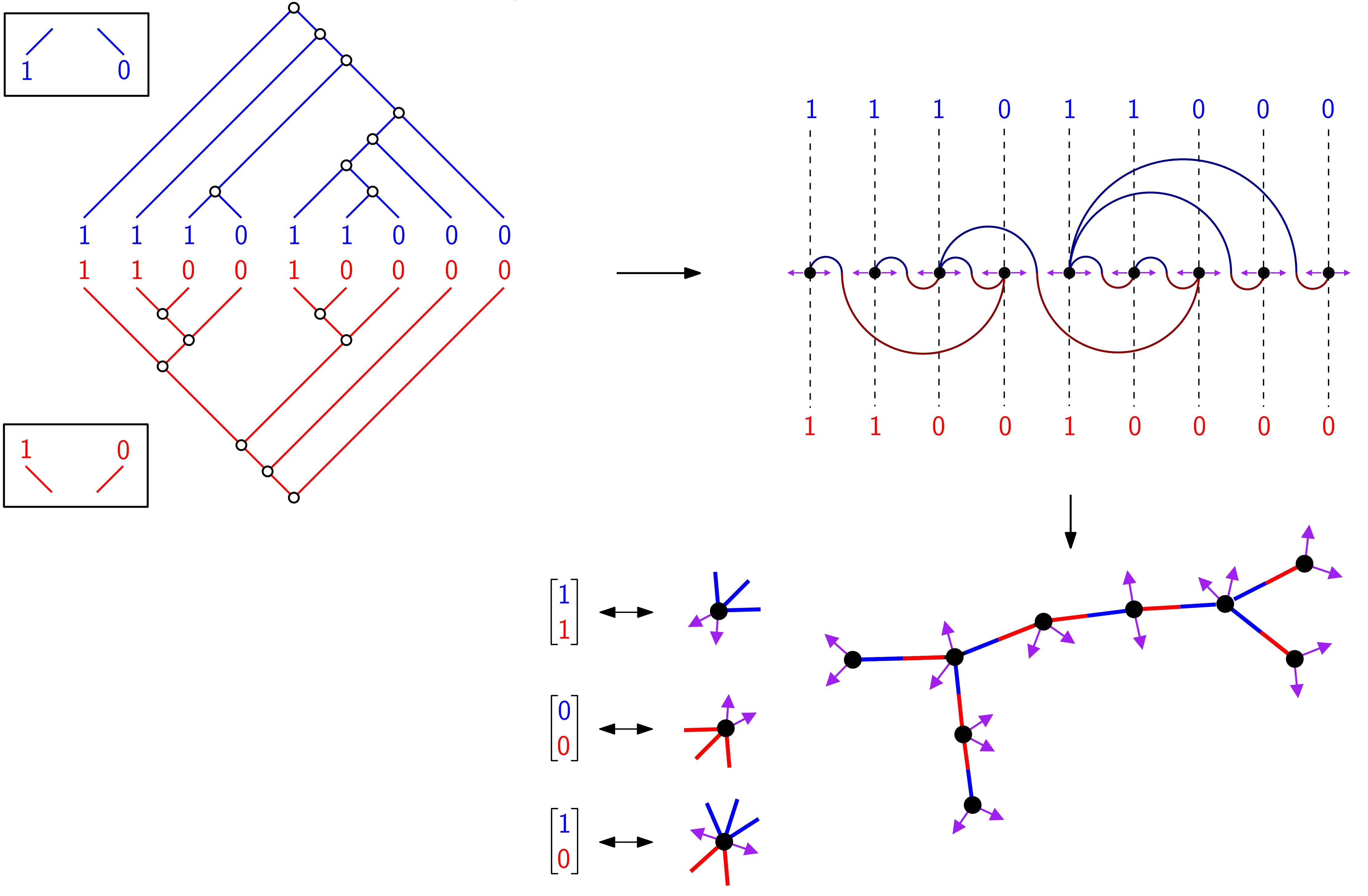
Control on canopy parameters



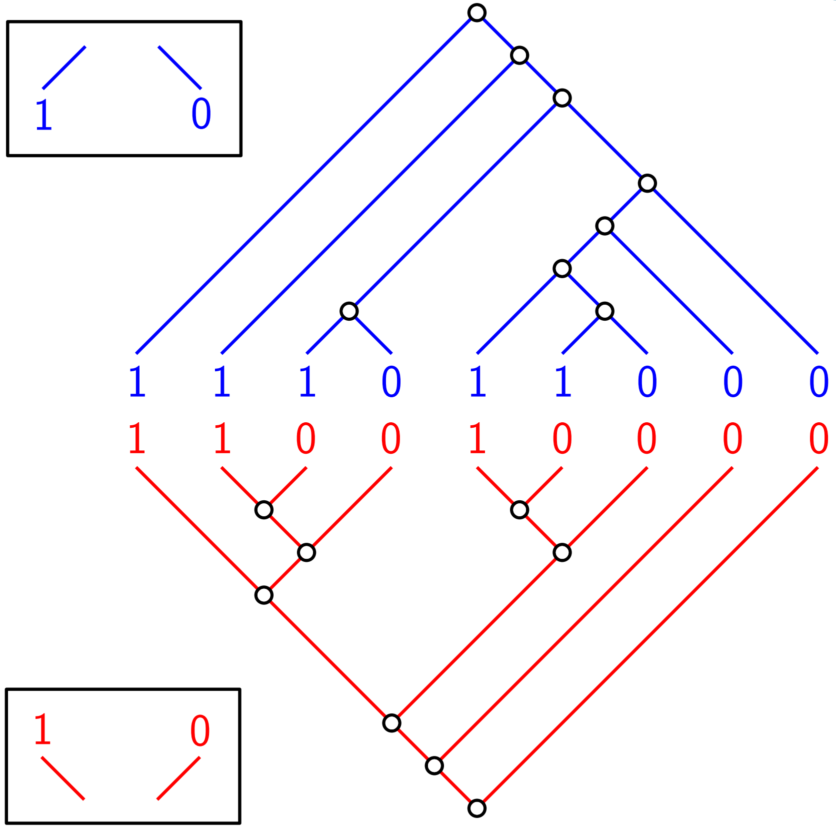
Control on canopy parameters

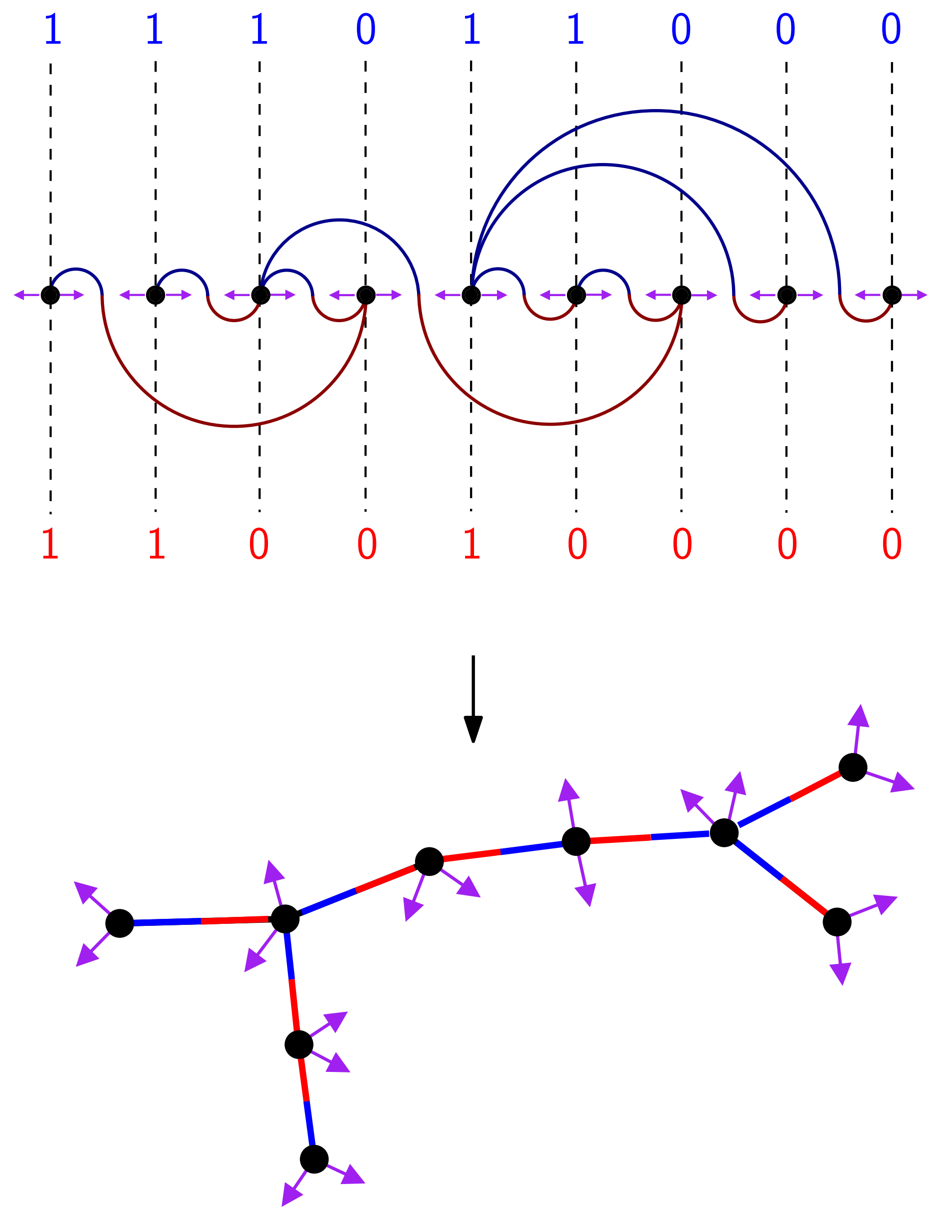


Control on canopy parameters



Control on canopy parameters

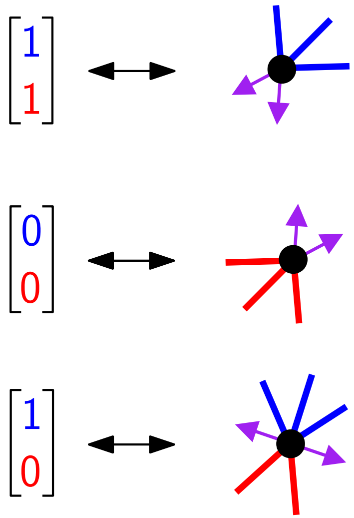




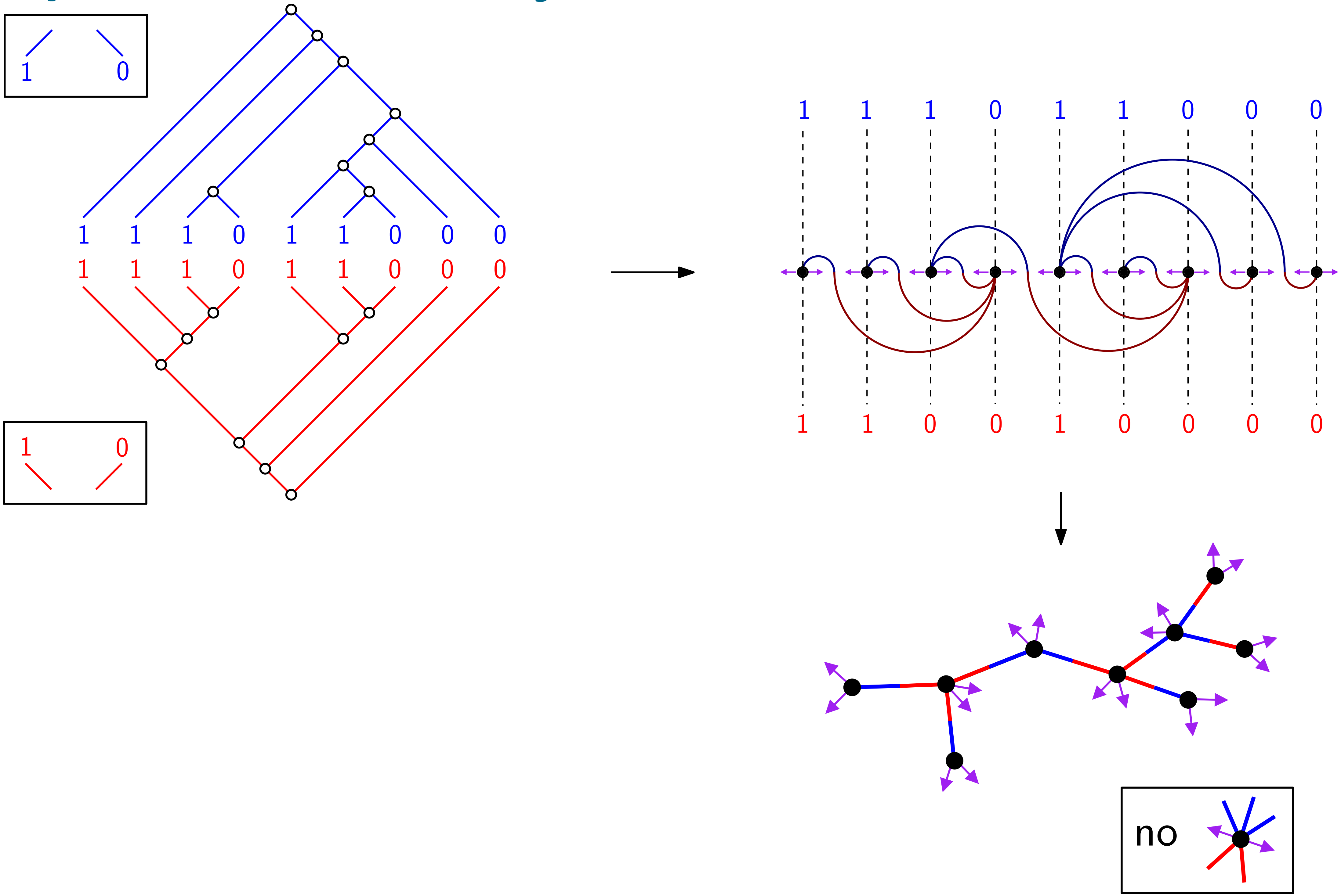
recover bivariate formula
[Bostan, Chyzak, Pilaud'23]

$b_{n,k} = \#$ intervals of size n
with $k+2$ common canopy-entries

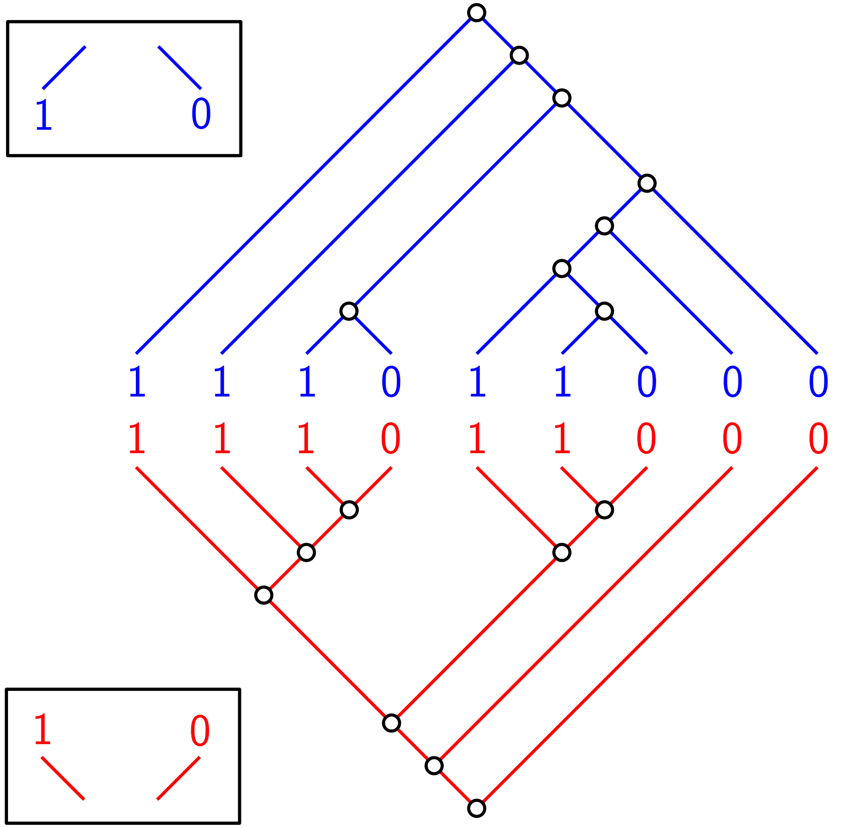
$$b_{n,k} = \frac{2}{n(n+1)} \binom{3n}{k} \binom{n}{k+2}$$

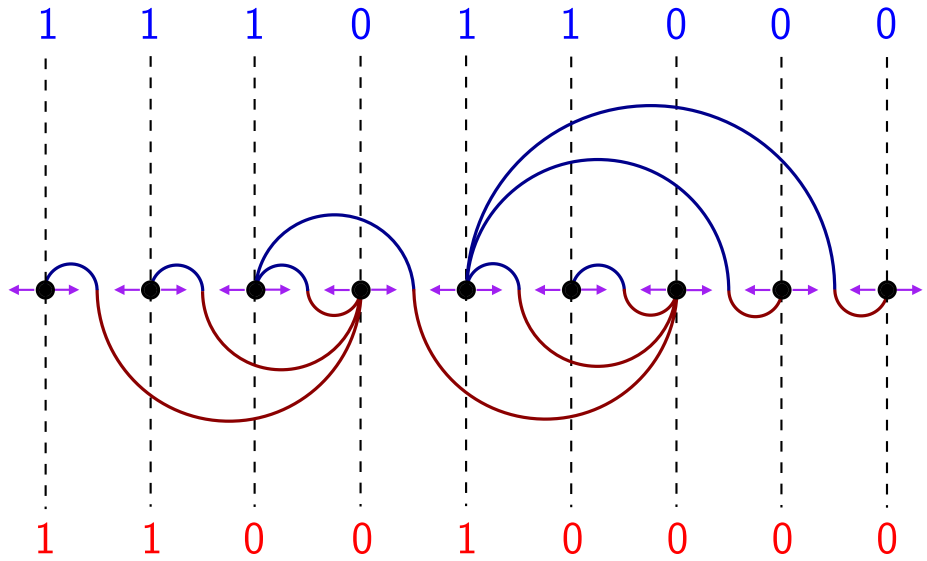


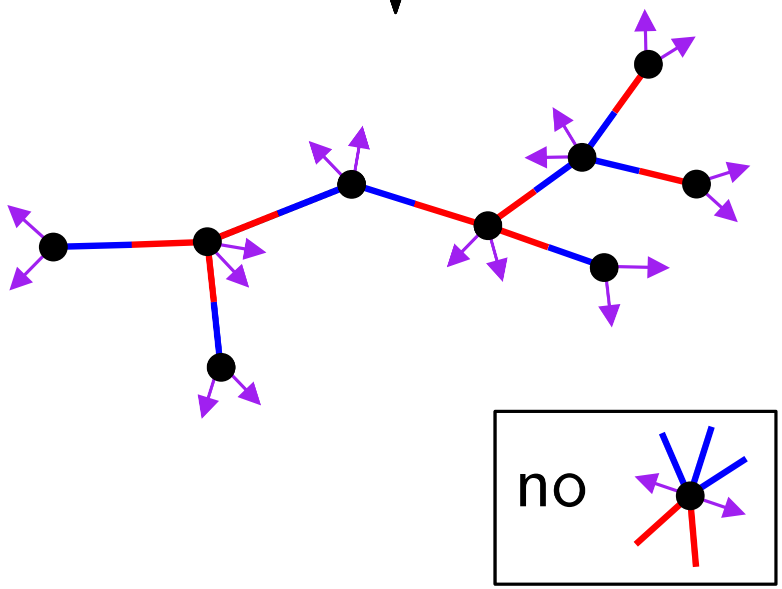
Specialization to synchronized intervals

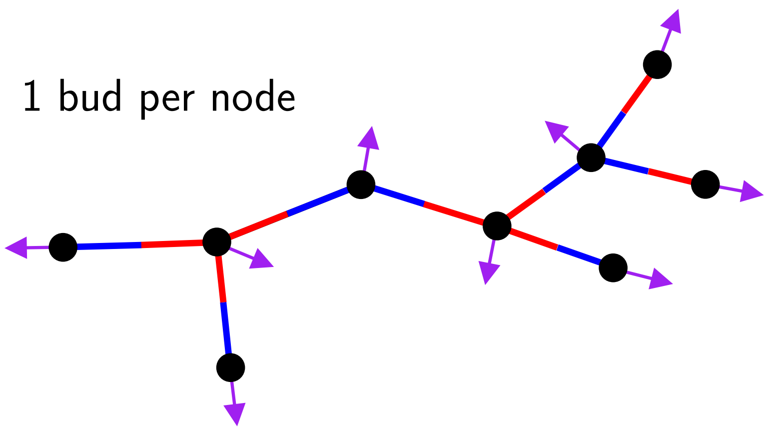


Specialization to synchronized intervals





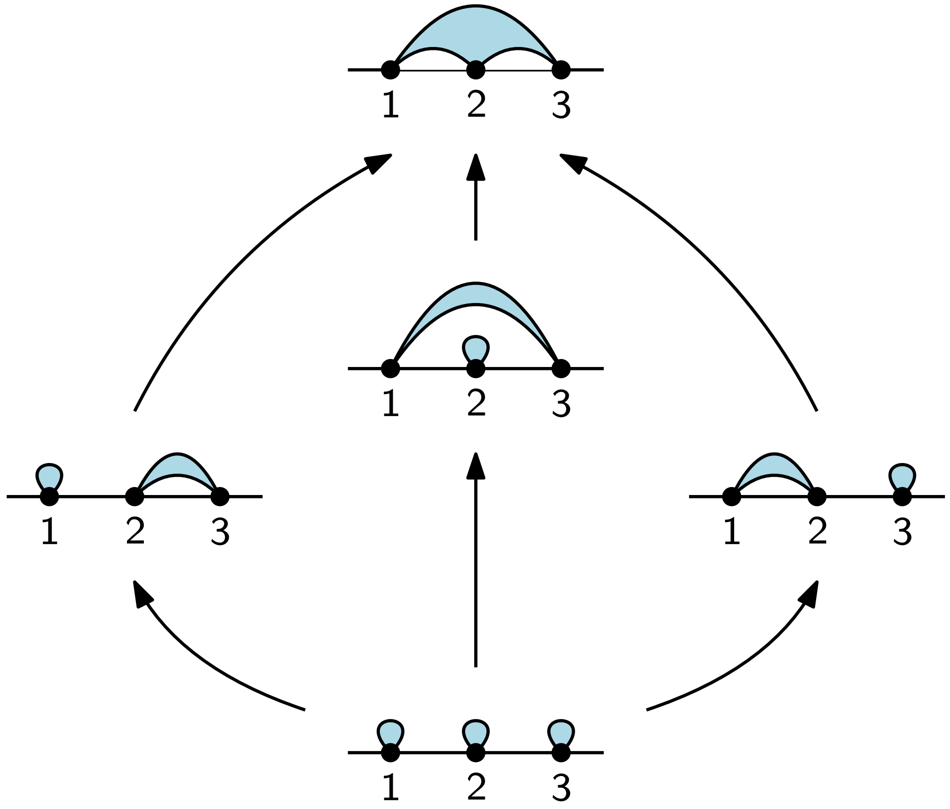




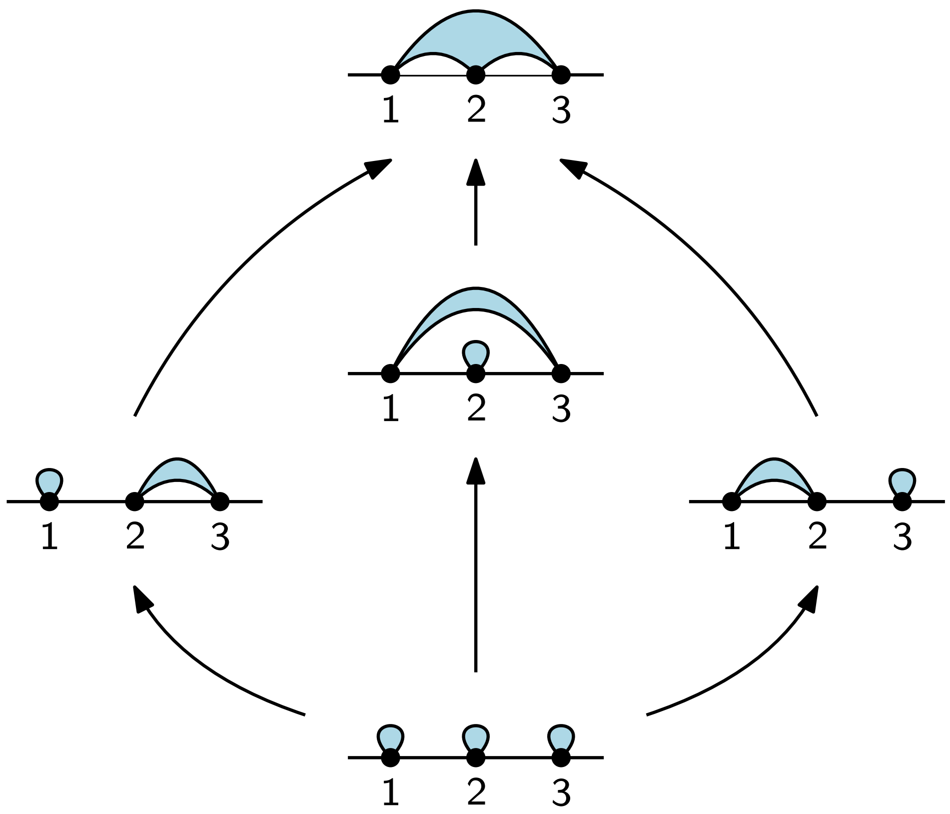
blossoming trees for simple quadrangulations

[F'07, Albenque-Poulalhon'13]

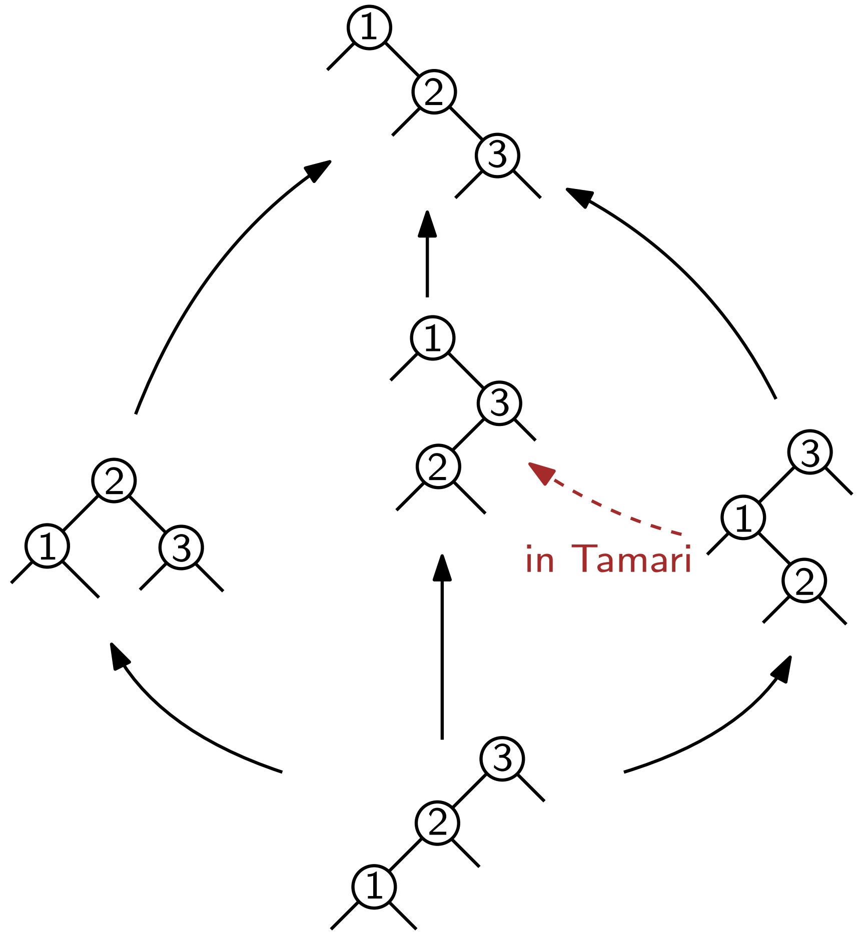
Specialization to Kreweras intervals



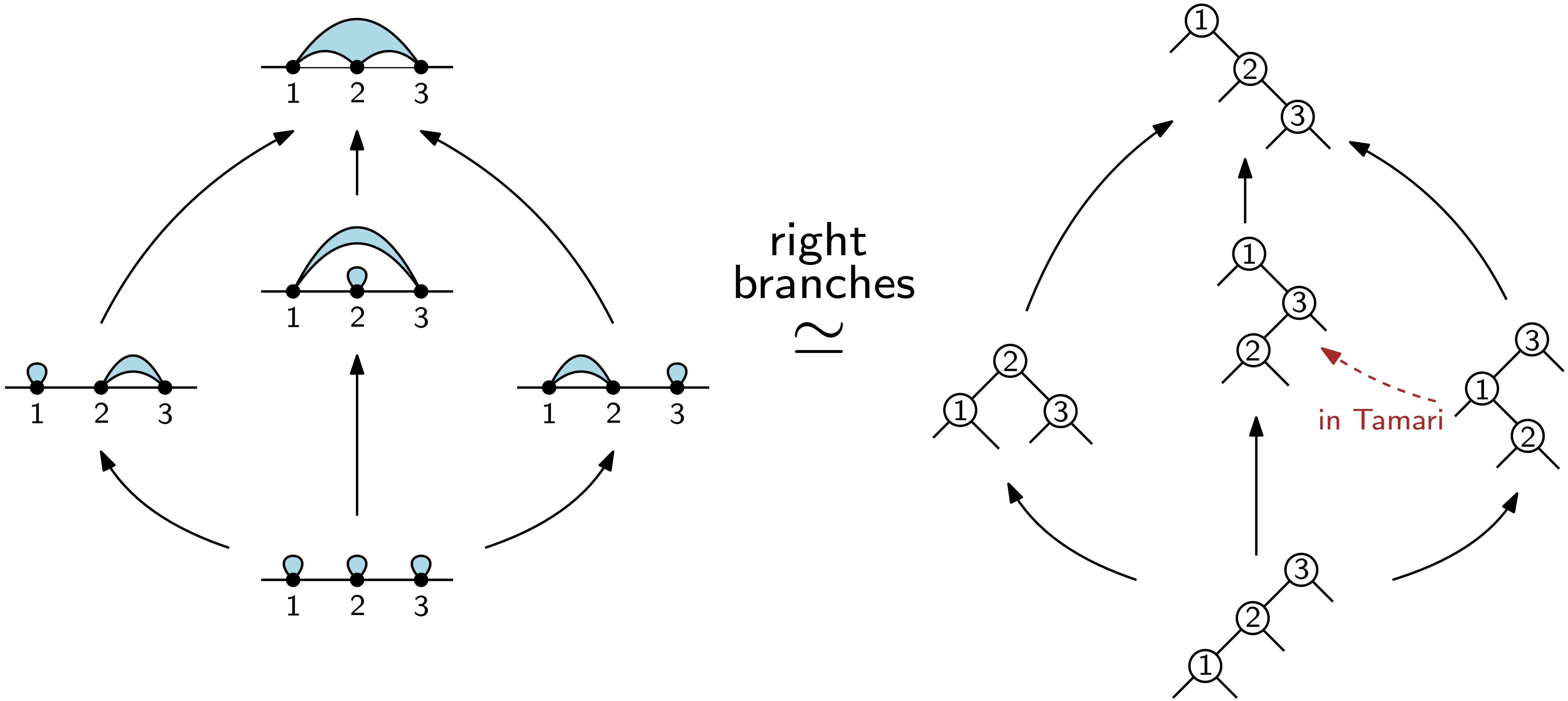
Specialization to Kreweras intervals



right branches
≃



Specialization to Kreweras intervals

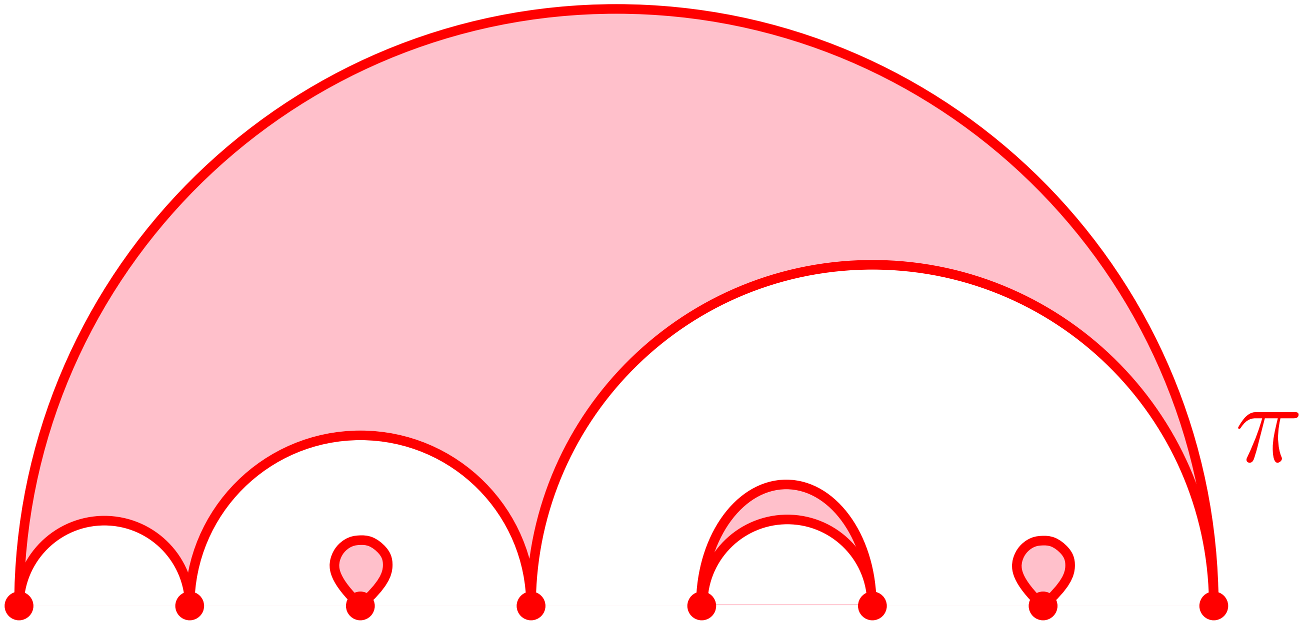


Theo: the number of Kreweras intervals of size n is $\frac{(3n)!}{n!(2n+1)!}$

[Kreweras'72, Bernardi-Bonichon'09]

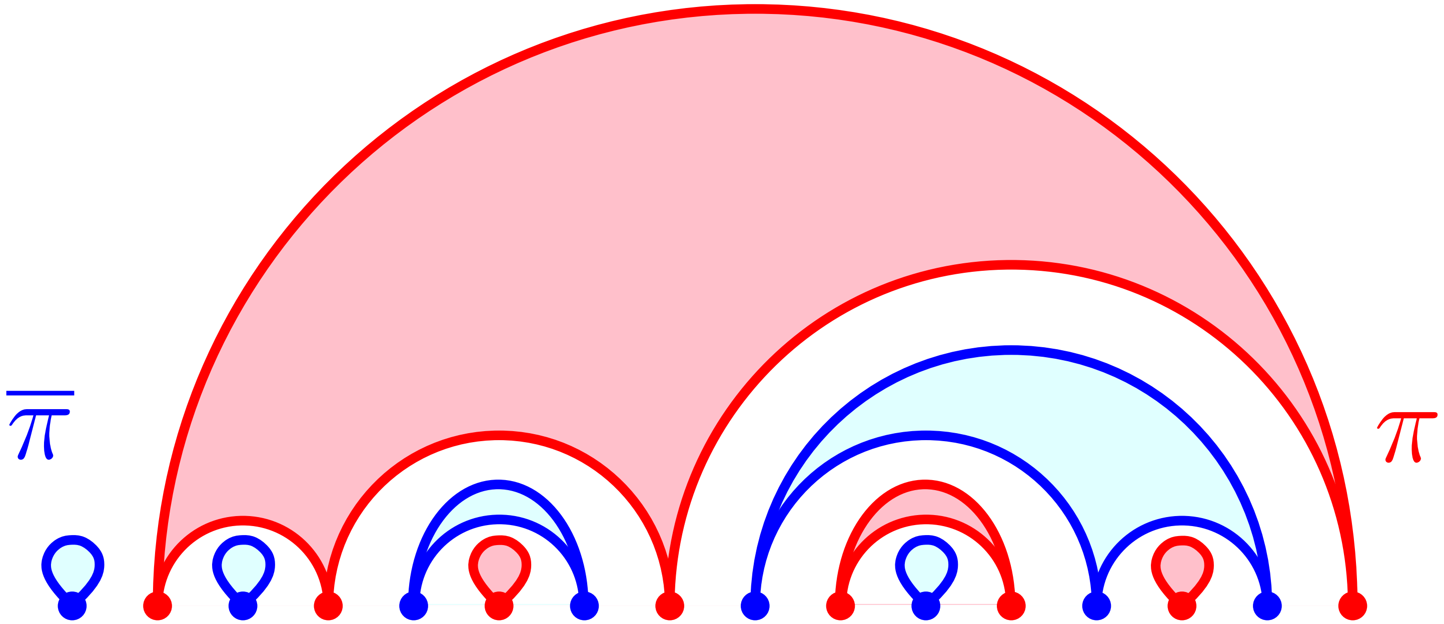
Kreweras complement

For π a non-crossing partition, let $\overline{\pi}$ be its Kreweras complement



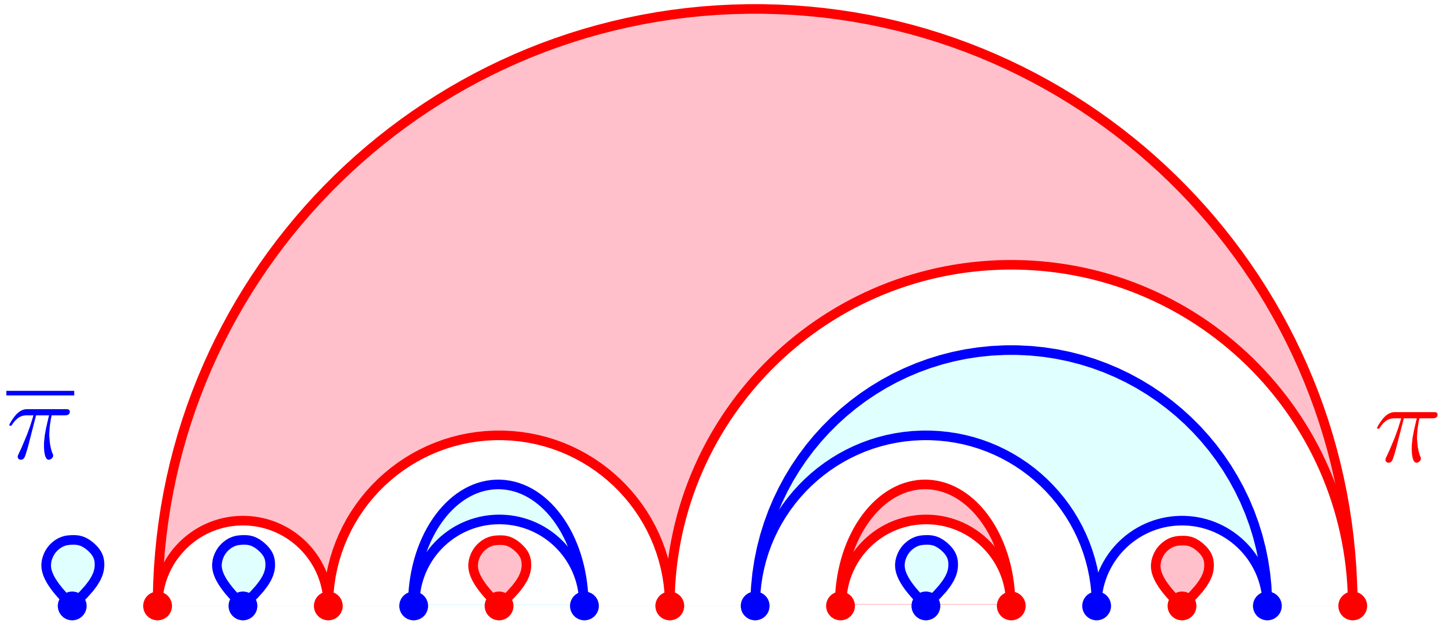
Kreweras complement

For π a non-crossing partition, let $\overline{\pi}$ be its Kreweras complement



Kreweras complement

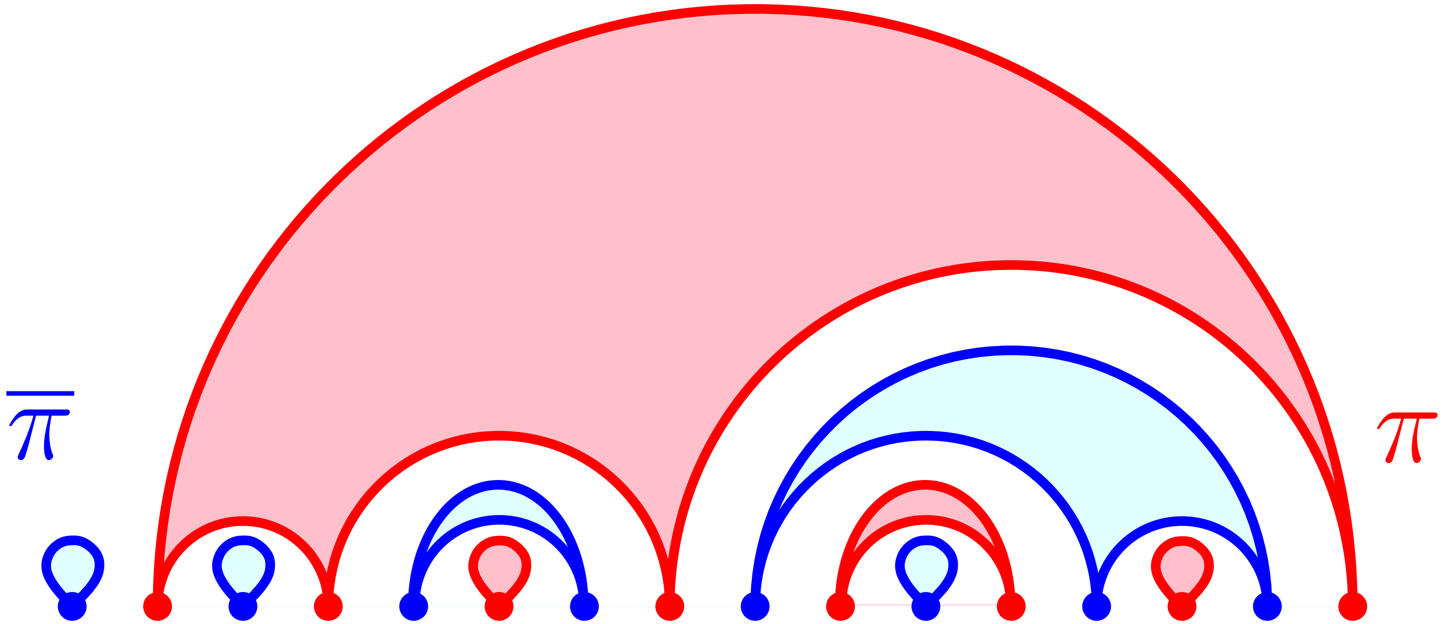
For π a non-crossing partition, let $\overline{\pi}$ be its Kreweras complement



Rk: $\pi \cup \overline{\pi}$ is a non-crossing partition

Kreweras complement

For π a non-crossing partition, let $\overline{\pi}$ be its Kreweras complement

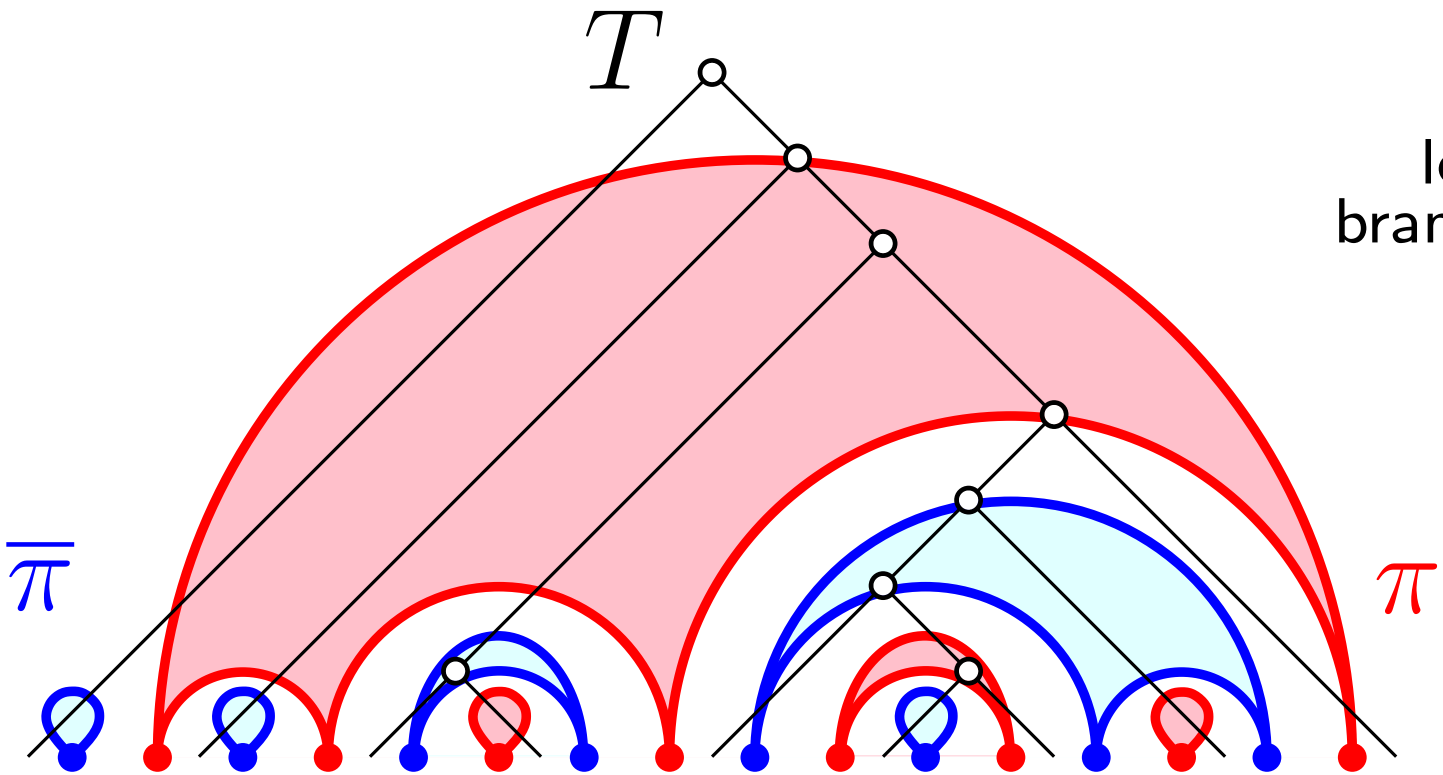


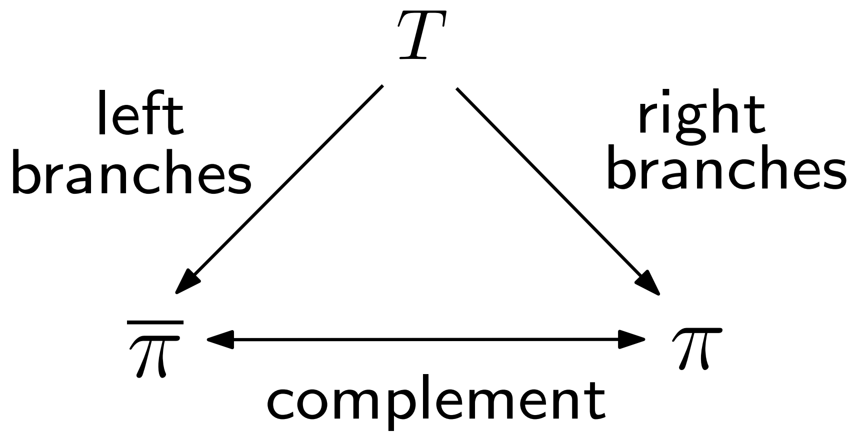
Rk: $\pi \cup \overline{\pi}$ is a non-crossing partition

more generally $\pi \leq \pi'$ iff $\pi \cup \overline{\pi'}$ is a non-crossing partition

Kreweras complement

For π a non-crossing partition, let $\overline{\pi}$ be its Kreweras complement

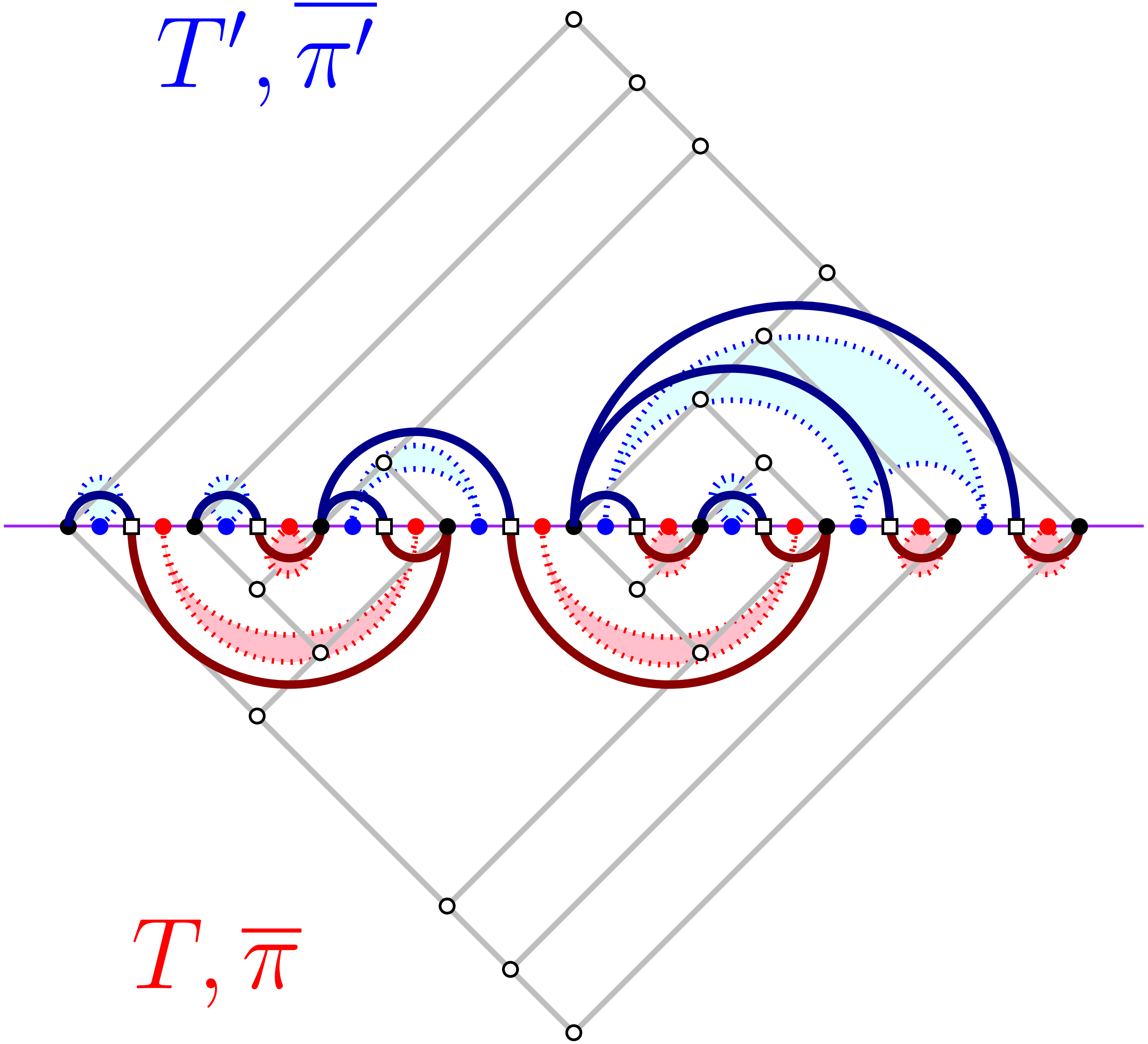




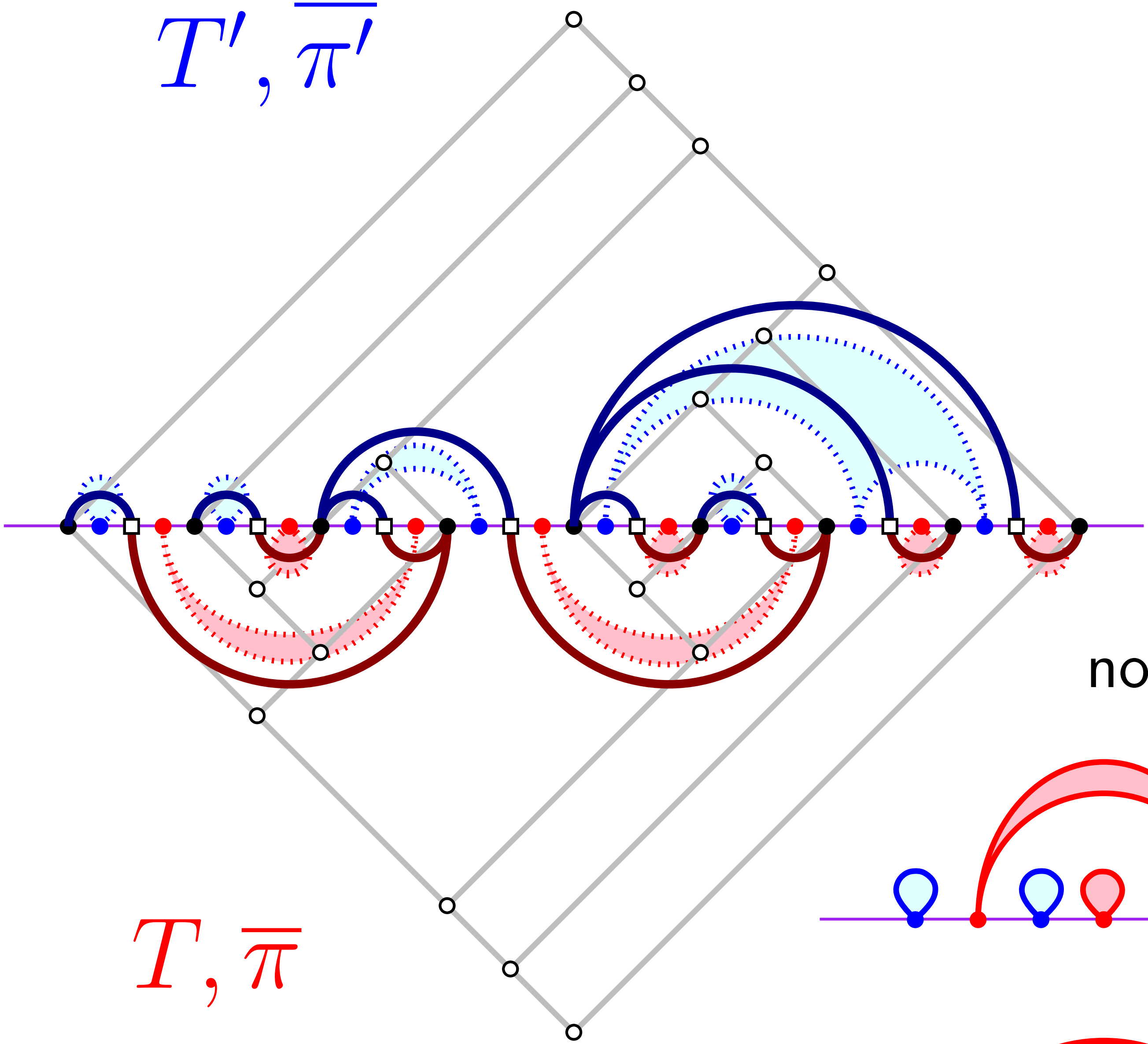
Rk: $\pi \cup \overline{\pi}$ is a non-crossing partition

more generally $\pi \leq \pi'$ iff $\pi \cup \overline{\pi'}$ is a non-crossing partition

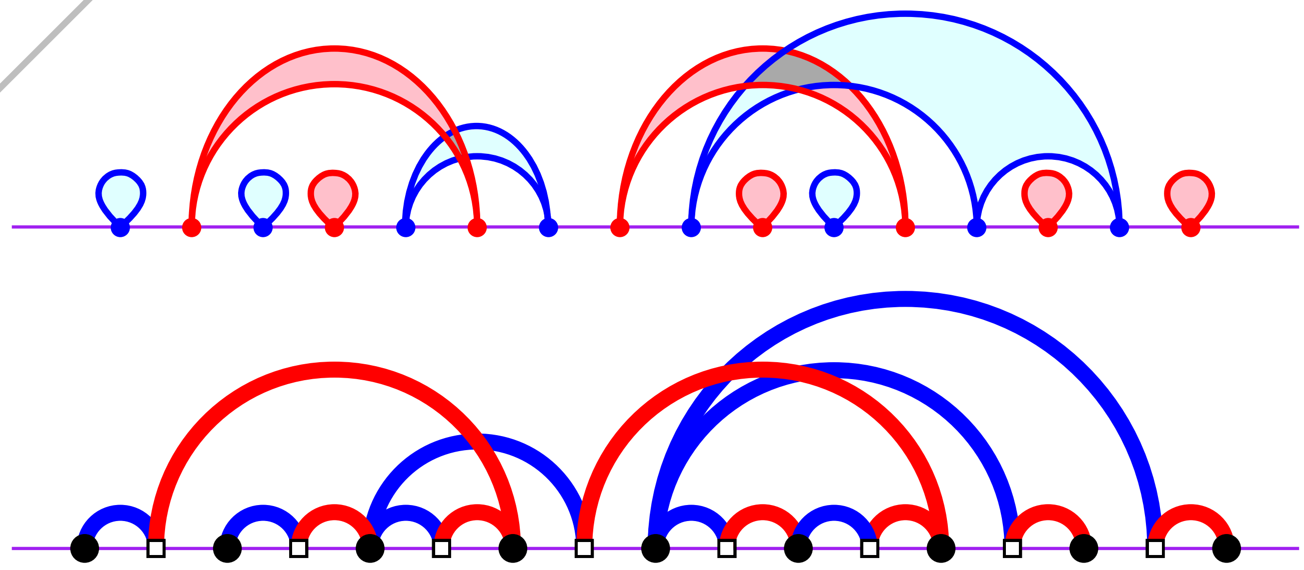
Meandering diagram via non-crossing partitions



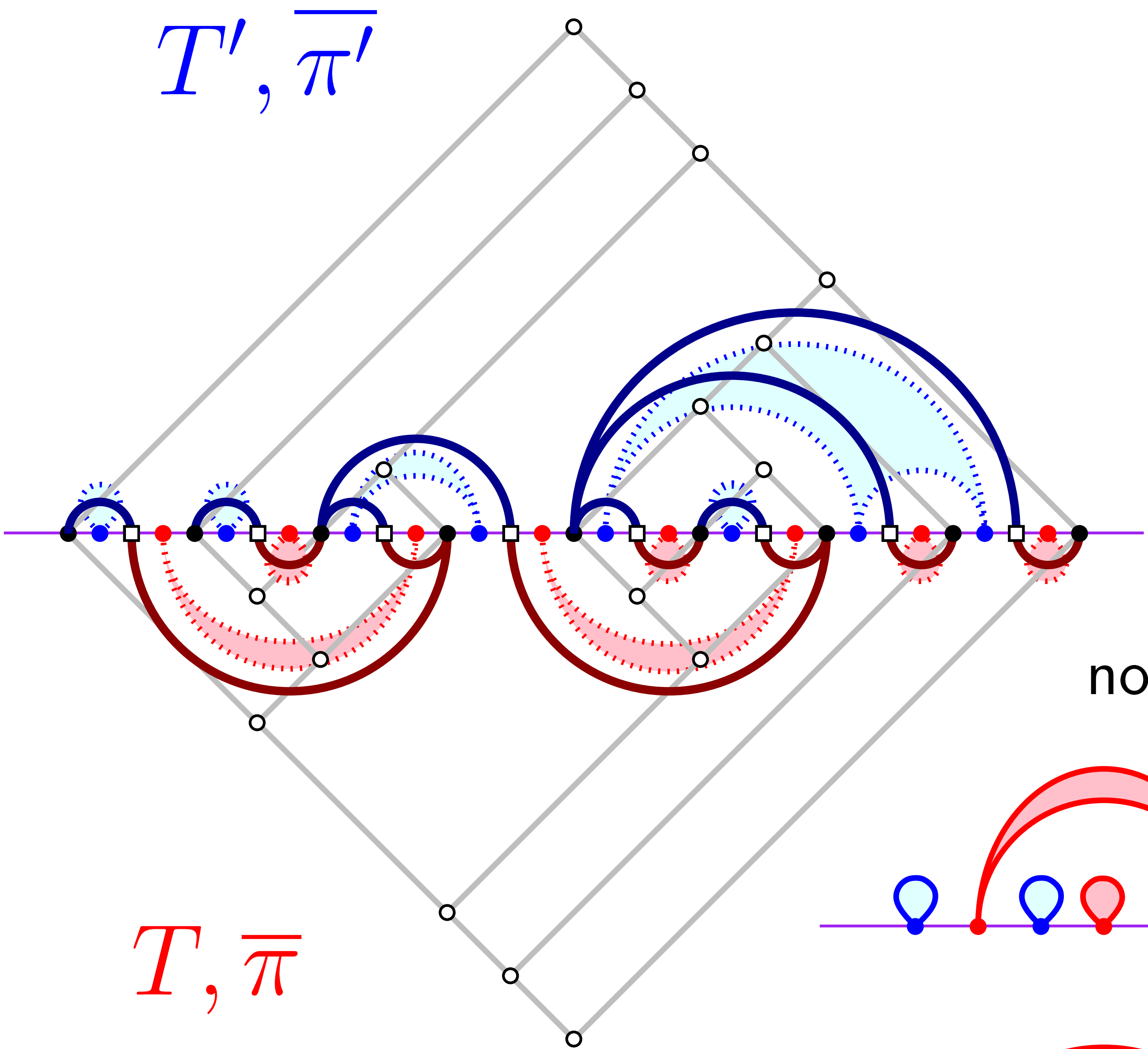
Meandering diagram via non-crossing partitions



not a Kreweras interval



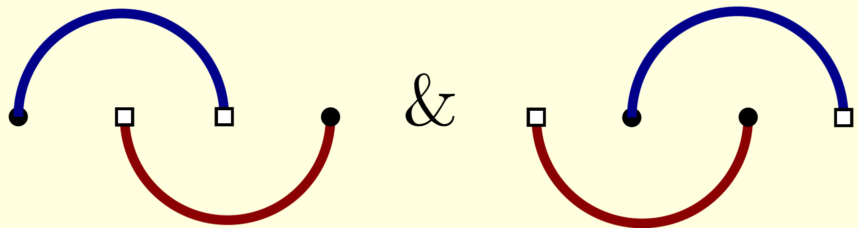
Meandering diagram via non-crossing partitions



Kreweras interval

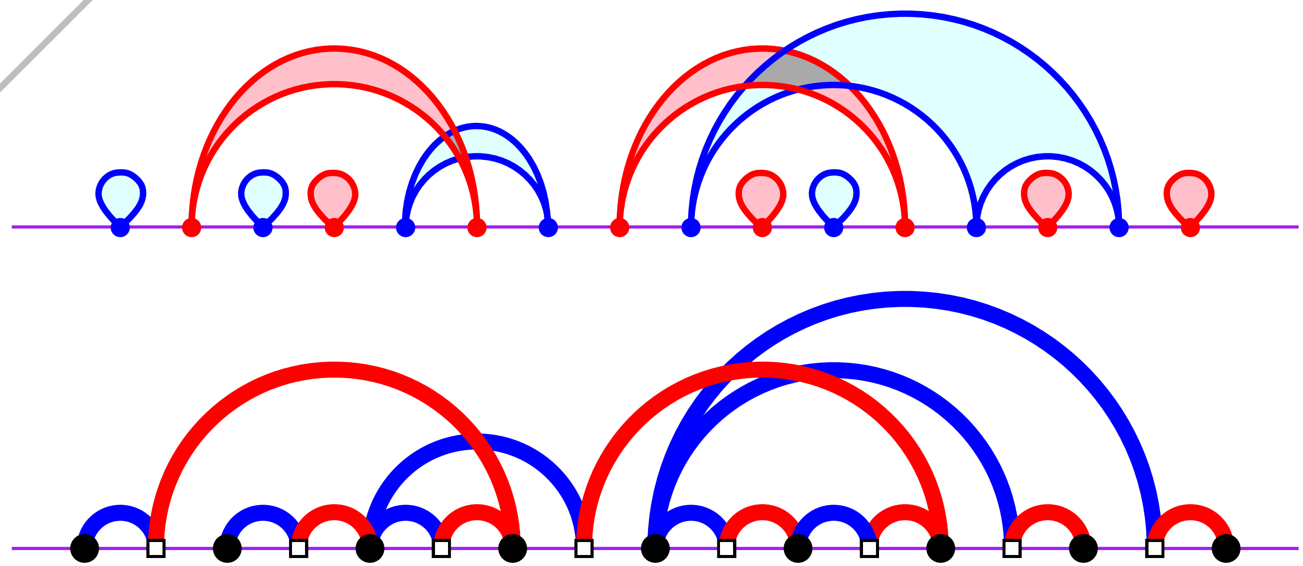
$\Updownarrow$

meandering diagram avoids

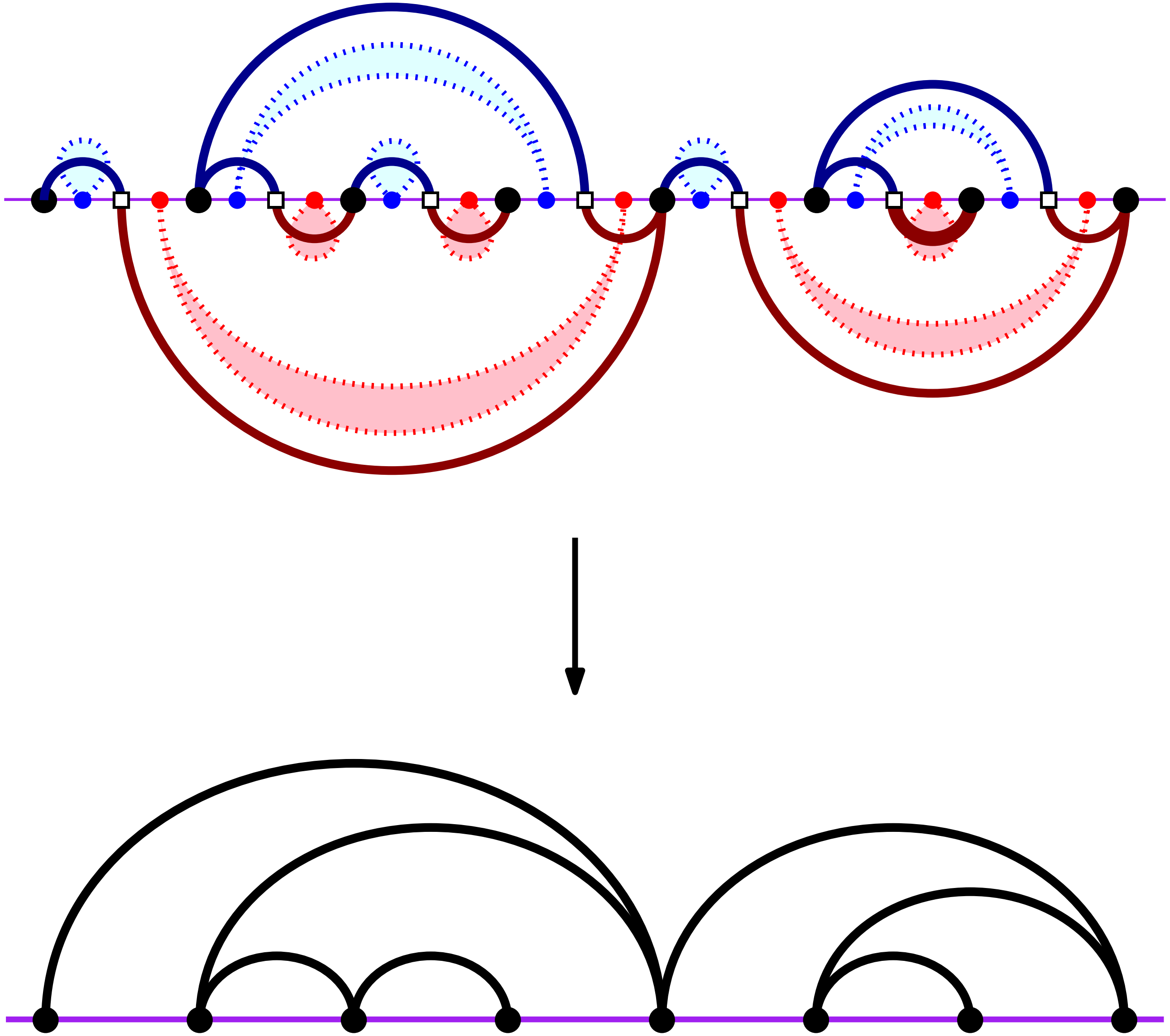


Tamari

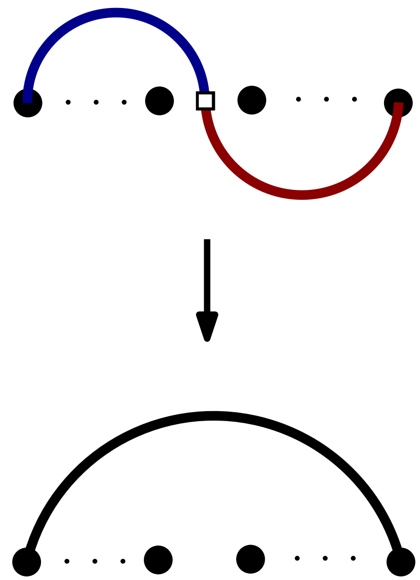
not a Kreweras interval



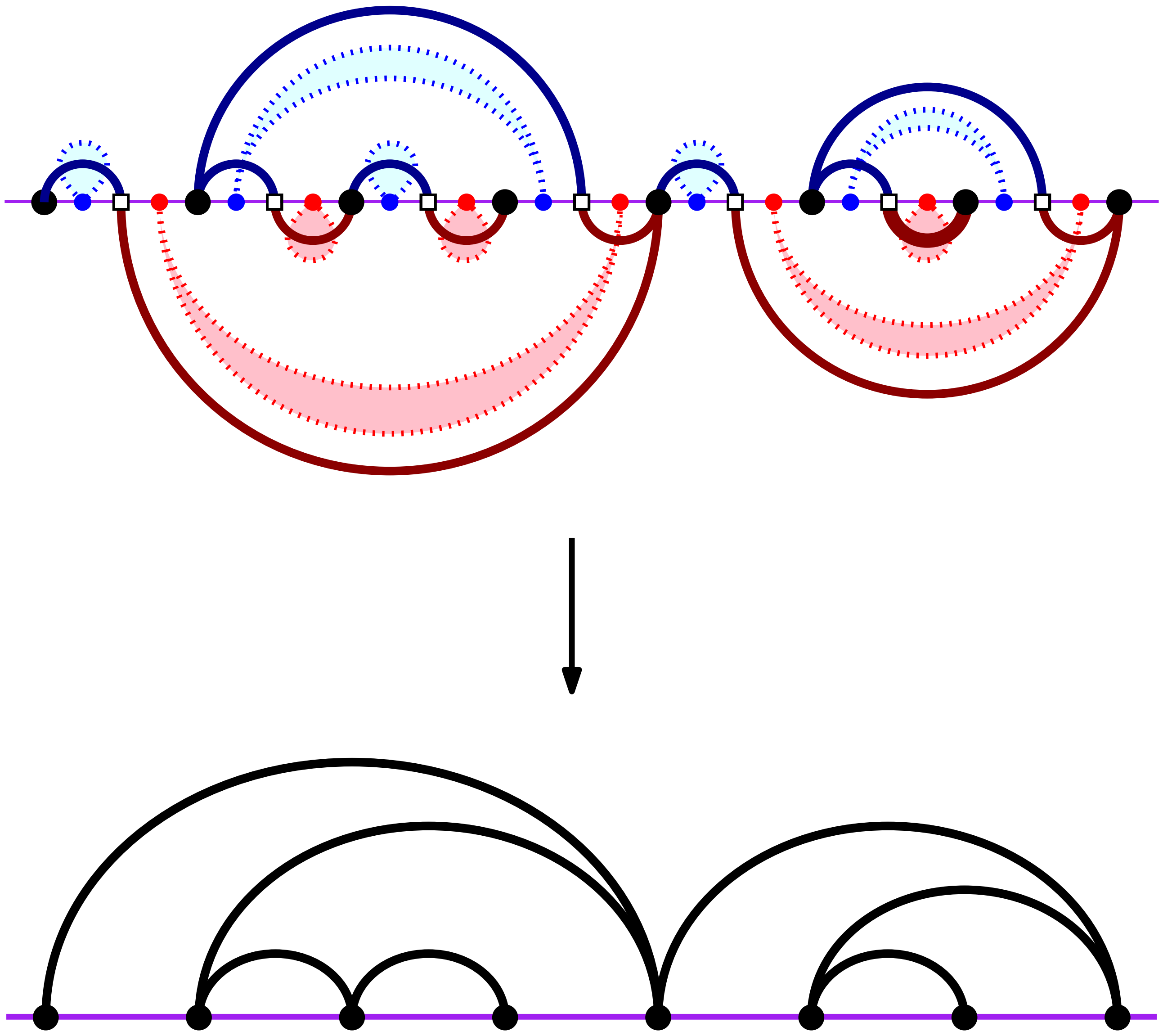
Kreweras meandering tree → non-crossing tree

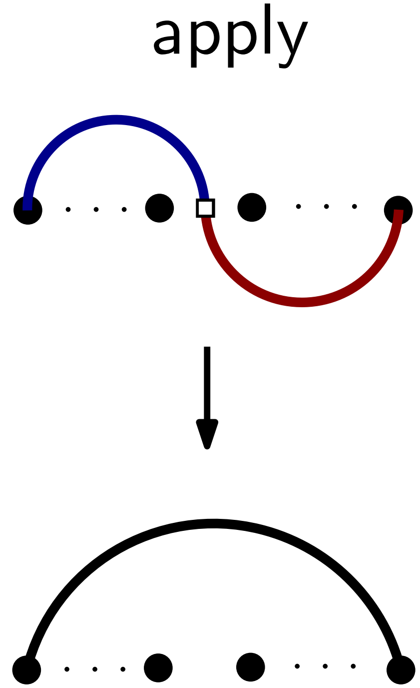


apply



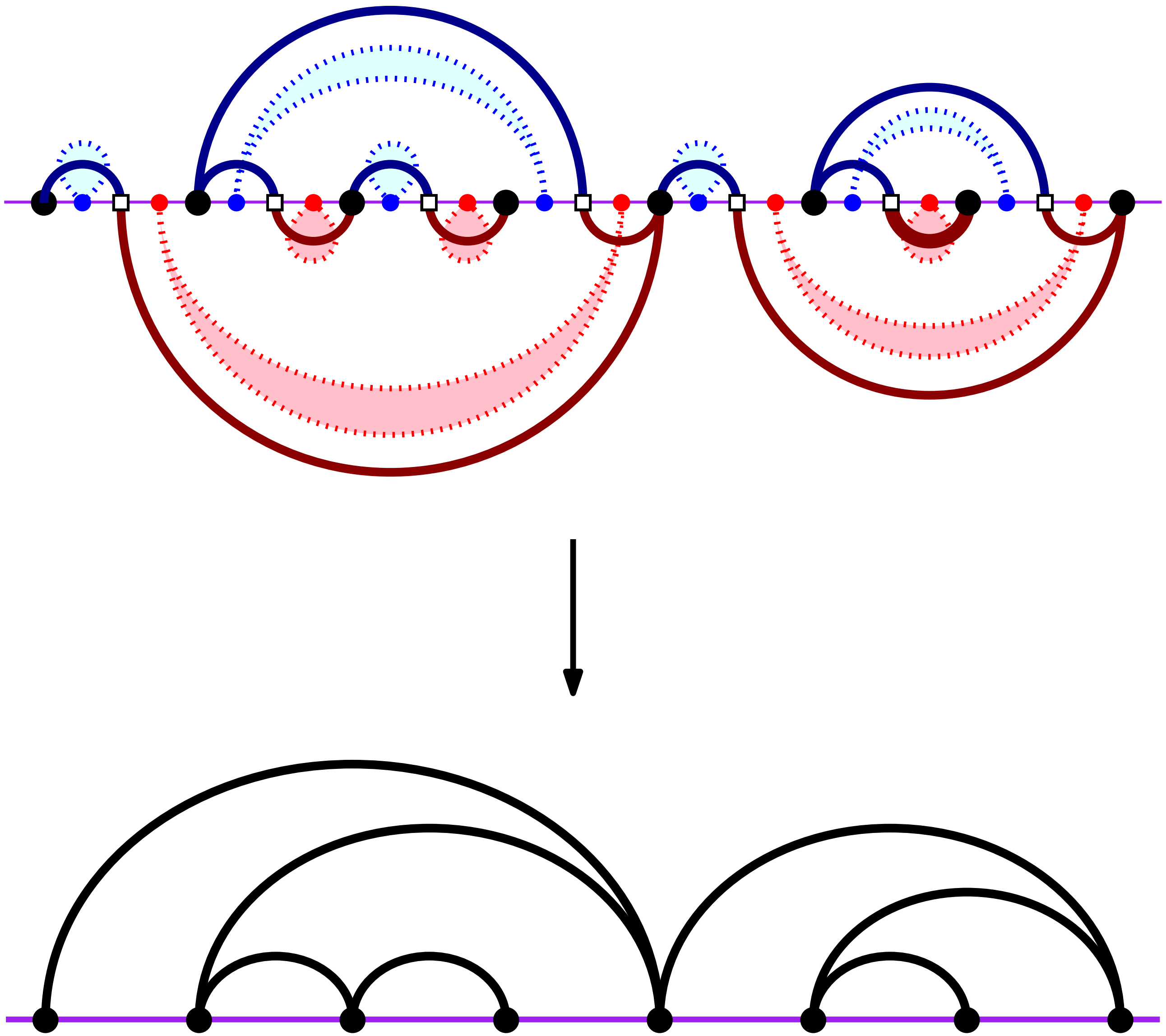
Kreweras meandering tree → non-crossing tree

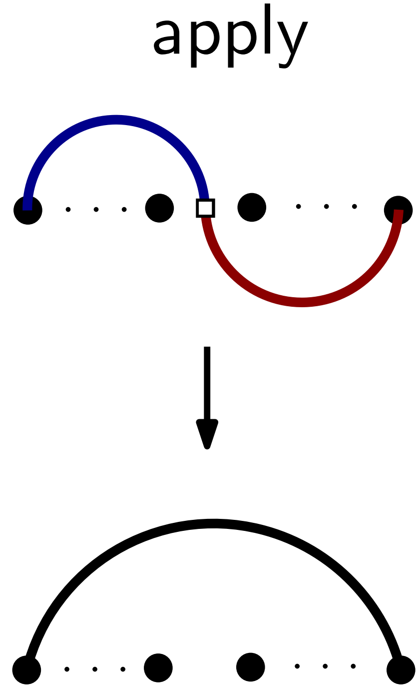




Recover **[Rognerud'18]** (obtained via interval-posets)

Kreweras meandering tree → non-crossing tree





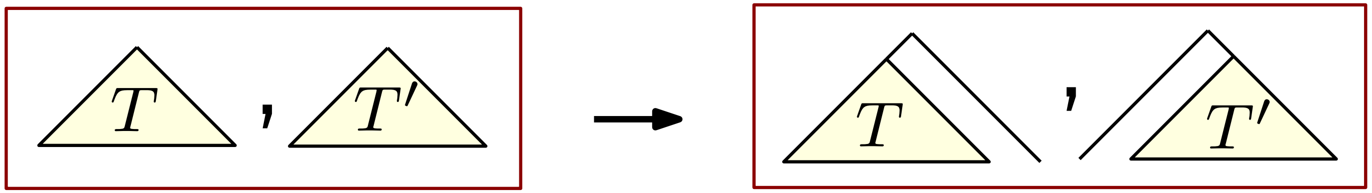
Recover **[Rognerud'18]** (obtained via interval-posets)

[Bernardi-Bonichon'09] Kreweras intervals ↔ stack triangulations

Modern and infinetely modern intervals

[Rognerud'18, Chapoton'06]

rise operator

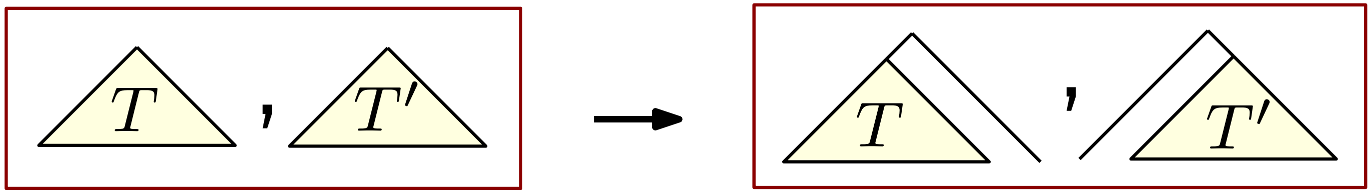


A Tamari interval (T, T') is **modern** if $\mathrm{rise}(T, T')$ is also a Tamari interval

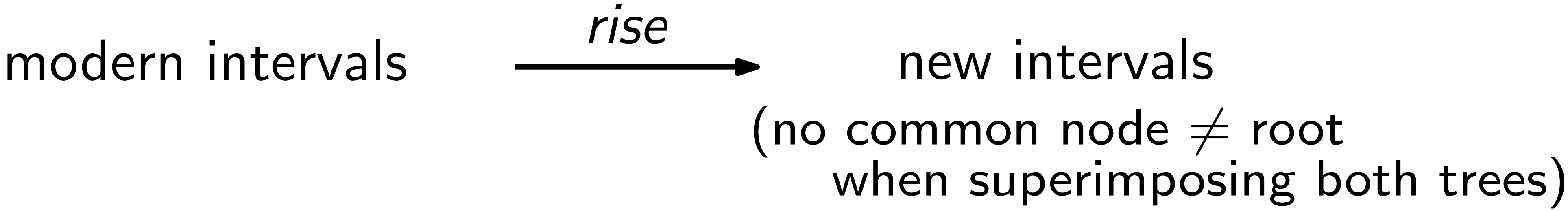
Modern and infinetely modern intervals

[Rognerud'18, Chapoton'06]

rise operator



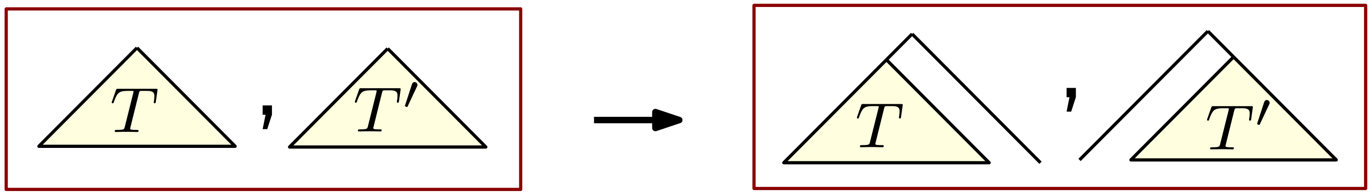
A Tamari interval (T, T') is **modern** if $\mathrm{rise}(T, T')$ is also a Tamari interval



Modern and infinetely modern intervals

[Rognerud'18, Chapoton'06]

rise operator



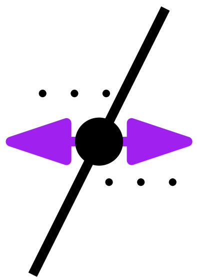
A Tamari interval (T, T') is **modern** if $\mathrm{rise}(T, T')$ is also a Tamari interval

modern intervals $\xrightarrow{\textit{rise}}$ new intervals

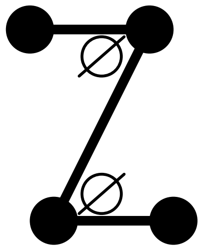
(no common node $\neq$ root
when superimposing both trees)

(T, T') is **infinitely modern** if $\mathrm{rise}^k(T, T')$ is a Tamari interval $\forall k \geq 0$

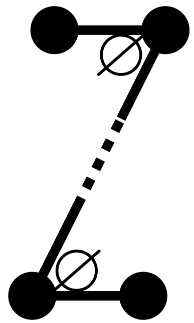
Subfamilies & forbidden patterns on blossoming trees



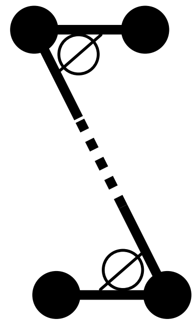
Synchronized



Modern

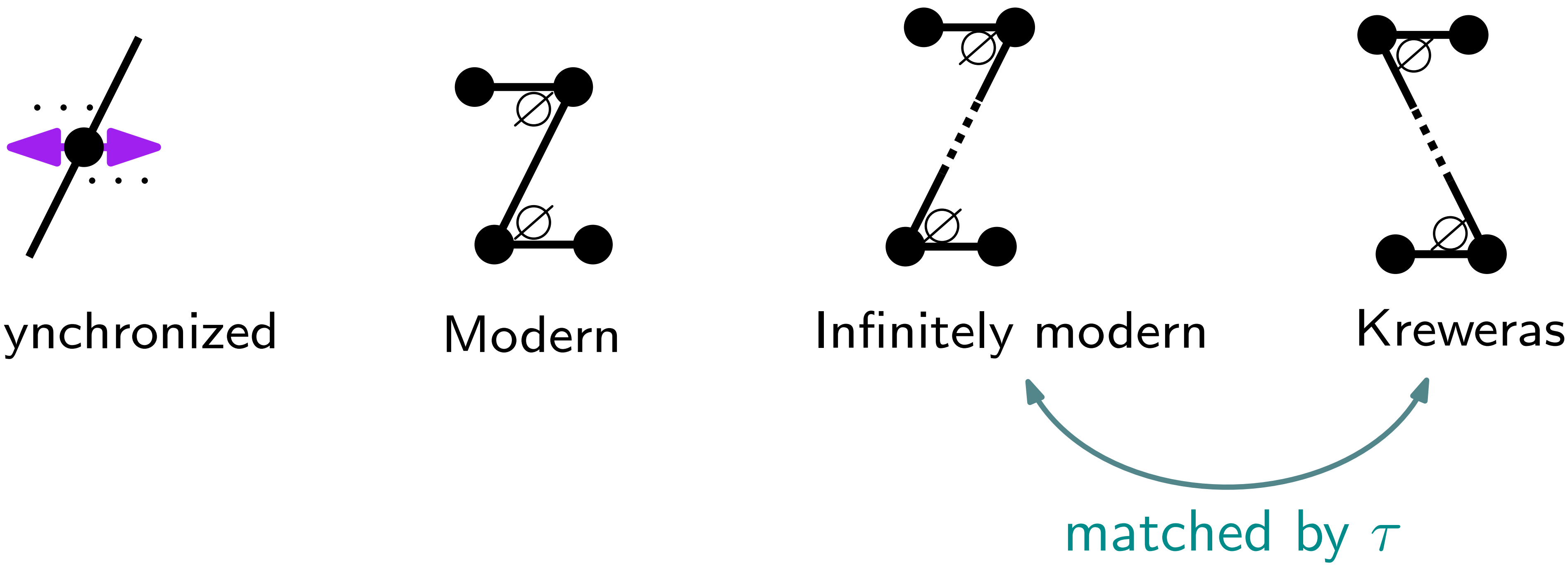


Infinitely modern



Kreweras

Subfamilies & forbidden patterns on blossoming trees

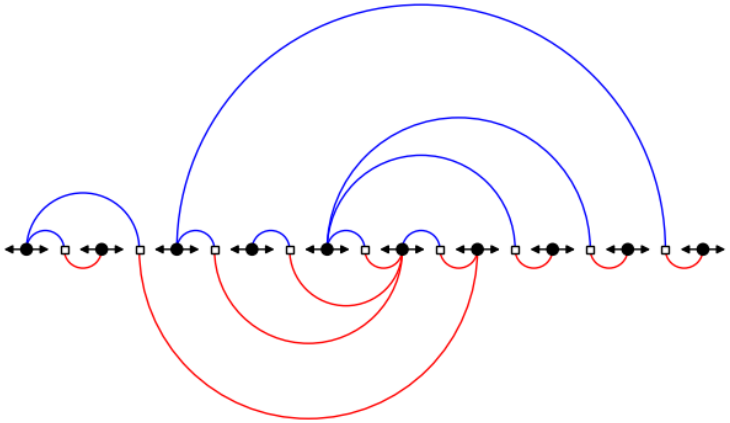


New involution τ on Tamari intervals: mirror of blossoming trees

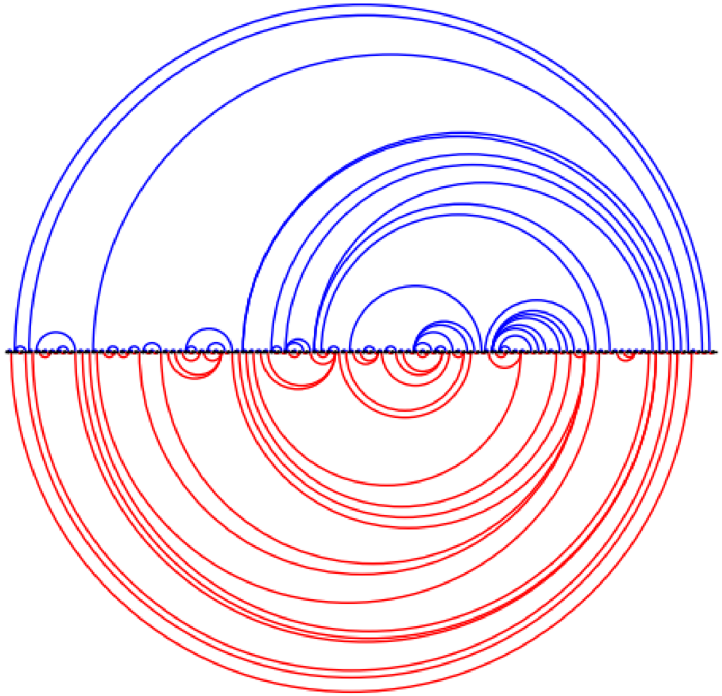
Counting results obtained from the bijection

Types	General size n	Self-dual size $2k$	Self-dual size $2k+1$
General	$\frac{2}{n(n+1)}\binom{4n+1}{n-1}$	$\frac{1}{3k+1}\binom{4k}{k}$	$\frac{1}{k+1}\binom{4k+2}{k}$
Synchronized	$\frac{2}{n(n+1)}\binom{3n}{n-1}$	0	$\frac{1}{k+1}\binom{3k+1}{k}$
Modern / new for size-1	$\frac{3\cdot 2^{n-1}}{(n+1)(n+2)}\binom{2n}{n}$	$\frac{2^{k-1}}{k+1}\binom{2k}{k}$	$\frac{2^{k}}{k+1}\binom{2k}{k}$
Modern and synchronized	$\frac{1}{n+1}\binom{2n}{n}$	0	$\frac{1}{k+1}\binom{2k}{k}$
Inf. modern / Kreweras	$\frac{1}{2n+1}\binom{3n}{n}$	$\frac{1}{2k+1}\binom{3k}{k}$	$\frac{1}{k+1}\binom{3k+1}{k}$

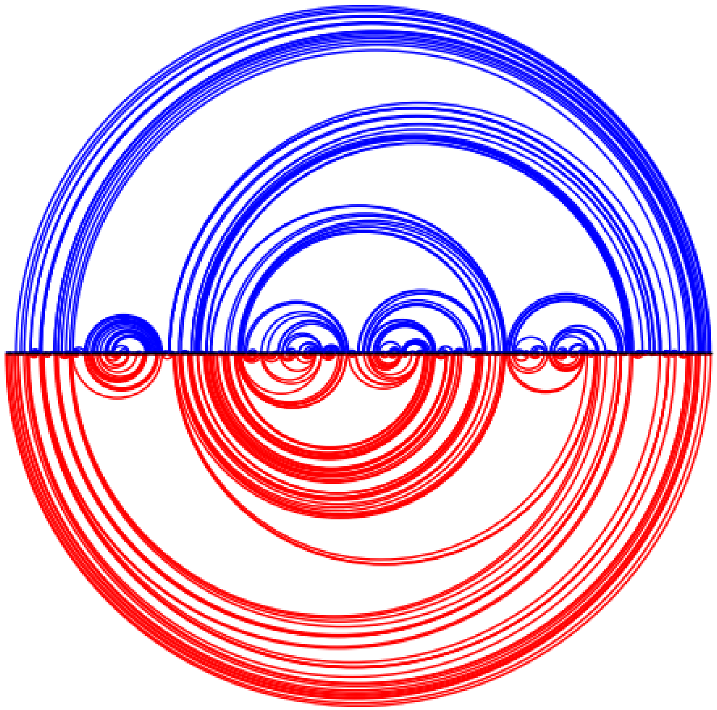
Random samples



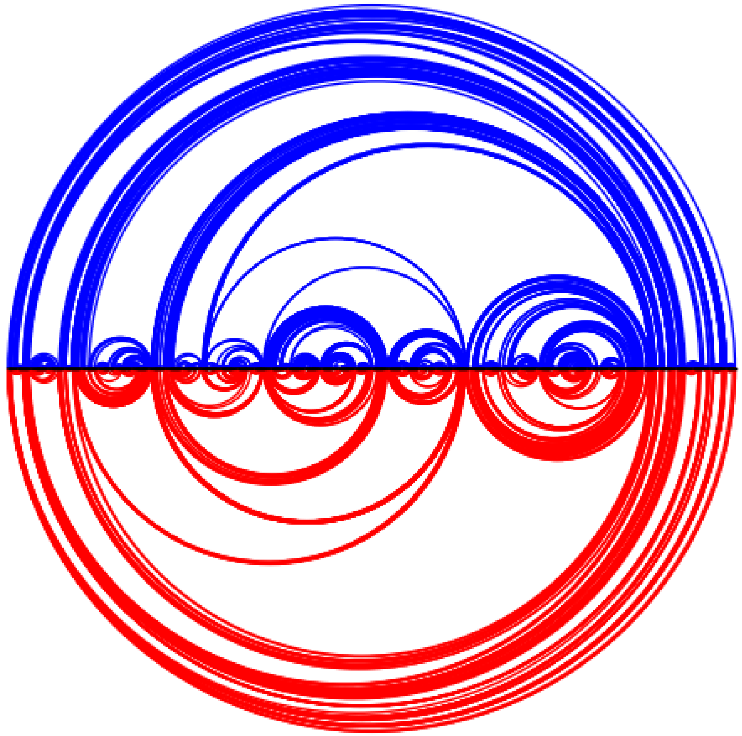
10^1



10^2



10^3



10^4