Bijections for d-angulated dissections

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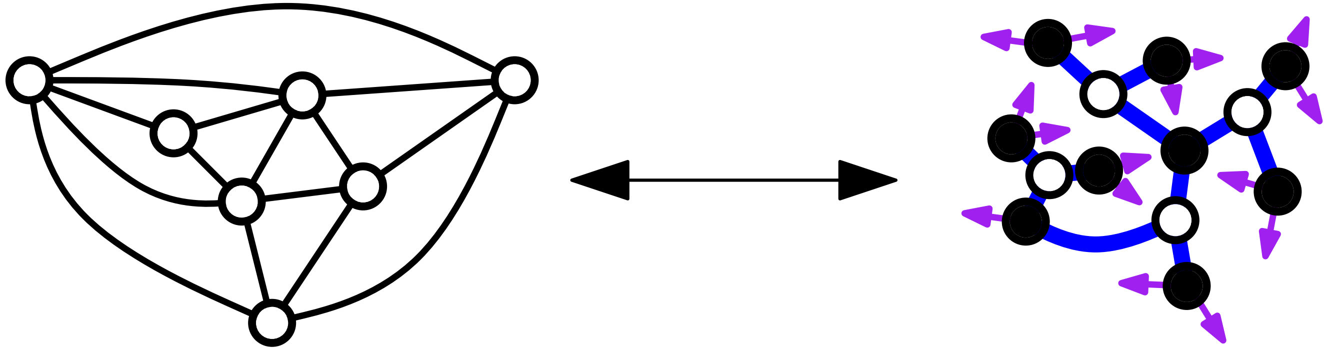
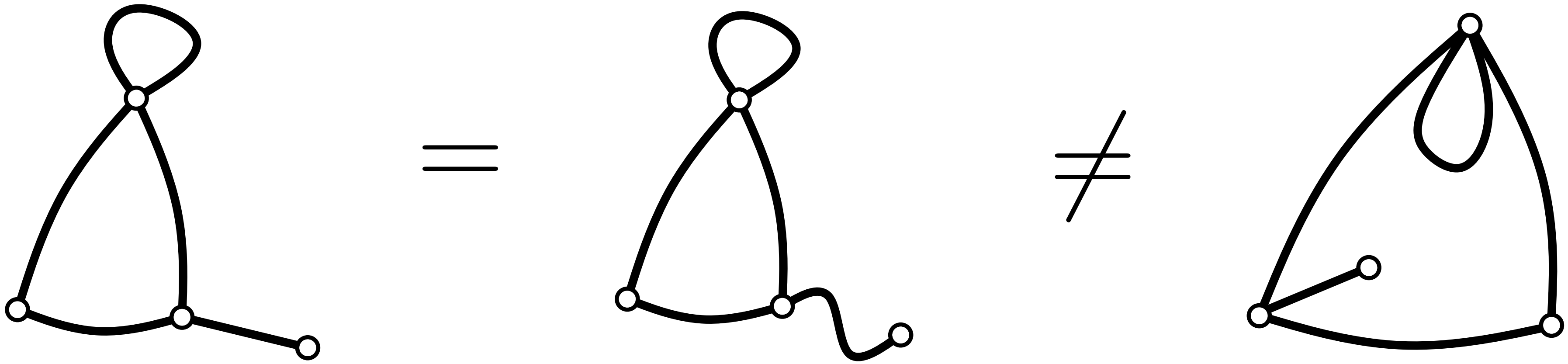
Joint work with Olivier Bernardi (Brandeis)



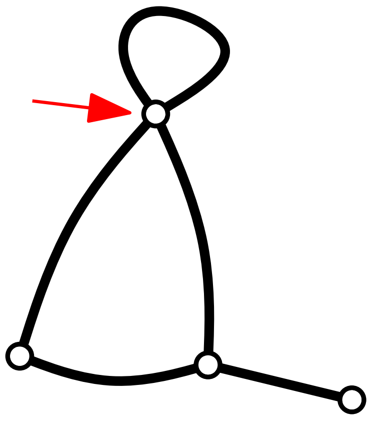
LIX, Nov. 2013

Planar maps

Def. Planar map = connected graph embedded in the plane up to isotopy



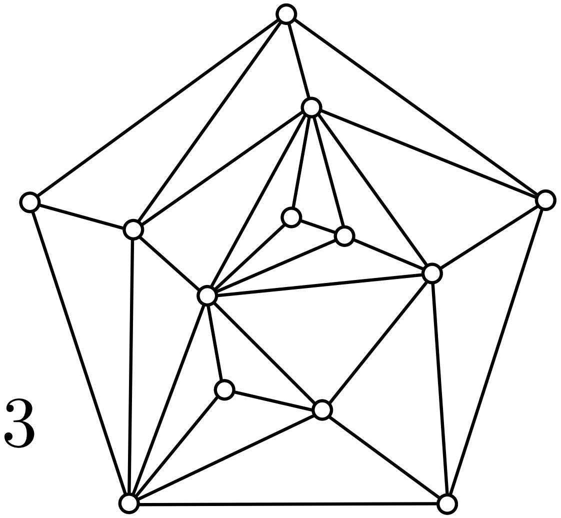
A map is rooted by marking a corner incident to the outer face



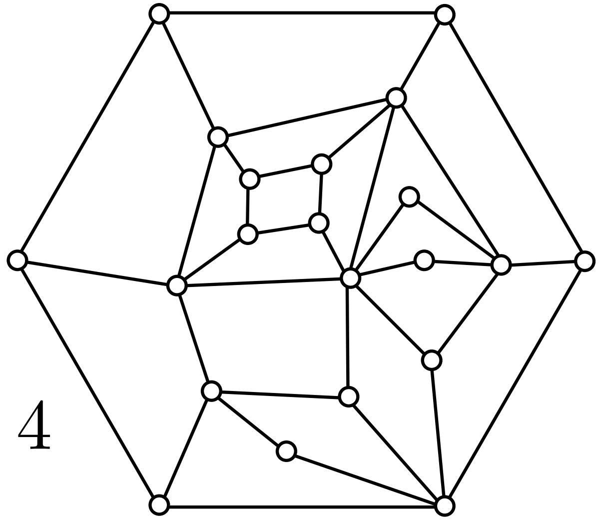
(k,d)-dissections. Irreducibility

(k,d)-dissection $=$ map with simple outer boundary of length k
inner faces of degree d, girth d

$k=5\ d=3$



$k=6\ d=4$



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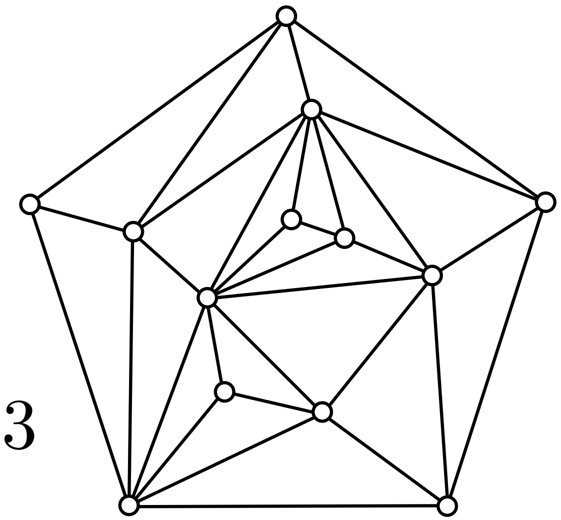

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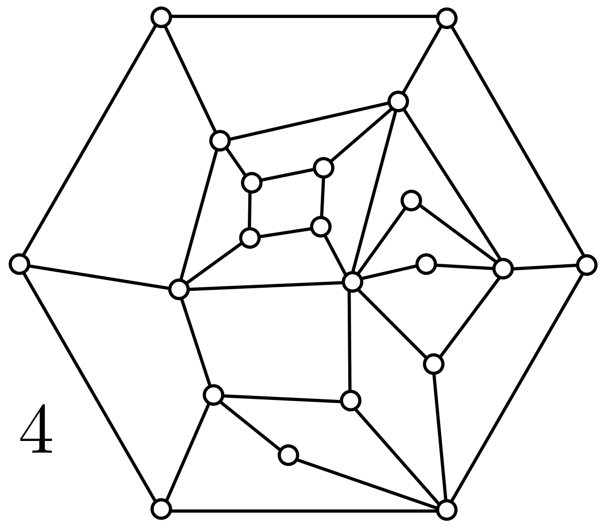
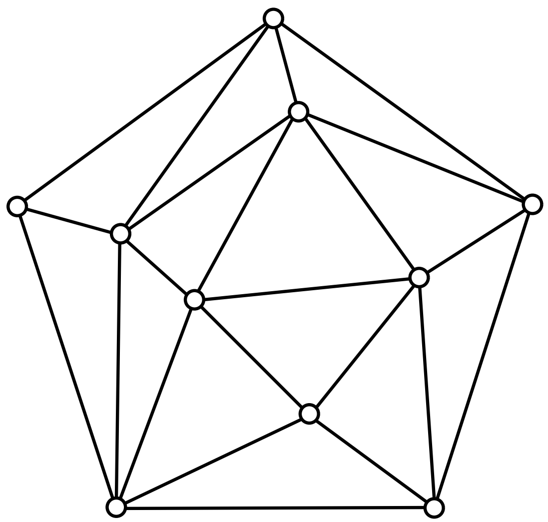
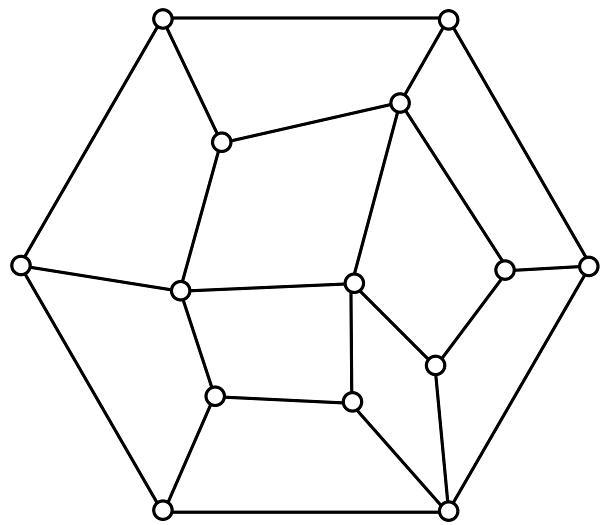


A (k, d)-dissection is **irreducible** if all non-facial cycles have length $> d$





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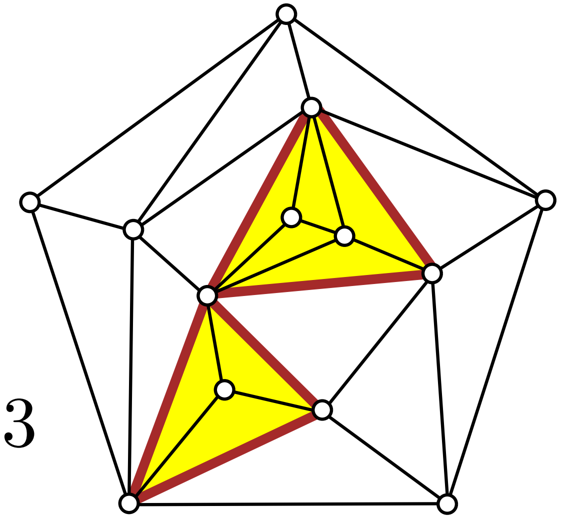

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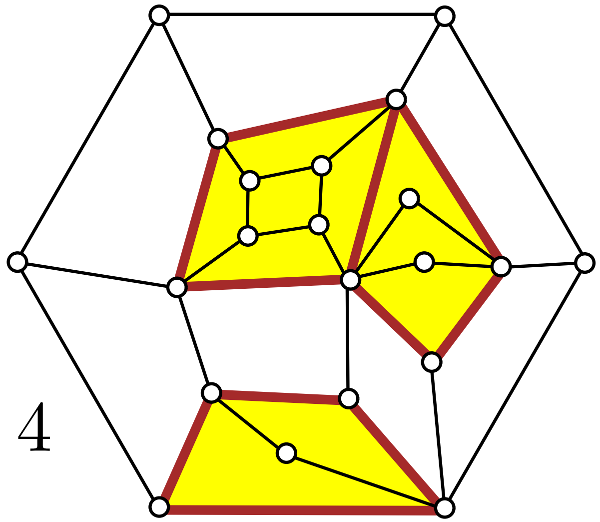

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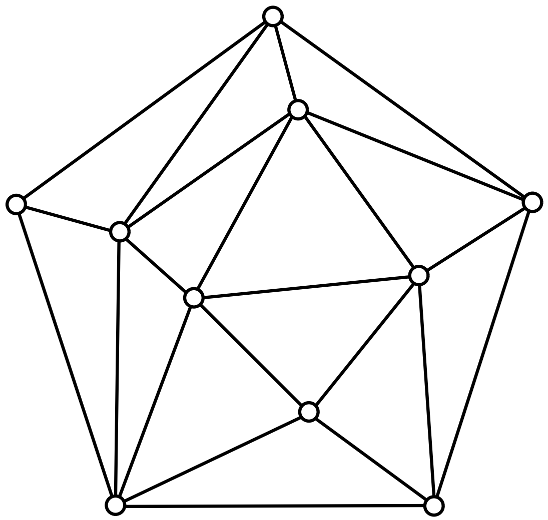

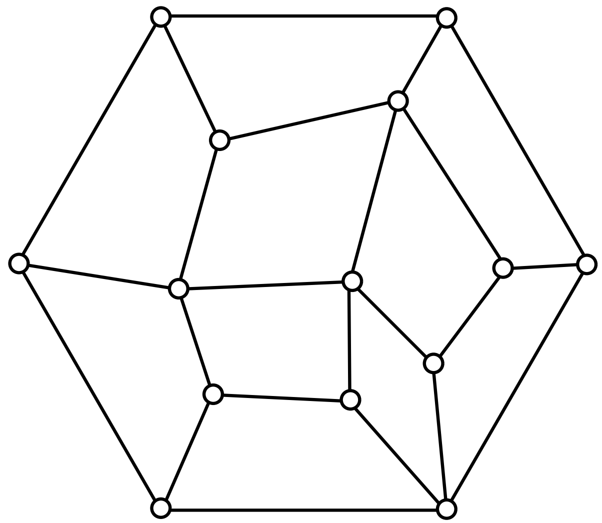

Counting formulas for dissections $(d=3,4)$

- # rooted k-outer triangulated dissections with n inner vertices

simple

$$\frac{2(2k-3)!}{(k-1)!(k-3)!}\frac{(4n+2k-5)!}{n!(3n+2k-3)!}$$

[Brown'64]
[Poulalhon,Schaeffer'06], [Albenque, Poulalhon'13]
[Bernardi,F'10]

irreducible $(k\geq 4)$

$$\frac{(2k-4)!}{(k-4)!(k-1)!}\frac{(3n+k-4)!}{n!(2n+k-2)!}$$

[Tutte'62]
[F'05] $k=4$
[Bouttier,Guitter'13]

- # rooted k-outer $(k=2p)$ quadrangulated dissections with n inner vertices

simple

$$\frac{3(3p-2)!}{(p-2)!(2p-1)!}\frac{(3n+3p-4)!}{n!(2n+3p-2)!}$$

[Brown'65]
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irreducible $(p\geq 3)$

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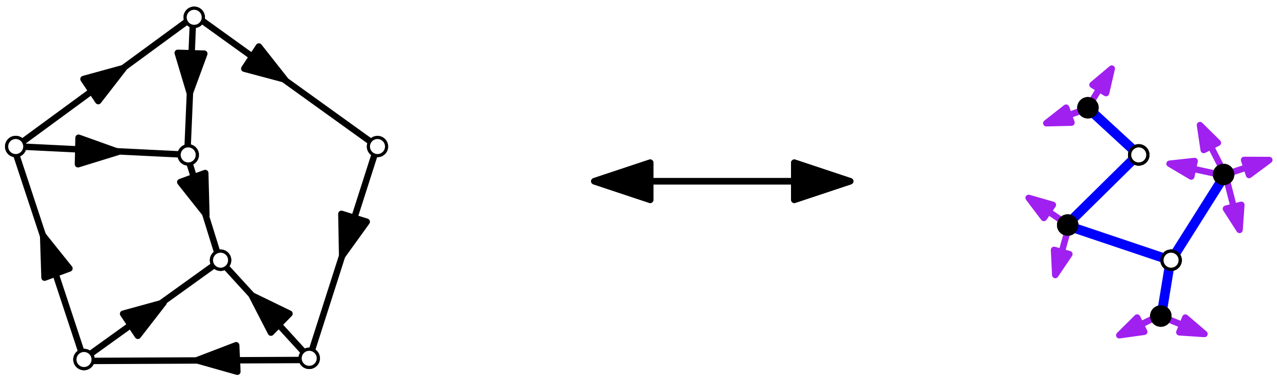
[Mullin, Schellenberg'68]
[F, Poulalhon, Schaeffer'05] $p=3$
[Bouttier,Guitter'13]

Outline

1. **Master bijection** between a class of **oriented maps** and a class of bicolored **decorated trees** (which are called mobiles).



2. Application to d-angulations of girth d (starting with $d = 3$)

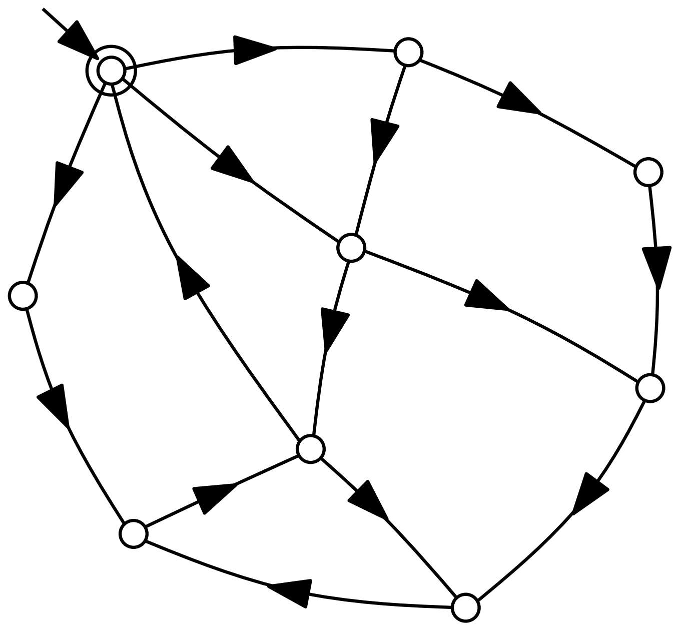
3. Application to d-angulated irreducible dissections

Master bijection between oriented maps and mobiles

Minimal accessible orientations

An **orientation** of a rooted plane map is called

- **accessible** if every vertex can be reached from the root-vertex
- **minimal** if there is no counterclockwise cycle



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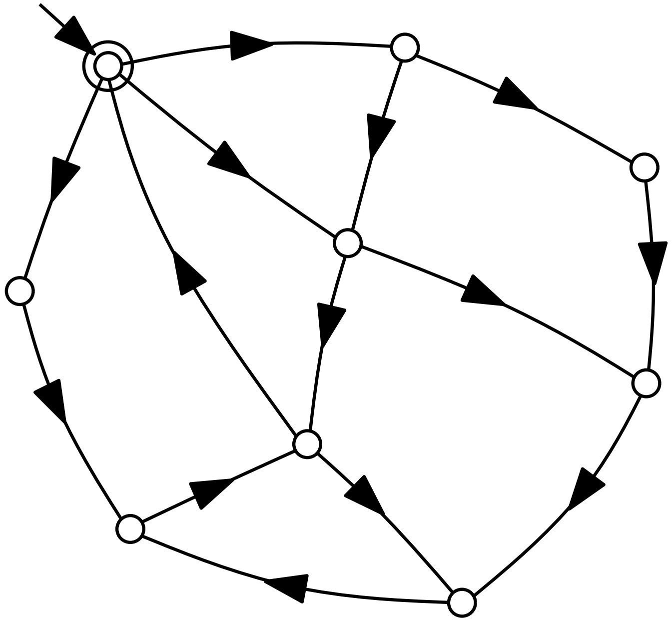
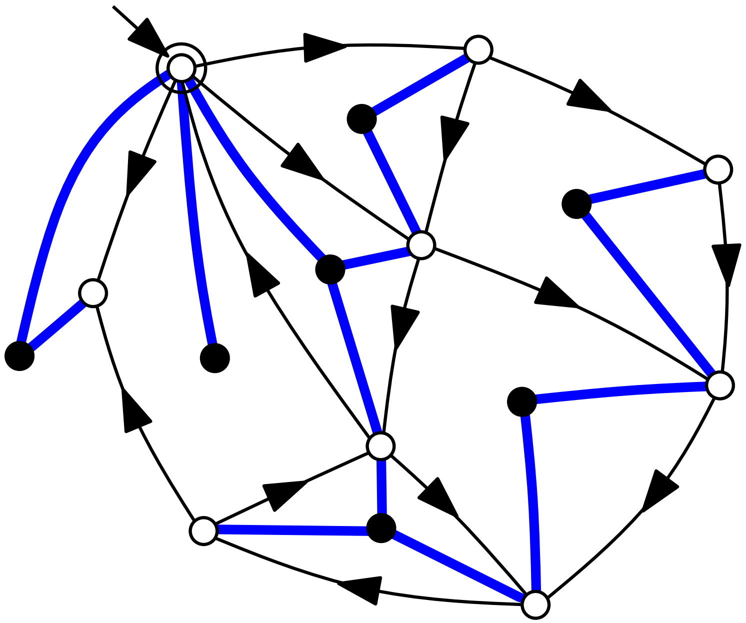
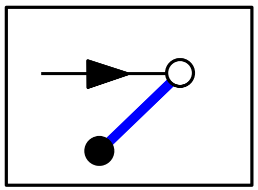
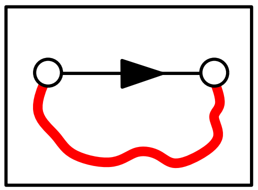


[Bernardi'06]

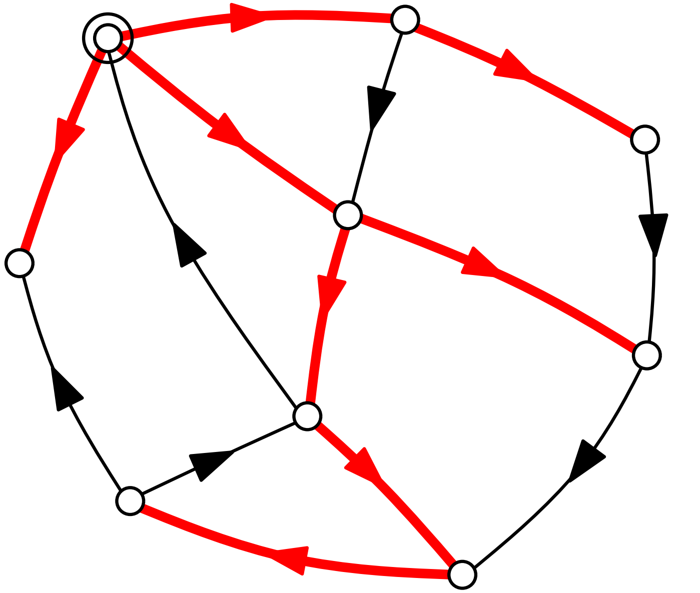
Mobile







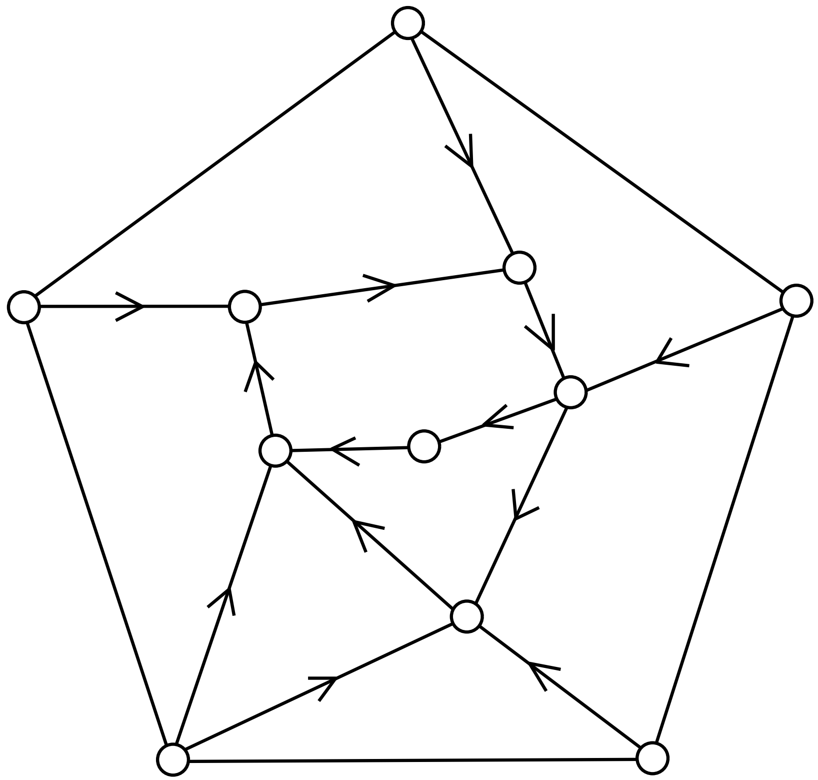
Canonical spanning tree



Families of orientations and mobiles

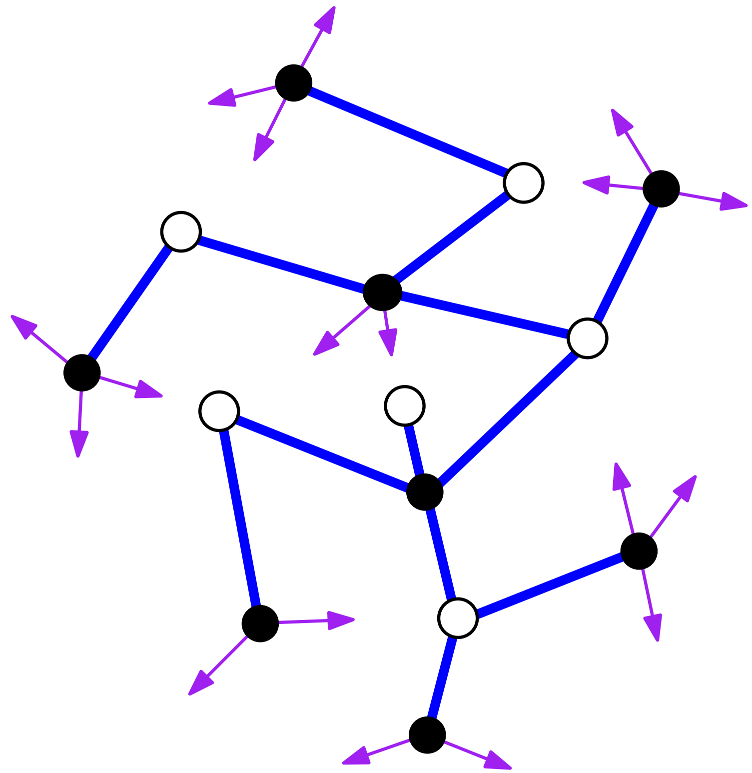
Let $\mathcal{O}$ be the set of **orientations** on planar maps such that:

- there is **no ccw circuit**
- Each inner vertex is **accessible** from the outer (unoriented simple) cycle
- the outer cycle is a **source**



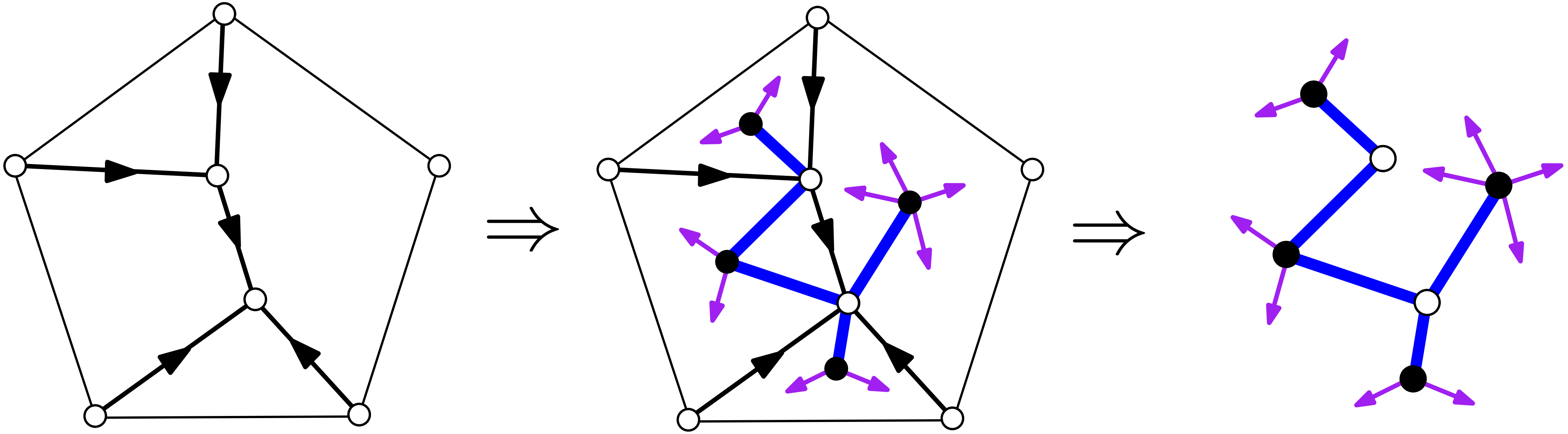
Let $\mathcal{M}$ be the set of **mobiles**, i.e.,

bipartite plane trees with **arrows** (called buds) at **black vertices**

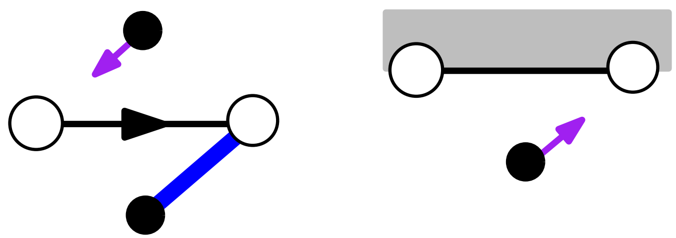


and with more buds than edges

From oriented maps to mobiles



Local rules



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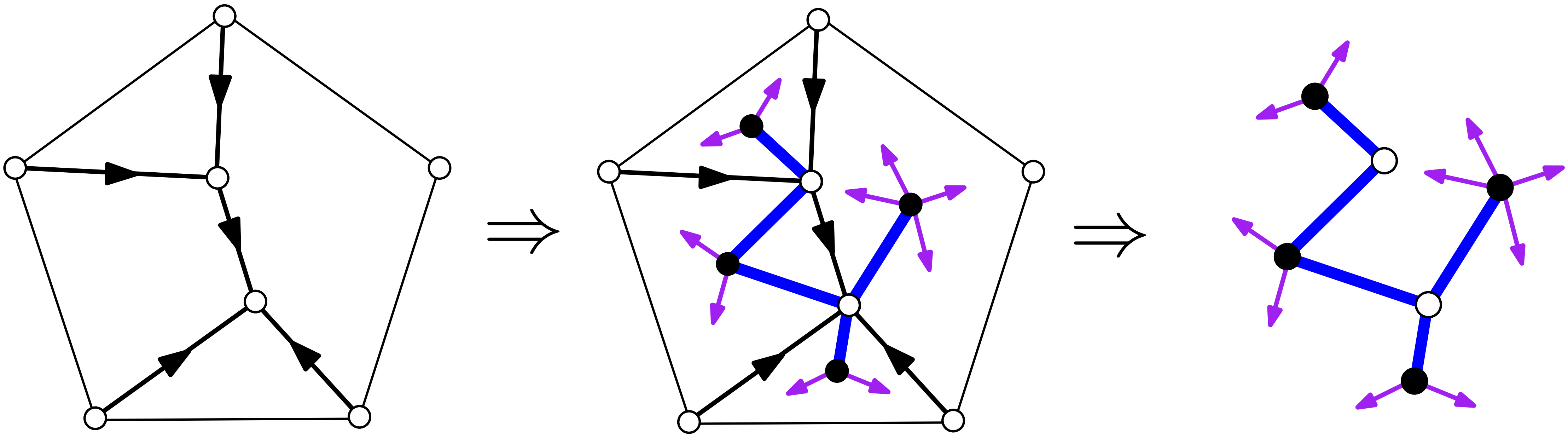
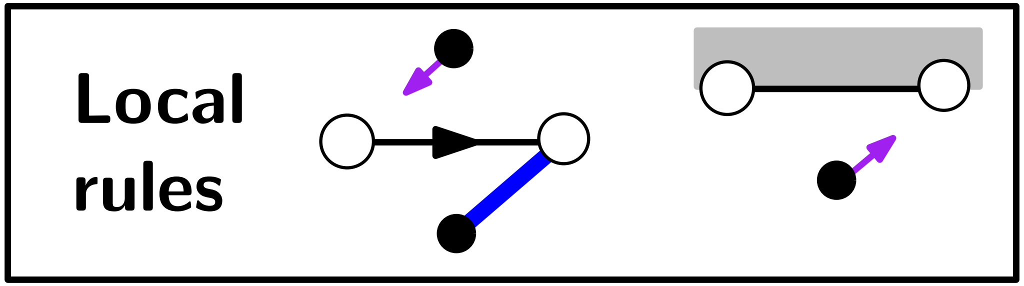
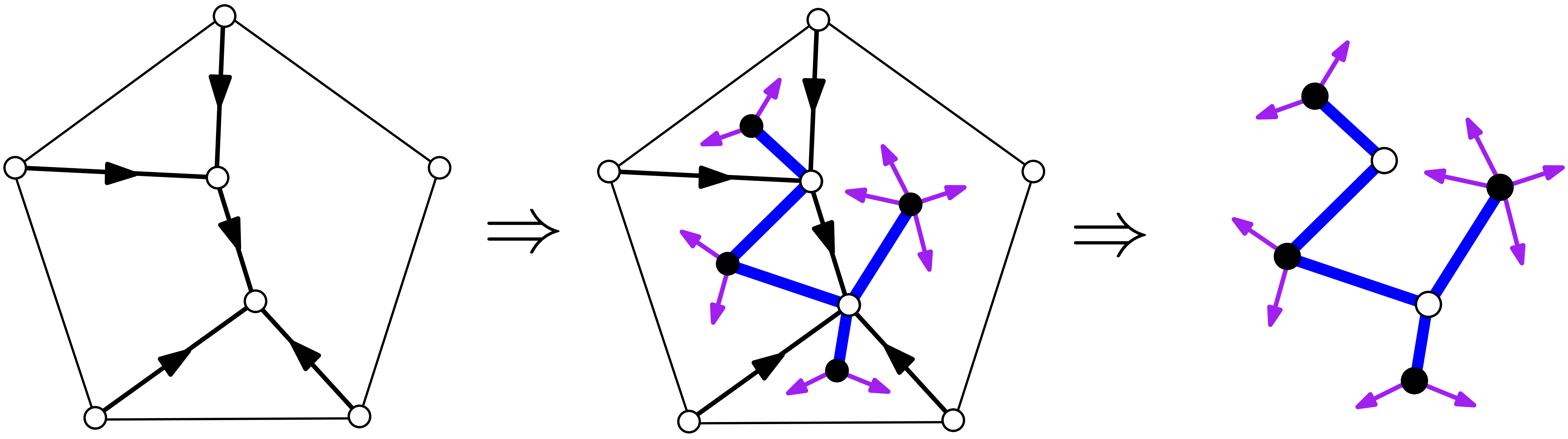


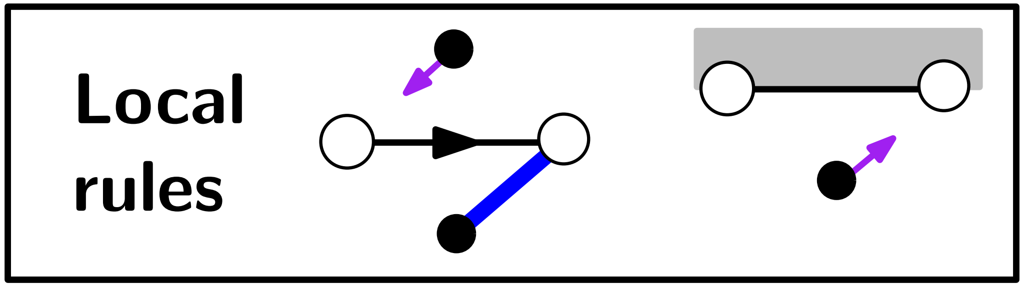


Theorem [Bernardi-F'10]: Φ is a **bijection** between $\mathcal{O}$ and $\mathcal{M}$.
Moreover,

degrees of internal faces $\longleftrightarrow$ degrees of black vertices
indegrees of internal vertices $\longleftrightarrow$ degrees of white vertices

From oriented maps to mobiles





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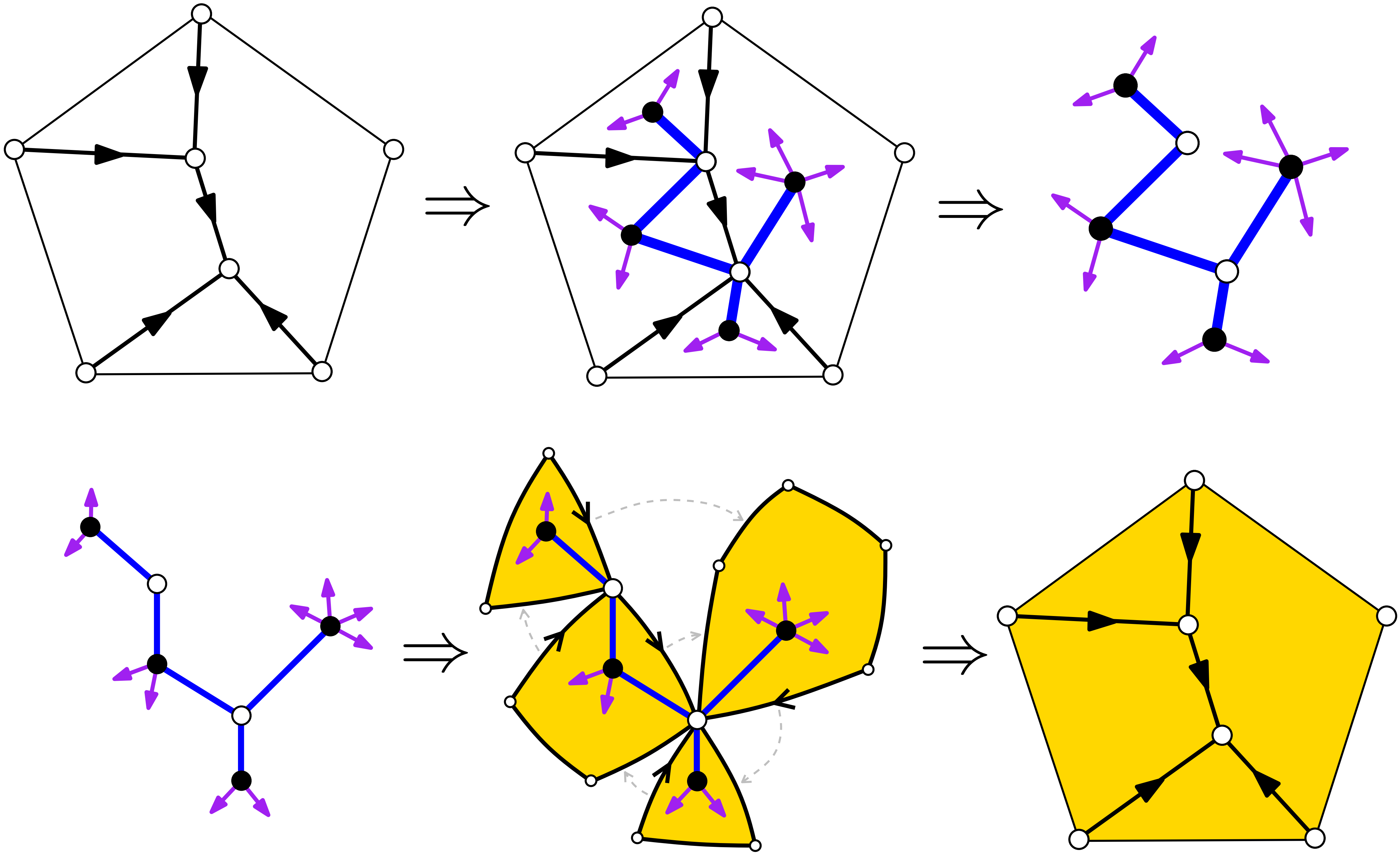
cf [Bernardi'07], [Bernardi-Chapuy'10]

And the inverse (closure) mapping



Using the master bijection for map enumeration

Scheme for the strategy

(1) Map family $\mathcal{C}$ identifies with a **subfamily** $\mathcal{O}_C$ of $\mathcal{O}$ with conditions on:

- Face degrees
- Vertex indegrees

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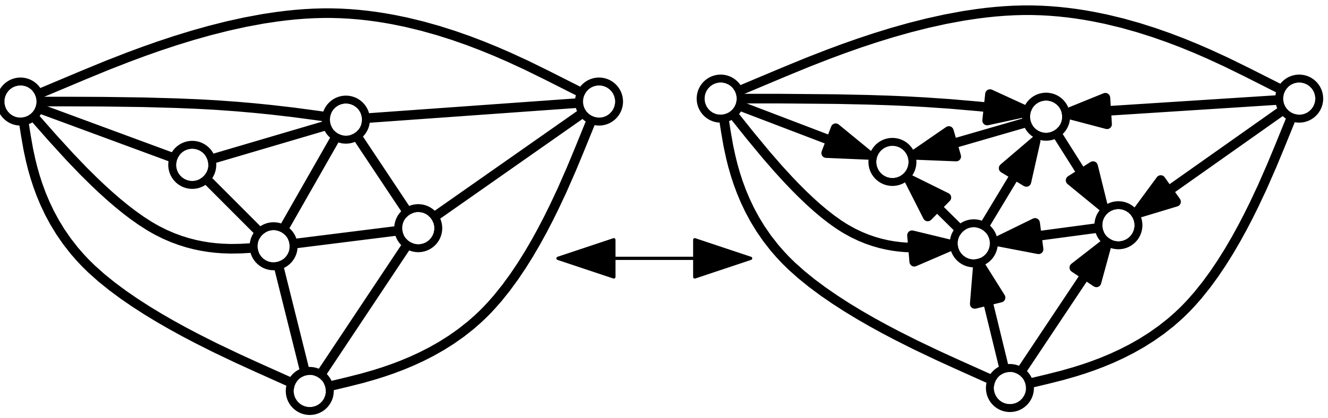
Example: $\mathcal{C}$ = Family of **simple triangulations**



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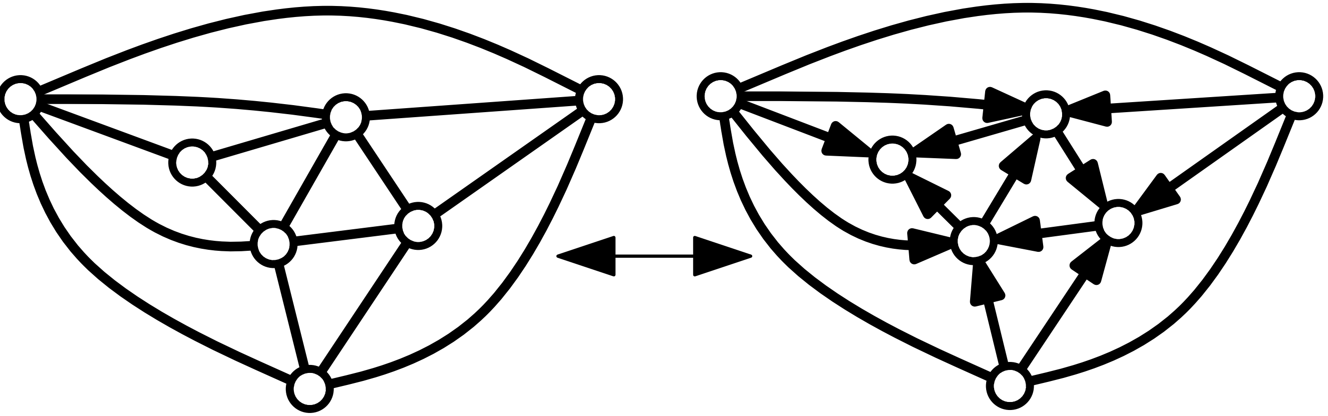

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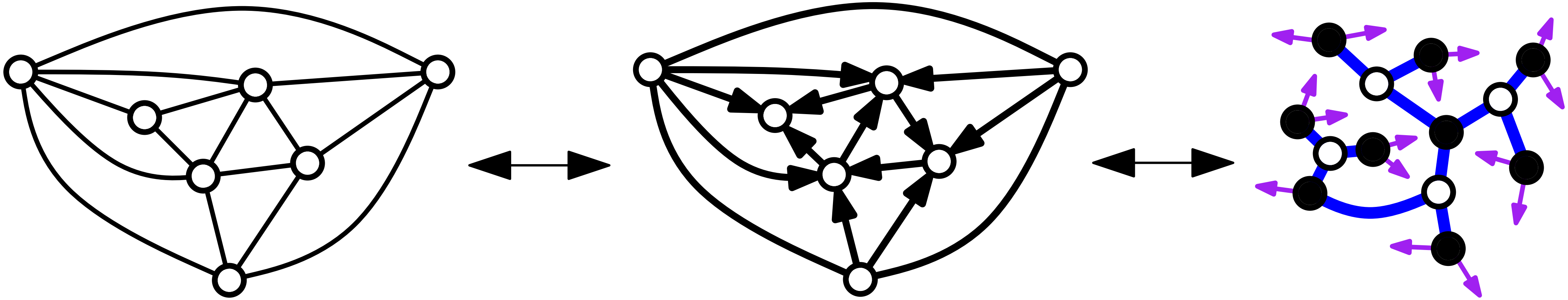
(2) **Specialize** the master bijection to the subfamily $\mathcal{O}_C$



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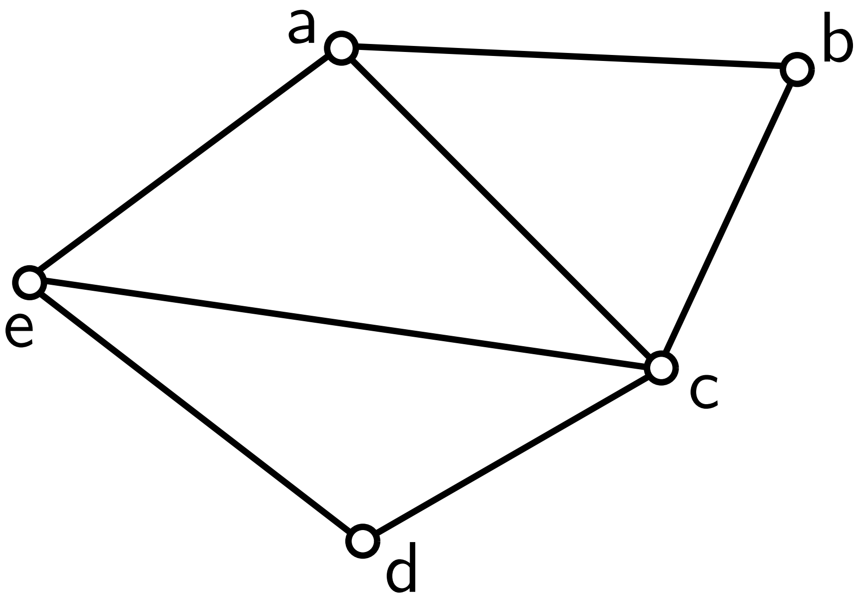
α-orientations

Let $G = (V, E)$ be a graph

Let α be a function from V to $\mathbb{N}$



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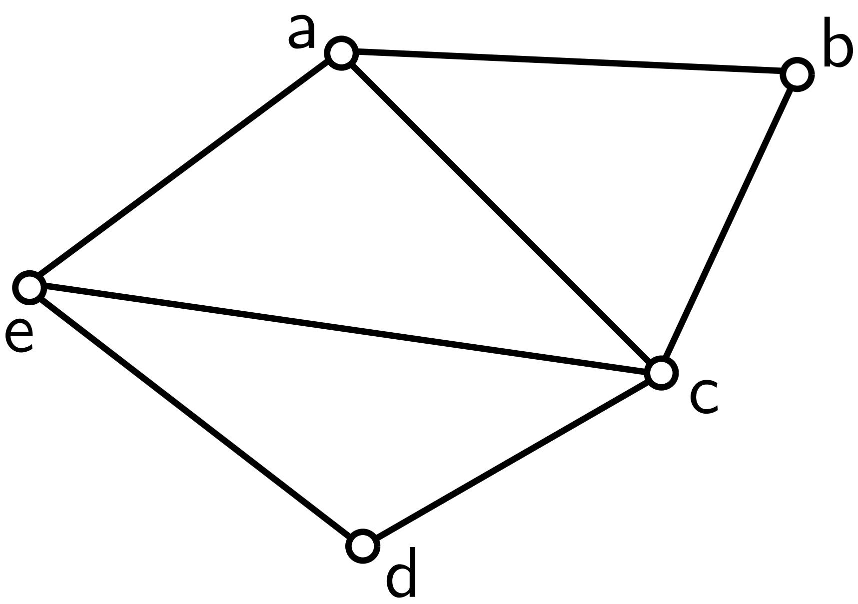

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Def: An **α-orientation** is an orientation of G where for each $v \in V$

$$\text{indegree}(v) = \alpha(v)$$

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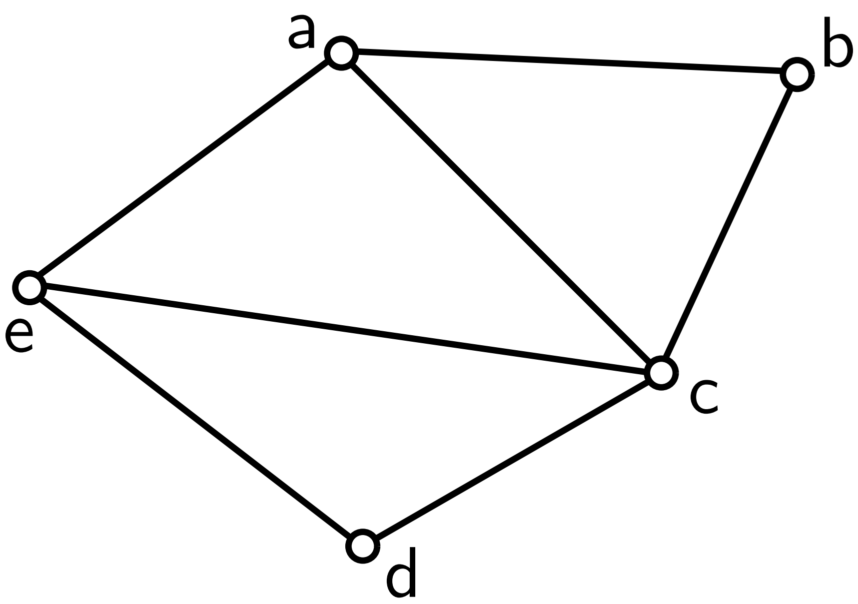

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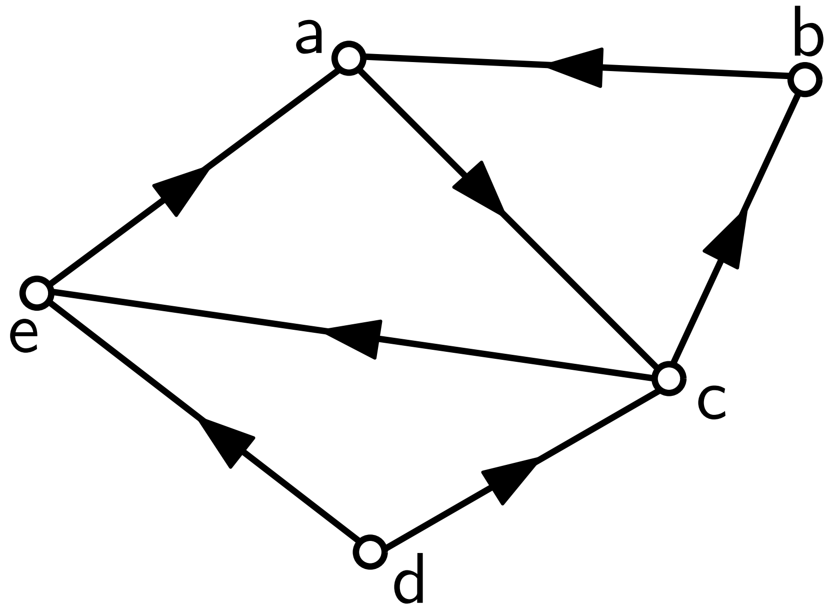

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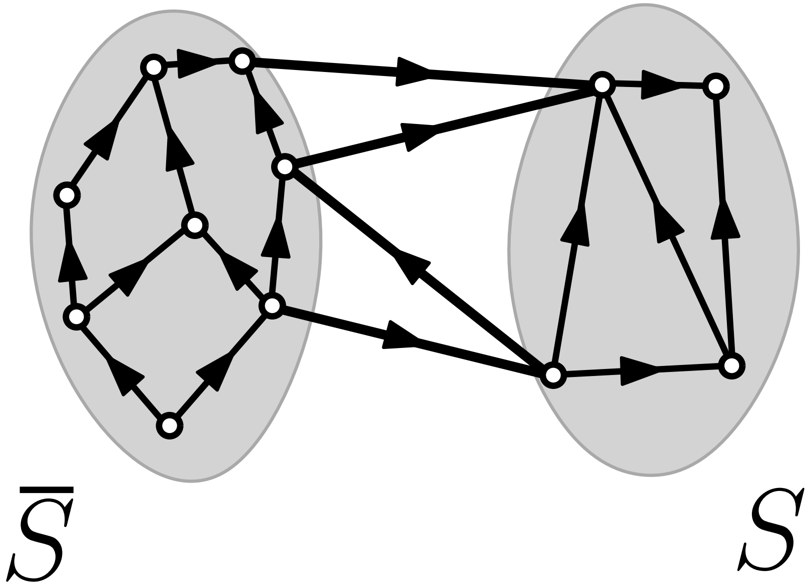
α-orientations: existence criterion

- If an α-orientation **exists**, then



(i) $\sum_{v \in V} \alpha(v) = |E|$

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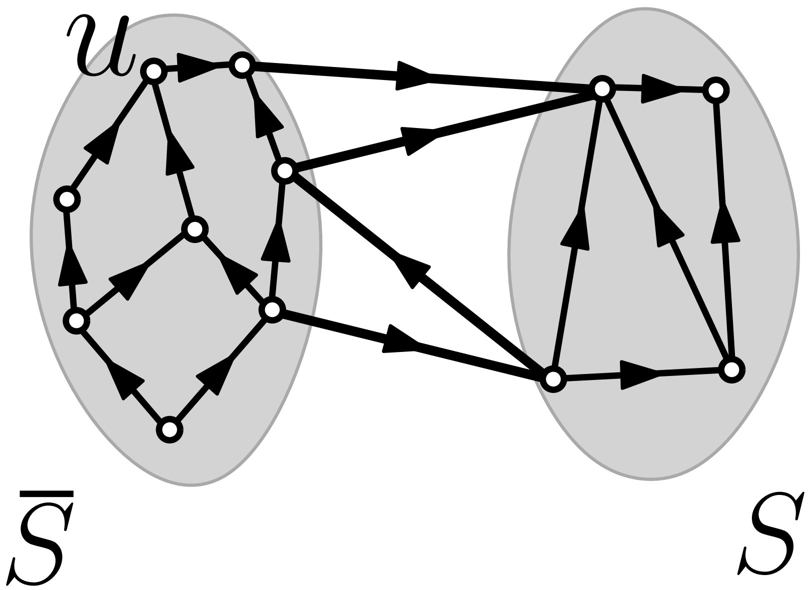

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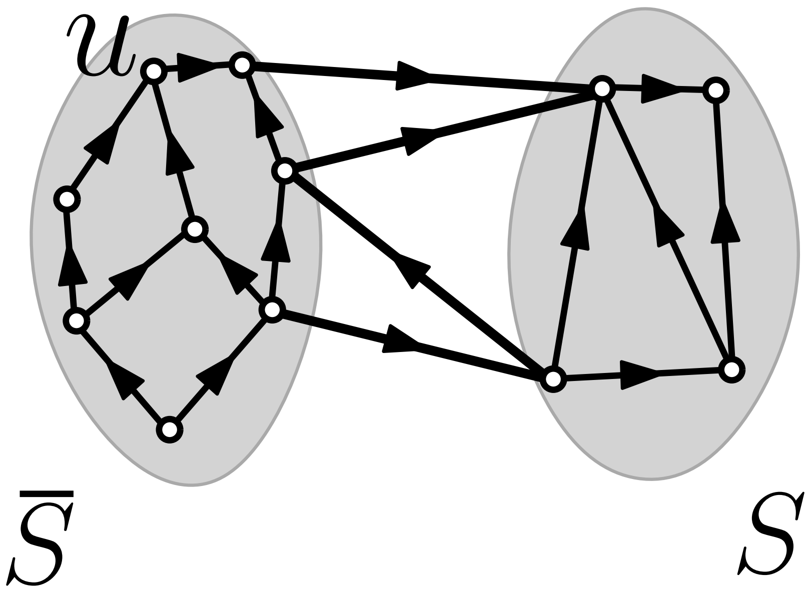

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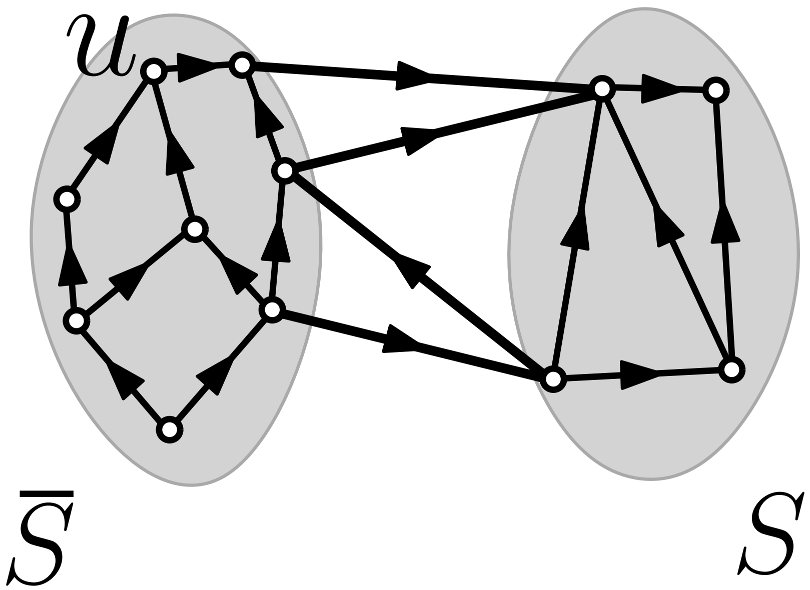

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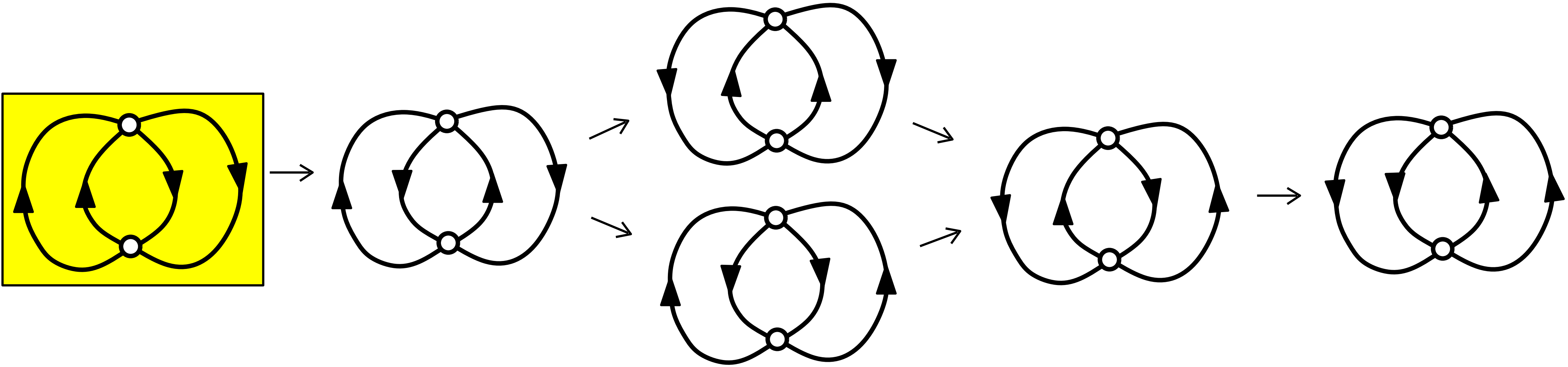
Lemma (folklore): The conditions are necessary **and sufficient**

$\Rightarrow$ accessibility from $u \in V$ just depends on α (not on which α-orientation)

α-orientations for plane maps

Fundamental lemma: If a plane map admits an α-orientation, then it admits a **unique** α-orientation **without ccw circuit**, called **minimal**



More precisely, the set of α-orientations is a **distributive lattice**
[Khueller et al'93], [Propp'93], [O. de Mendez'94], [Felsner'03]

Example: simple triangulations

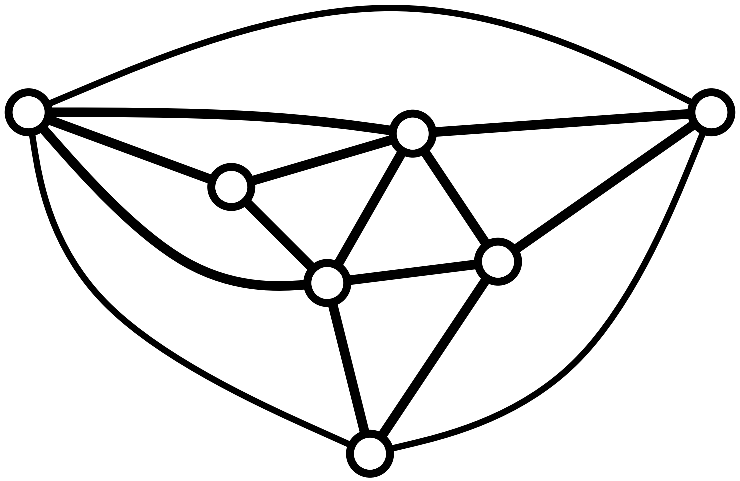
Triangulations

Fact: A triangulation with n internal vertices has $3n$ internal edges.

Proof: The numbers v, e, f of vertices edges and faces satisfy:

- Incidence relation: $3f = 2e$.
- Euler relation: $v - e + f = 2$. □

call **3-orientation** such an α-orientation



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Natural candidate for indegree function:

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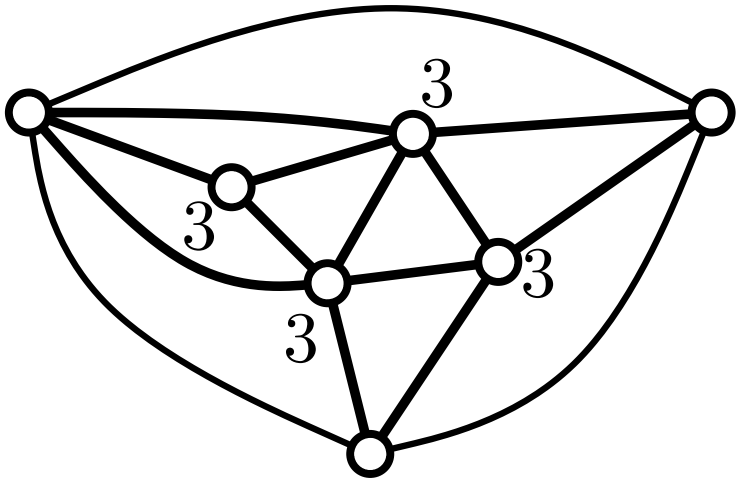

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Natural candidate for indegree function:

$$\alpha: \begin{array}{l} v \mapsto 3 \text{ for each internal vertex } v. \\ v \mapsto 1 \text{ for each external vertex } v. \end{array}$$

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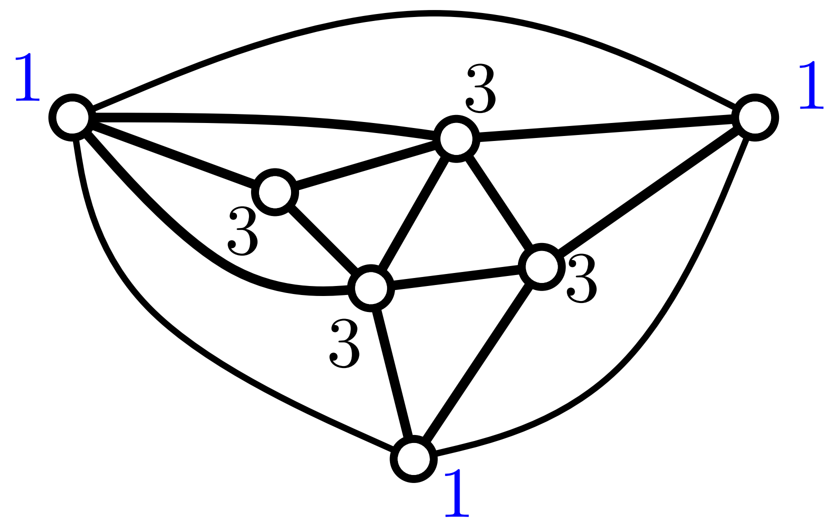
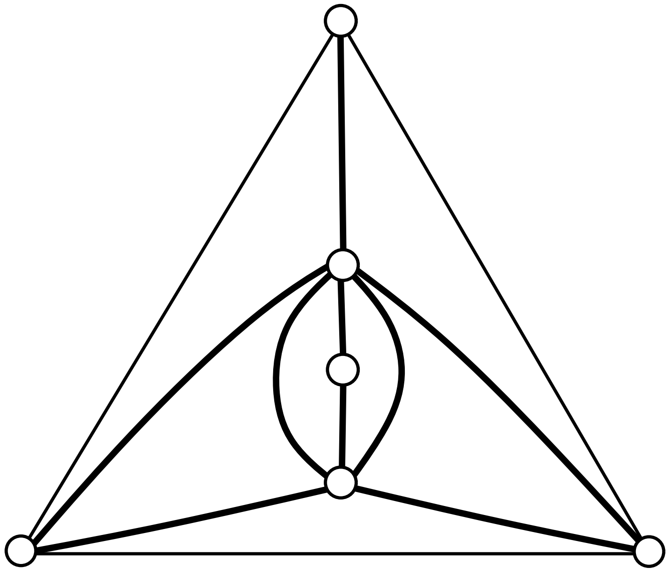
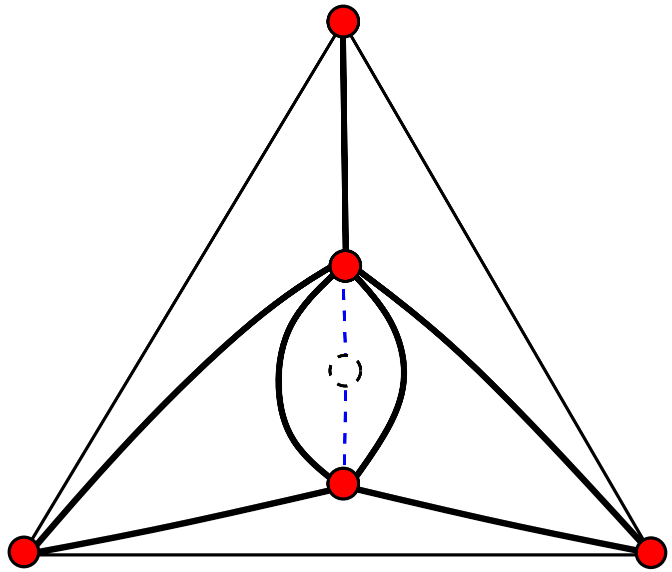


Triangulations

Fact: A triangulation admitting a 3-orientation is simple



$\Rightarrow$



k internal vertices

$3k+1$ internal edges

Triangulations

Thm [Schnyder 89]: A simple triangulation admits a 3-orientation, and any 3-orientation is accessible from the outer boundary

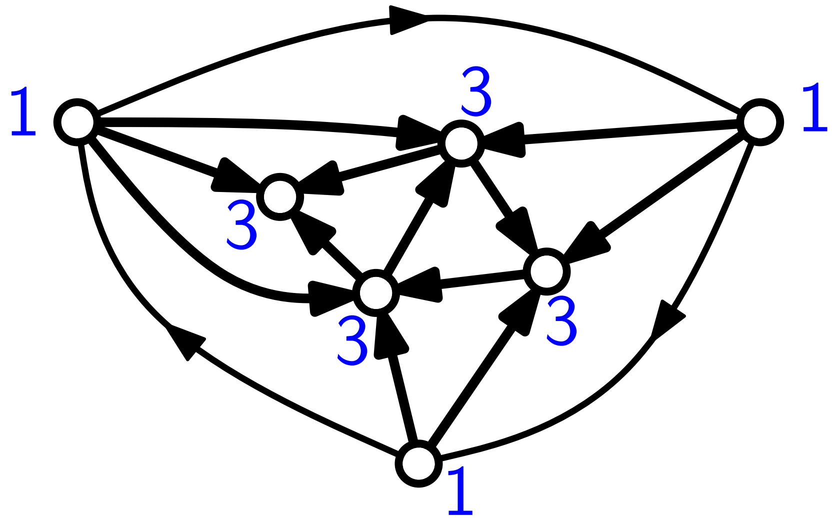
New (easier) proof: Any simple planar graph $G = (V, E)$ satisfies

$$|E| \leq 3|V| - 6 \quad \text{(Euler relation)}$$

Hence $\forall S \subseteq V$, $|E_S| \leq \alpha(S)$,

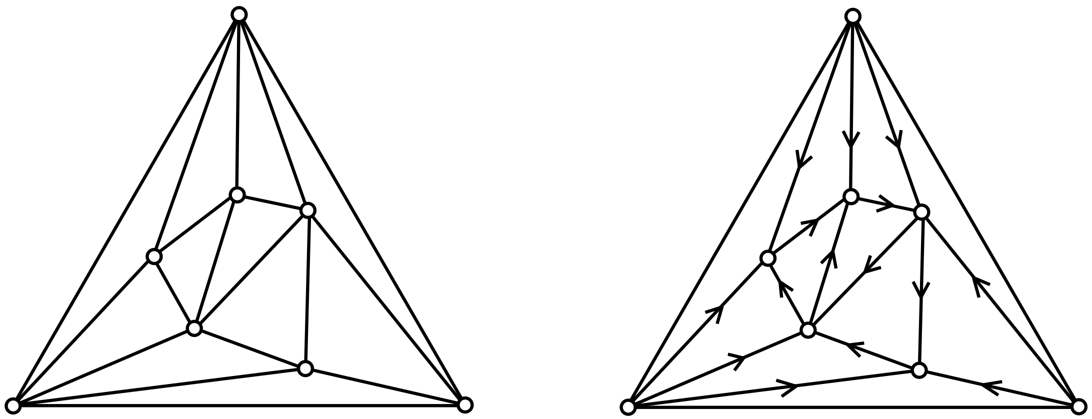
with strict inequality when S misses at least one outer vertex

hence the existence/accessibility conditions are satisfied.$\square$

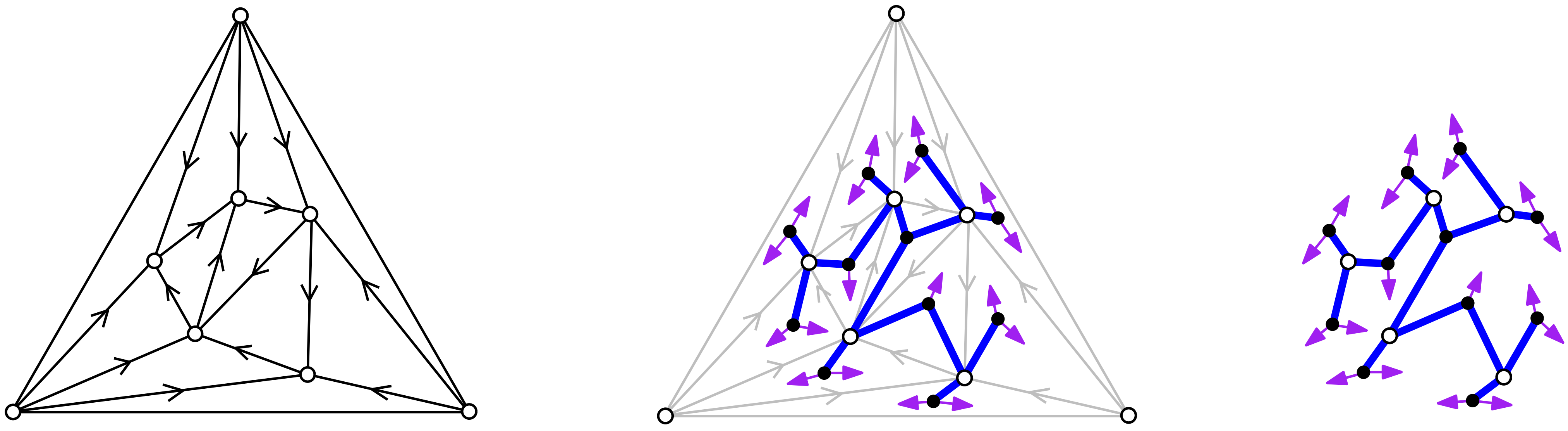


Triangulations

- From the lattice property (**taking the min**) we have
  family of simple triangulations $\leftrightarrow$ **subfamily $\mathcal{F}$ of $\mathcal{O}$** where:
  - faces have degree 3
  - inner vertices have outdegree 3



- From the **master bijection specialized to $\mathcal{F}$**, we have
  $\mathcal{F}$ $\leftrightarrow$ **subfamily of mobiles** where all vertices have **degree 3**

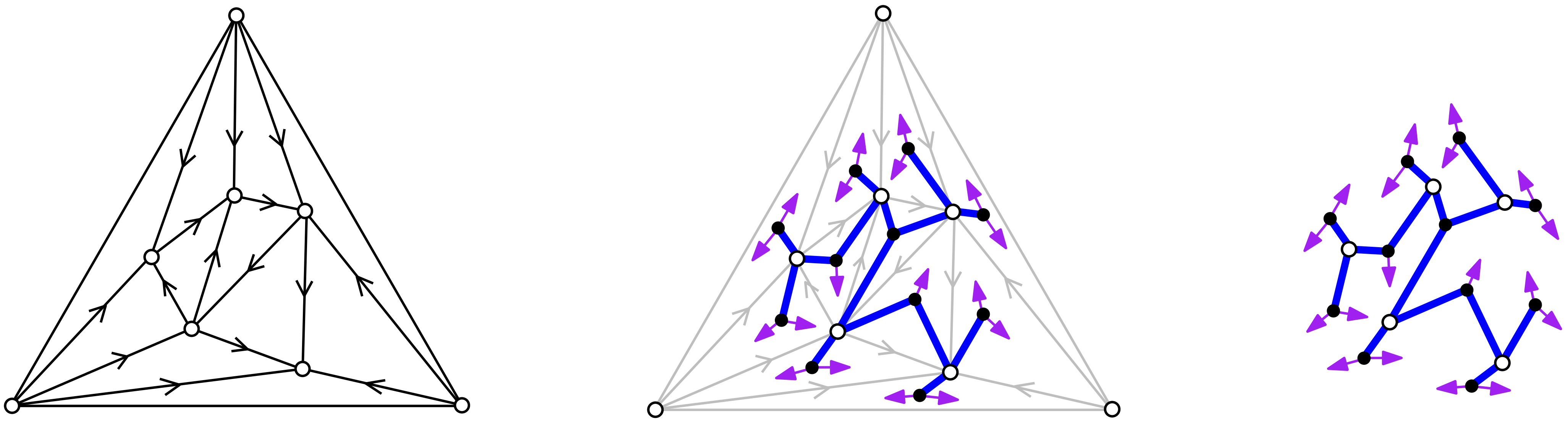


[F, Poulalhon, Schaeffer'08], other bijection in [Poulalhon, Schaeffer'03]

Triangulations

Counting: The generating function of mobiles with vertices of degree 3 rooted on a white corner is $T(x) = U(x)^3$, where $U(x) = 1 + xU(x)^4$.

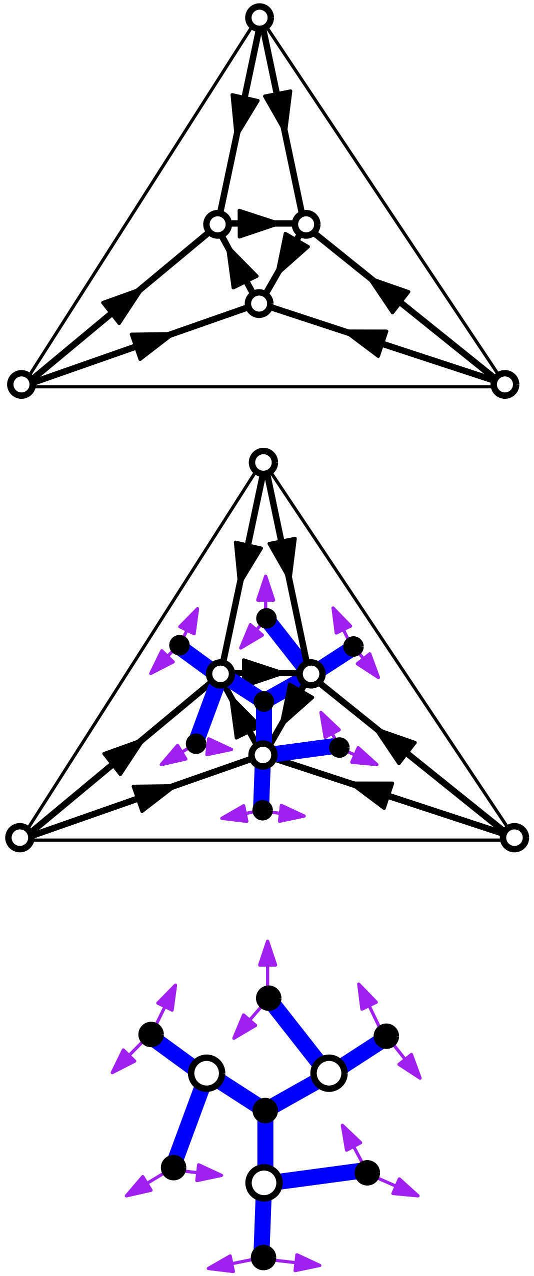
Consequently, the number of (rooted) simple triangulations with $2n$ faces is $\dfrac{1}{n(2n-1)}\dbinom{4n-2}{n-1}$.



Triangulations: two constructions

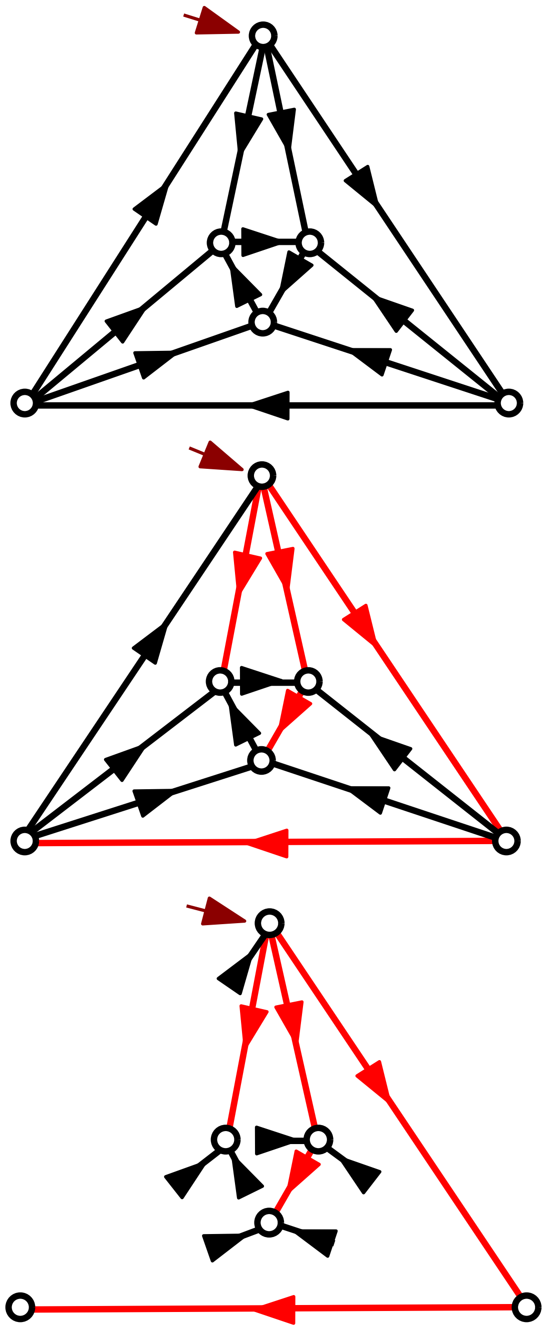
mobiles

[FuPoSc'08], [Bernardi-F'10]



blossoming trees

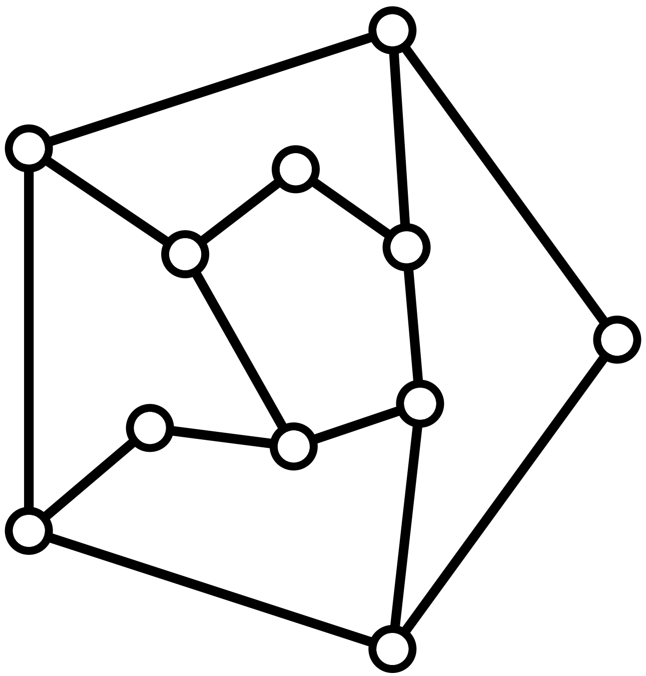
[PoSc'03], [AlPo'13]



Generalization to d-angulations of girth d

d-angulations of girth d

Fact: A d-angulation with $(d-2)n$ internal vertices has dn internal edges.

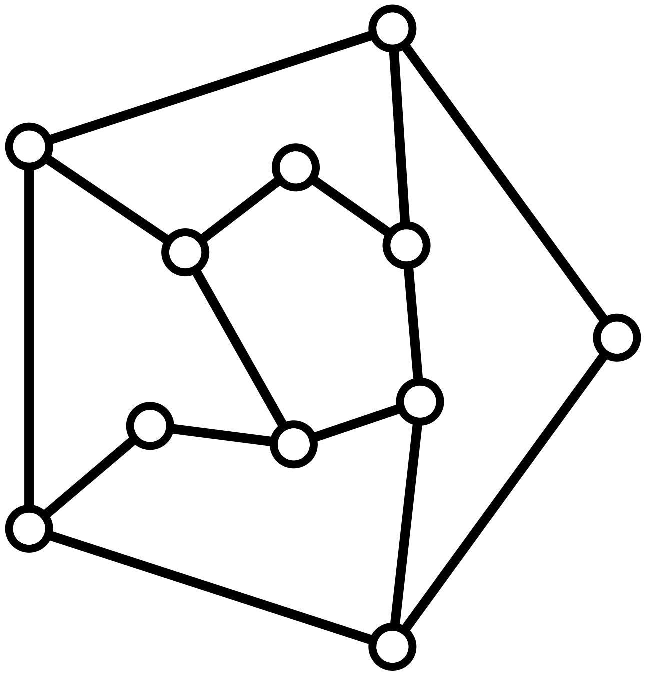


d-angulations of girth d

Fact: A d-angulation with $(d-2)n$ internal vertices has dn internal edges.

Natural candidate for indegree function:

$$\alpha : \; v \mapsto \frac{d}{d-2} \text{ for each internal vertex } v\ldots$$

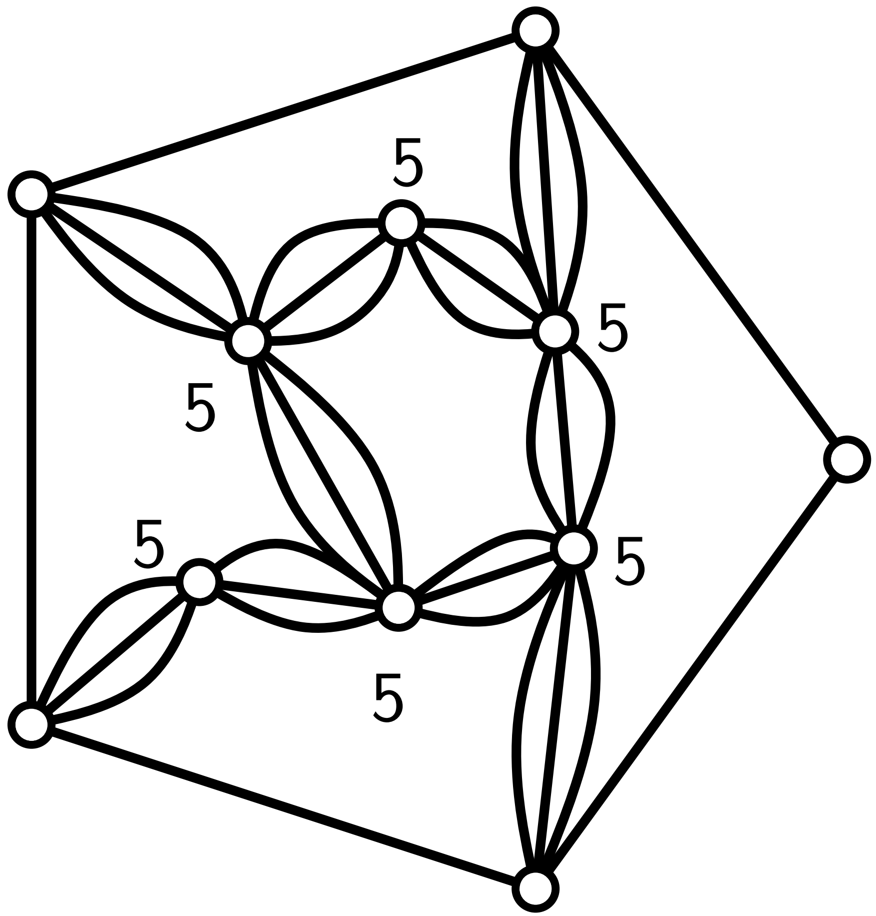


d-angulations of girth d

Fact: A d-angulation with $(d-2)n$ internal vertices has dn internal edges.

Idea: We can look for an orientation of $(d-2)G$ with indegree function

$$\alpha : \; v \mapsto d \text{ for each internal vertex } v.$$



d-angulations of girth d

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Idea: We can look for an orientation of $(d-2)G$ with indegree function

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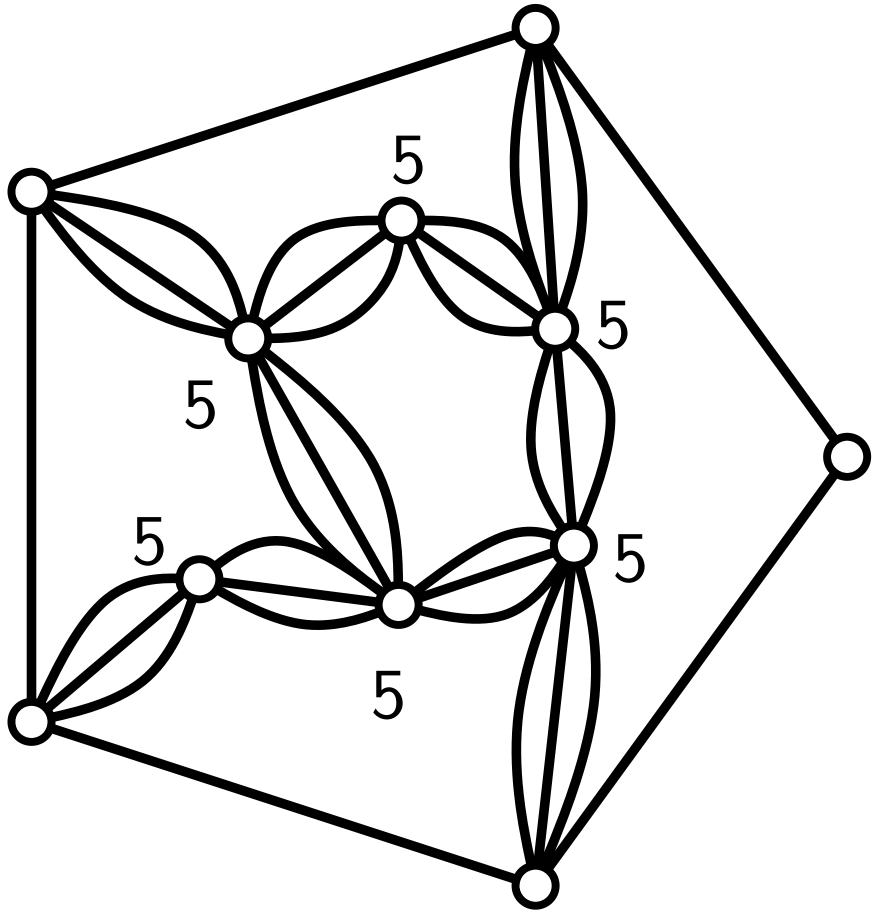
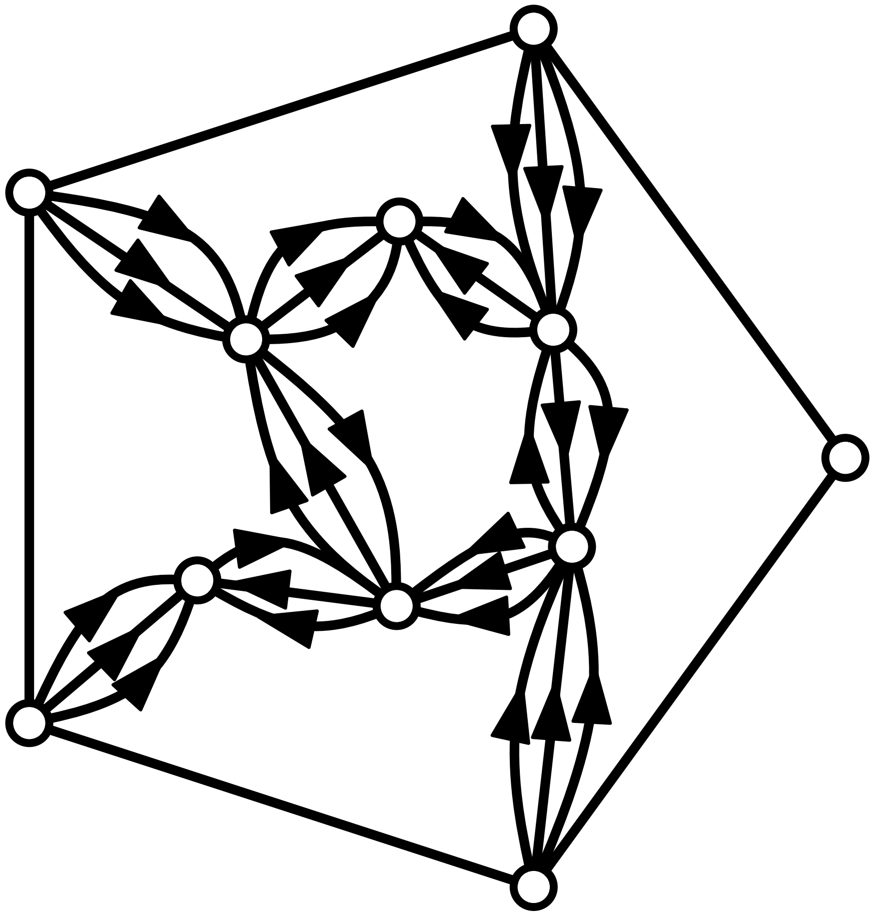
call $d/(d-2)$-orientation such an orientation

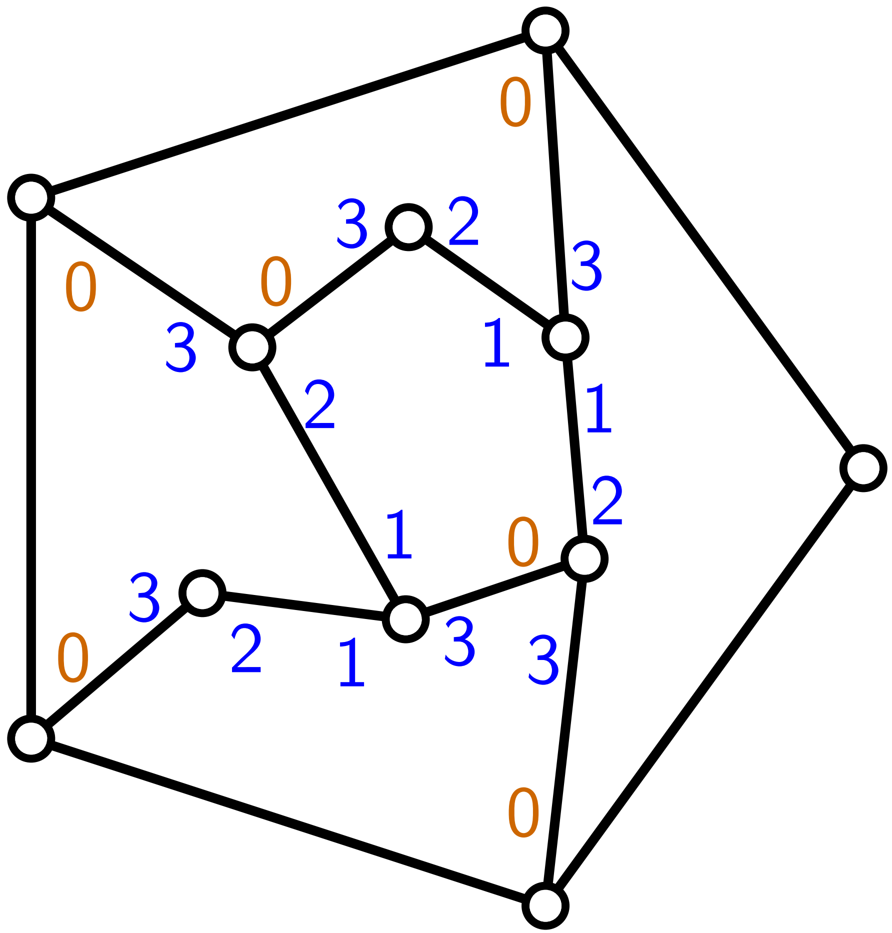


d-angulations of girth d

Thm [Bernardi-F'10]: Let G be a d-angulation. Then $(d{-}2)G$ admits a $d/(d-2)$- orientation if and only if G has girth d.

$d = 5$



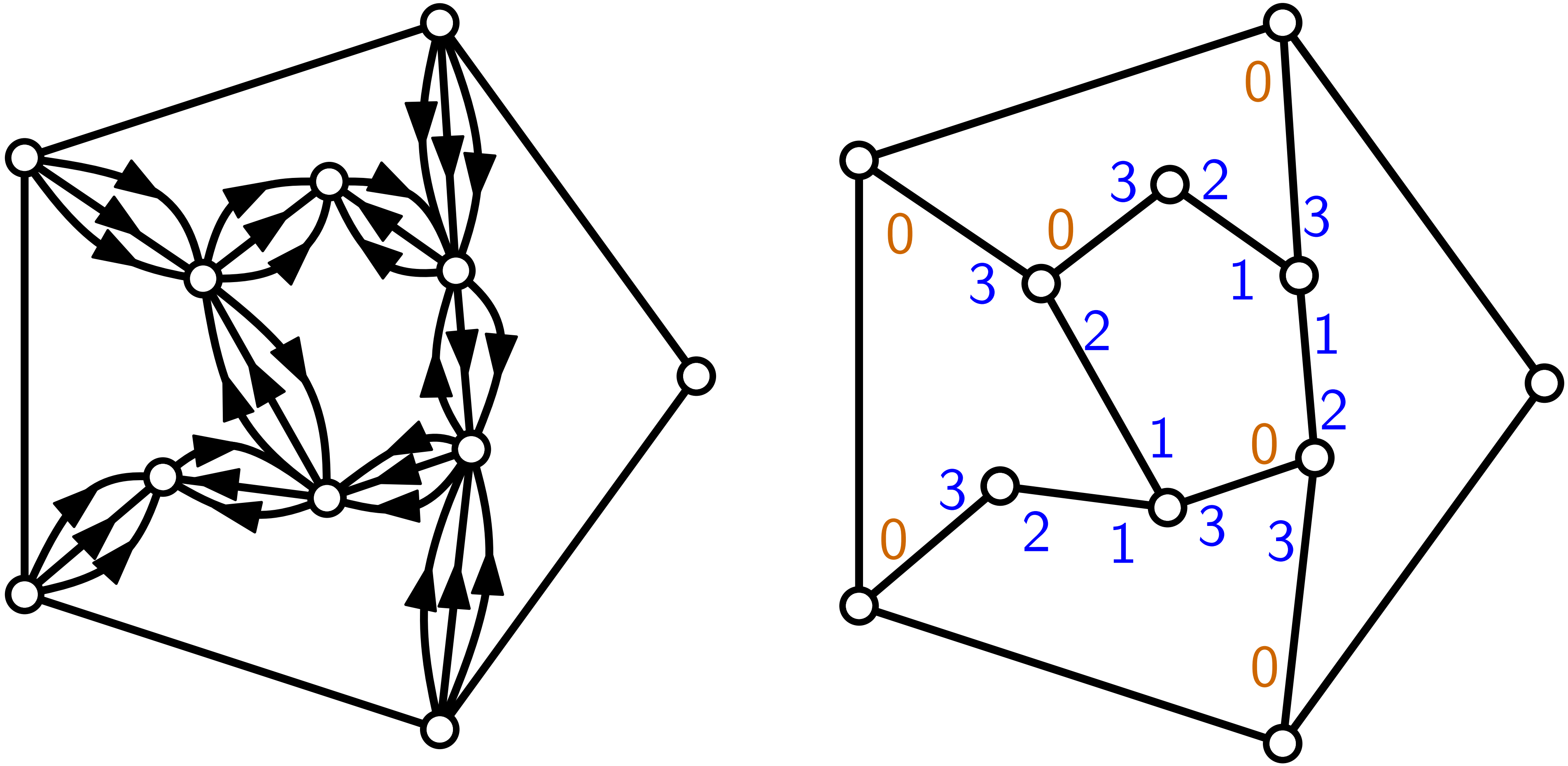


d-angulations of girth d

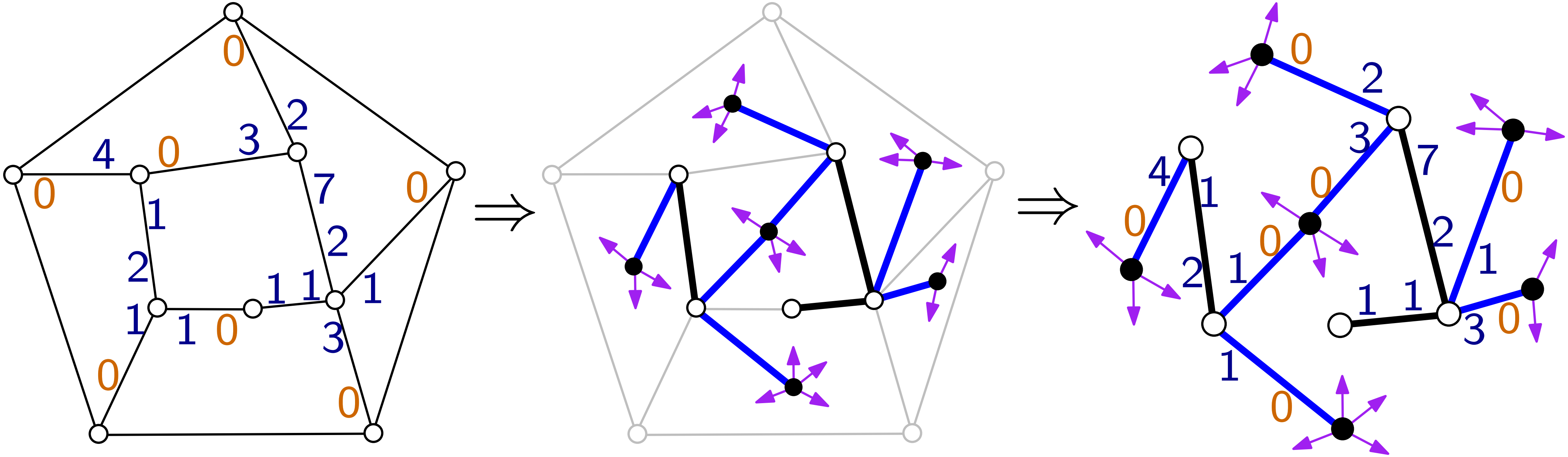
Thm [Bernardi-F'10]: Let G be a d-angulation. Then $(d-2)G$ admits a $d/(d-2)$- orientation if and only if G has girth d.

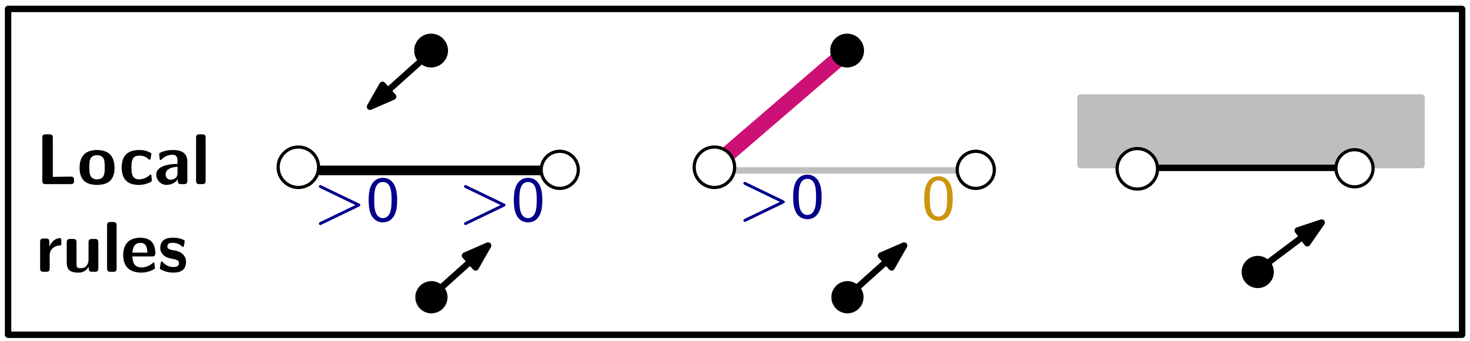
Proof: Similar to $d = 3$. Uses the fact that a planar graph $G = (V, E)$ of girth at least d satisfies $(d-2)|E| \leq d|V| - 2d$

$d = 5$



Master bijection in the flow-formulation



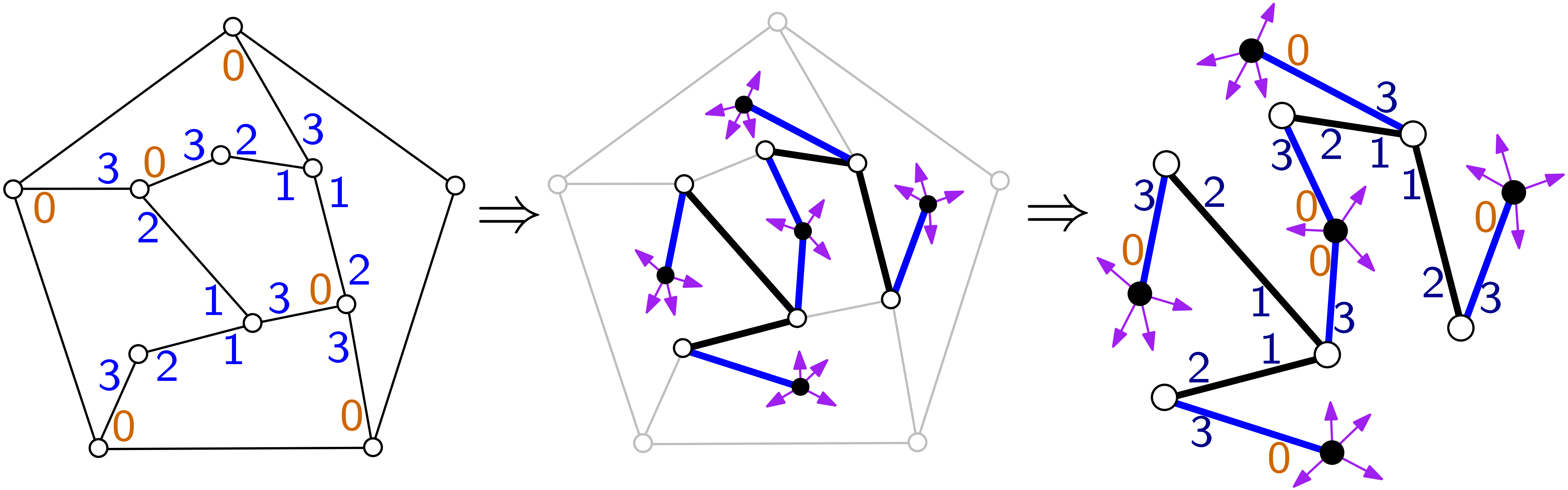


degrees of inner faces ⟷ degrees of black vertices

total flows at inner vertices ⟷ total weights at white vertices

total flows at inner edges ⟷ total weights at edges

Specialization to d-angulations of girth d



Bijection **d-angulations of girth d** $\leftrightarrow$ **weighted mobiles** such that

- each black vertex has degree d
- each white vertex has total weight d
- each edge has total weight $d-2$ (weight>0 at $\circ$, weight$=0$ at $\bullet$)

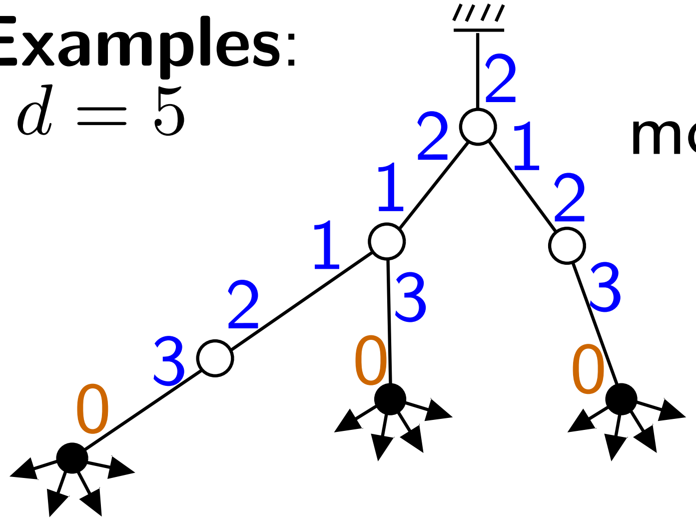
[Albenque, Poulalhon'13]: other bijection (with blossoming trees)

Generating function expression

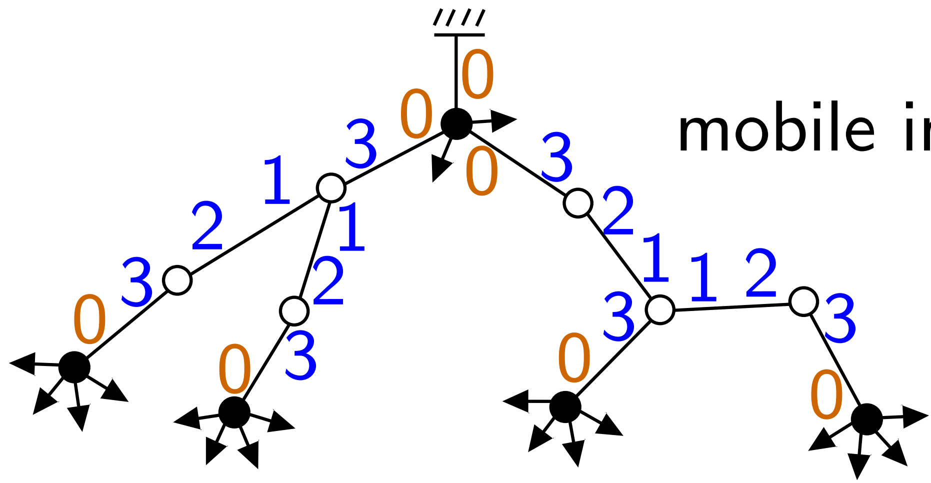
For $i \in [0..d]$, $\mathcal{L}_i$:= family of such mobiles with a **root-leg of weight i**

Let $L_i(x)$ be the GF of $\mathcal{L}_i$ where x marks black nodes

Examples:
$d = 5$



mobile in $\mathcal{L}_2$



mobile in $\mathcal{L}_0$

For $d \geq 3$, $F_d(x) :=$ GF of (rooted) d-angulations of girth d by inner faces

- **Bijection** when an **inner face** is **marked**

$$\Rightarrow \boxed{F'(x) = (1 + L_{d-2})^d}$$

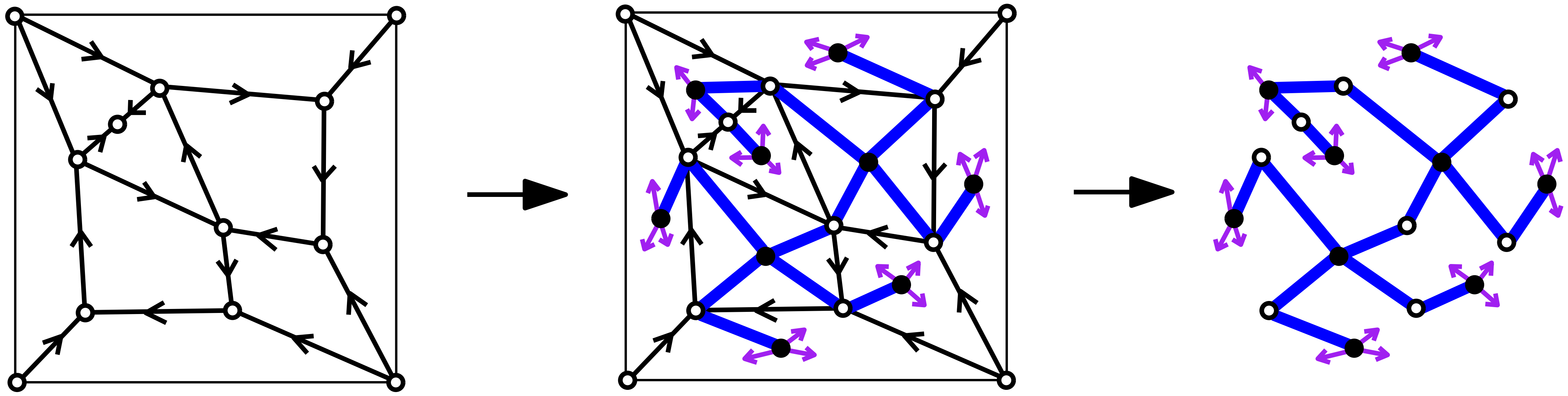
- **Root-decomposition** of mobiles in $\mathcal{L}_i \Rightarrow (L_0, L_1, \ldots, L_d)$ are given by

$$\left\{ \begin{array}{lcl} L_0 & = & x \cdot (1 + L_{d-2})^{d-1}, \\ L_d & = & 1, \\ L_i & = & \sum_{j>0} L_{d-2-j} L_{i+j} \quad \text{for } i = 1..d-1 \end{array} \right.$$

Simplification in the bipartite case

- For d even, $d = 2b$, we have $\dfrac{d}{d-2} = \dfrac{b}{b-1}$
- Can work with $b/(b-1)$-orientations:
  - edges have weight $b-1$
  - vertices have indegree b

Example: $b = 2$, simple quadrangulations



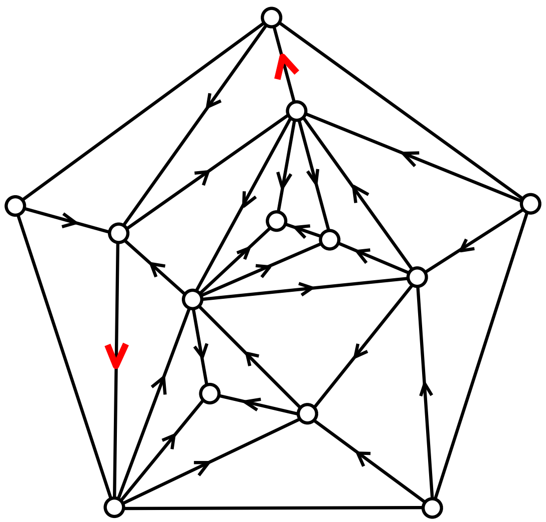
recover a bijection of Schaeffer (1999)

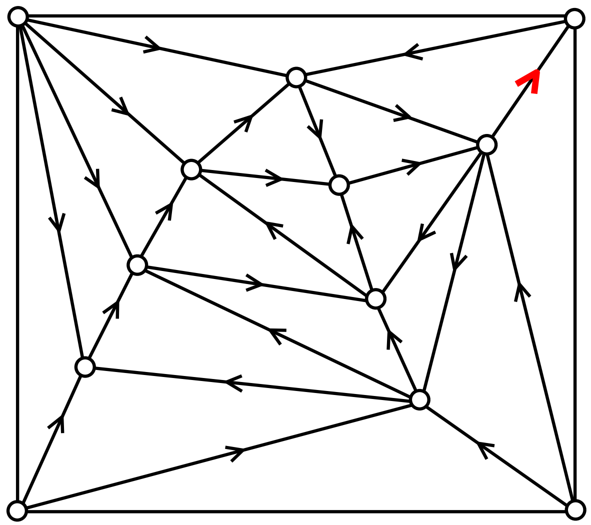
Bijections for irreducible $(4,3)$-dissections (triangulated case)

3-orientations for triangulated dissections

Def: A 3-orientation of a $(k, 3)$-dissection is an orientation of the inner edges where all inner vertices have indegree 3

Rk: Euler relation $\Rightarrow$ $k - 2$ inner edges point to the boundary





again minimal means "no ccw cycle" (not unique for $k \geq 4$)

Characterizing irreduciblity on orientations

Co-accessibility: from every inner vertex one can reach the outer boundary

For a $(k, 3)$ dissection D endowed with a (any) 3-orientation O,

D is irreducible iff O is co-accessible

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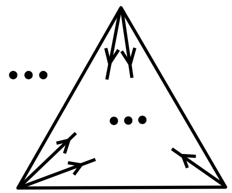
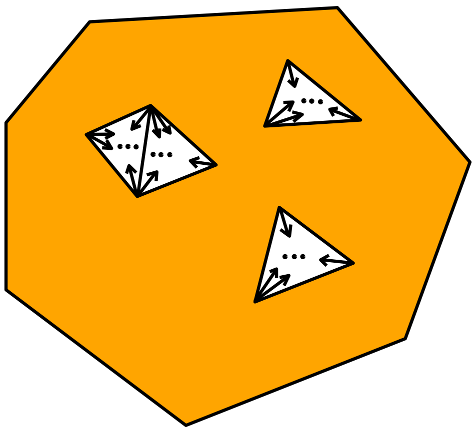
Proof:

D not irreducible $\Rightarrow$ O not co-accessible



can not reach the 3-cycle
from a vertex inside the 3-cycle

O not co-accessible $\Rightarrow$ D not irreducible



In orange the complex induced by vertices that can reach the outer boundary

The holes have to be triangular

Characterizing irreduciblity on orientations

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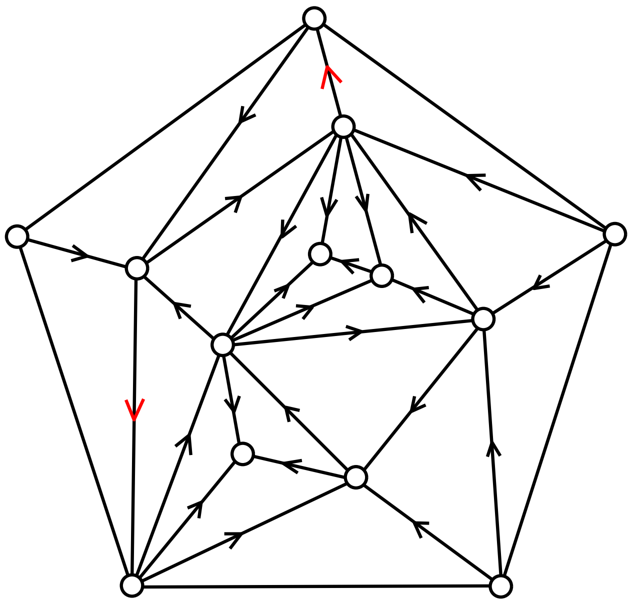
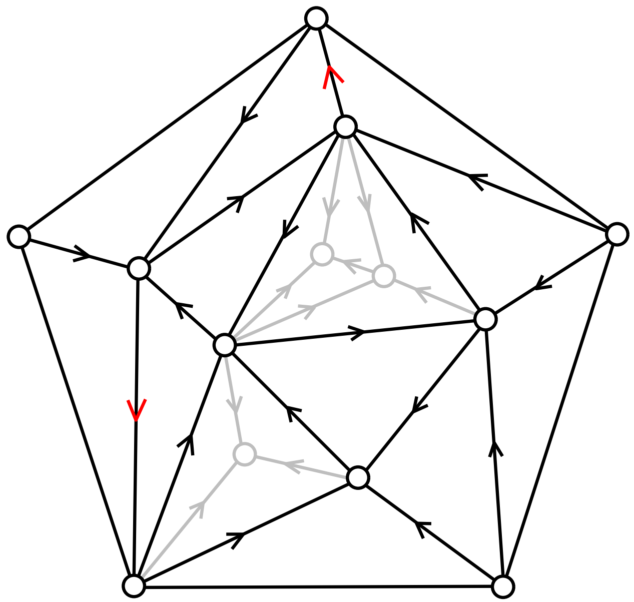
For a $(k,3)$ dissection D endowed with a (any) 3-orientation O,

D is irreducible iff O is co-accessible

Rk: also gives a simple algorithm to extract the irreducible core

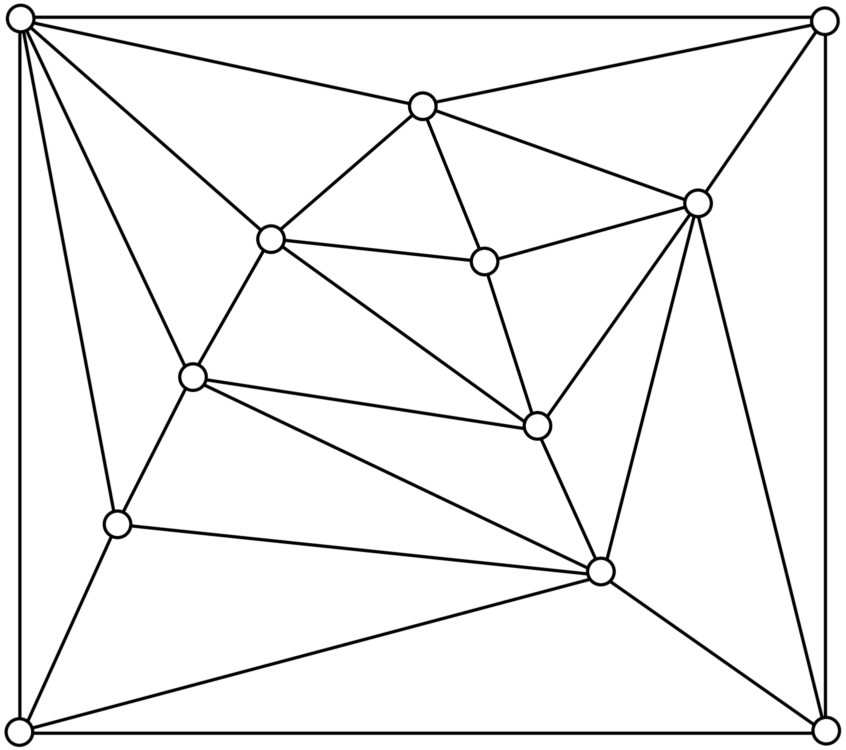




in gray the vertices (and incident edges) that can not reach the outer boundary

Irreducible 4-outer triangulation $\rightarrow$ ternary tree

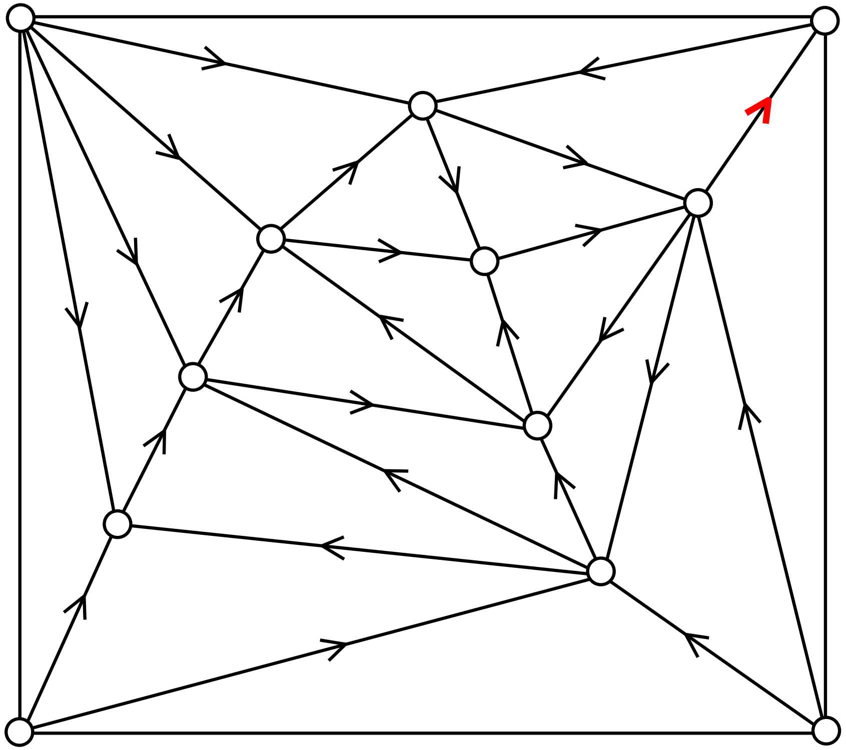
Let T be an irreducible 4-outer triangulation



Irreducible 4-outer triangulation → ternary tree

Let T be an irreducible 4-outer triangulation

Endow T with a minimal 3-orientation



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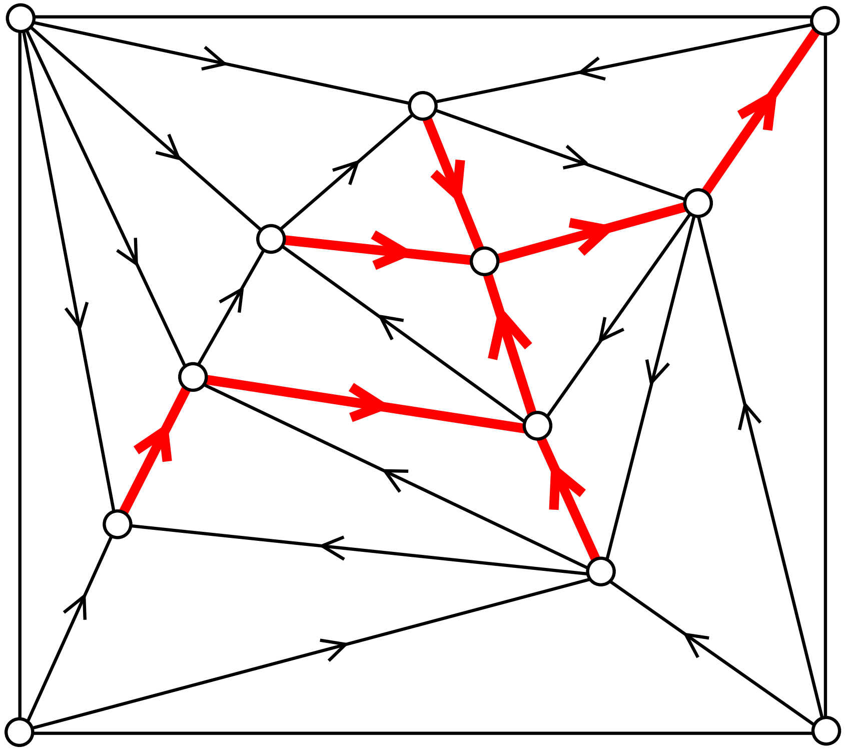

In red the canonical spanning tree (spanning inner vertices)

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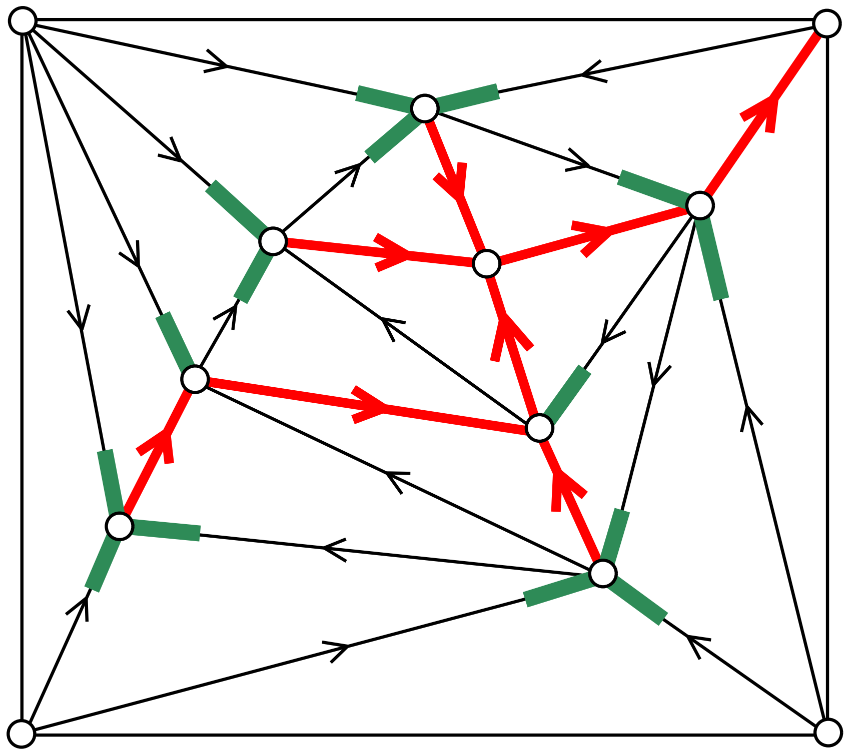

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In green the other ingoing half-edges

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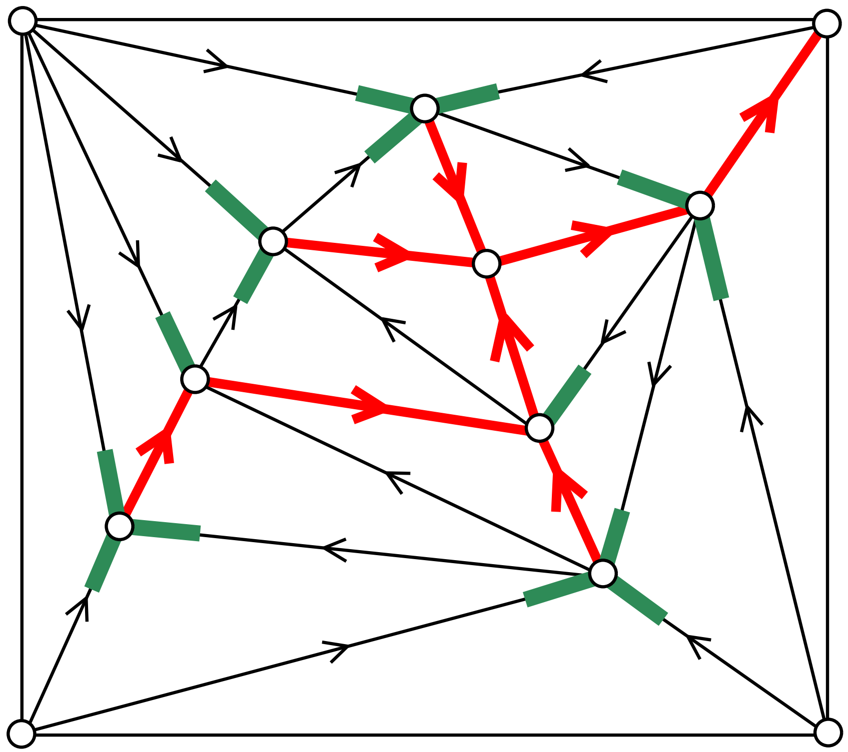

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The red-green graph is a rooted ternary tree

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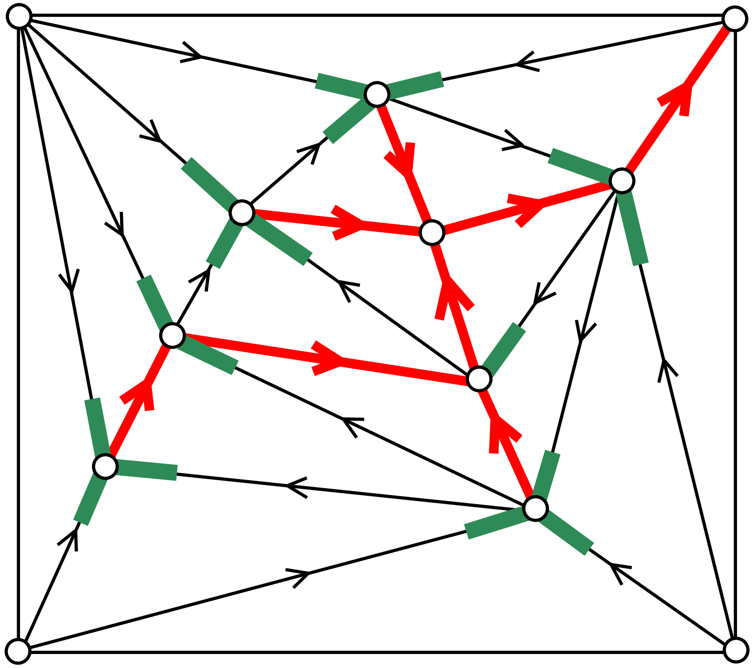

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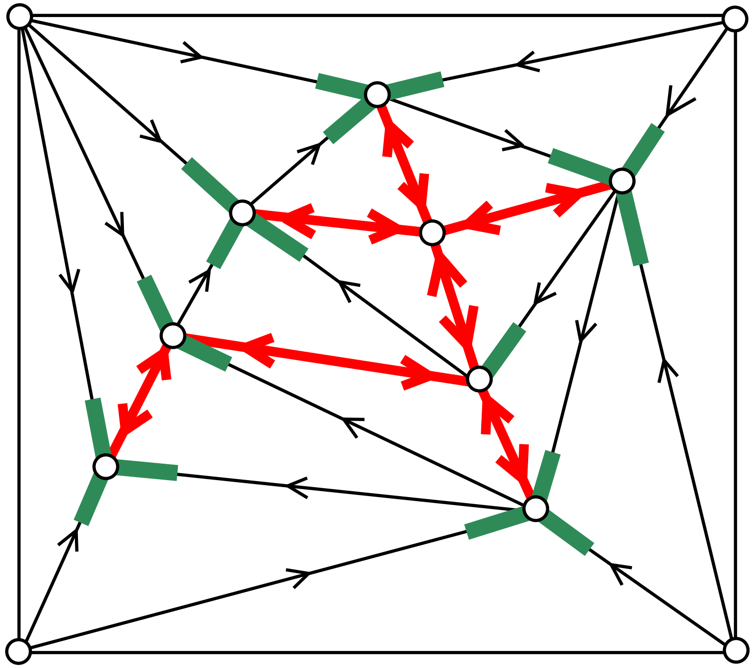
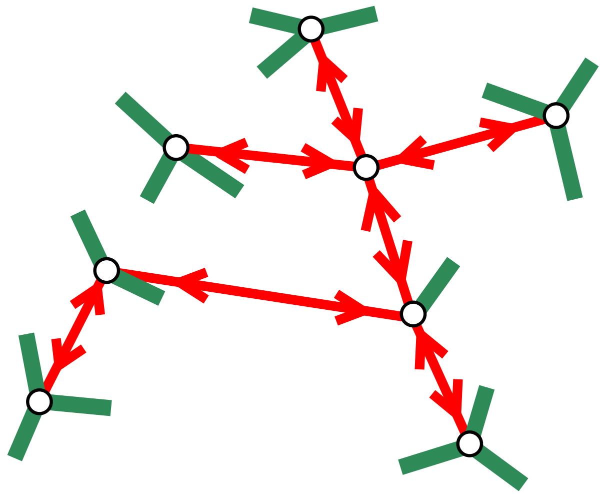


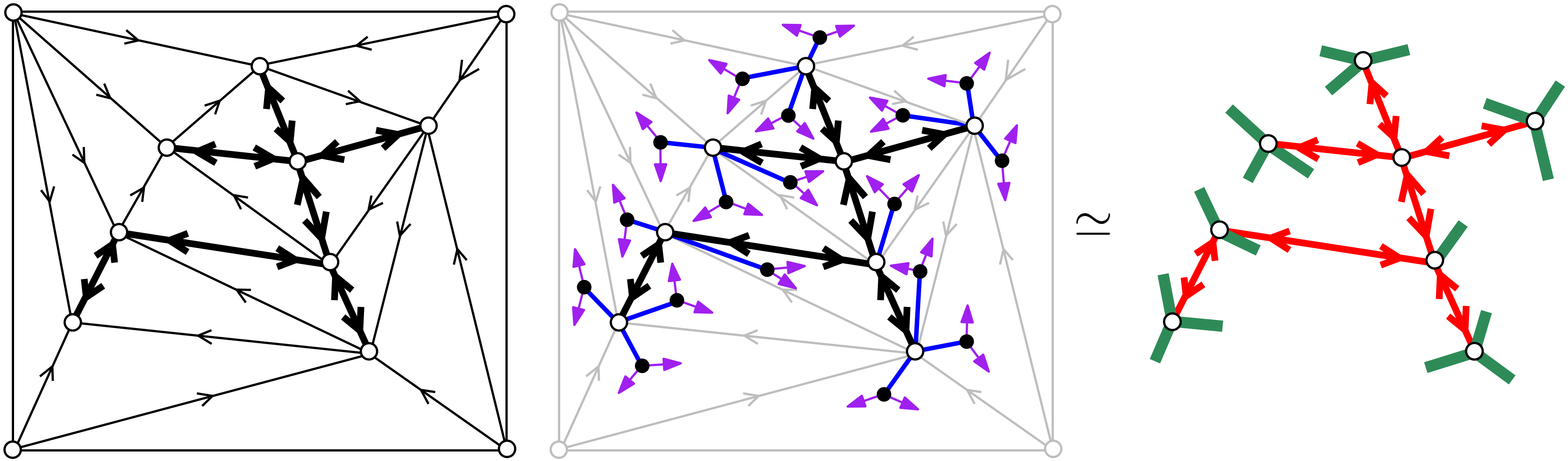
return the edge to the root in the canonical spanning tree

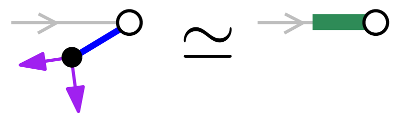
bi-orient the other red edges



ternary tree (unrooted)

Seeing the mapping as a mobile construction

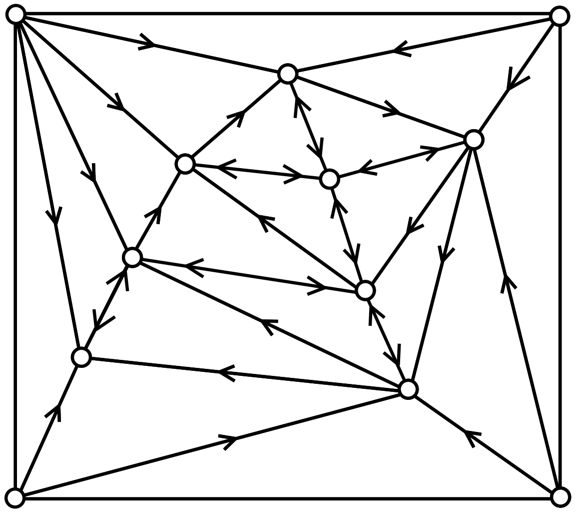




Uniqueness of the orientation

We have proved the existence of a minimal orientation such that

- Inner edges are directed or bidirected
- Inner (resp. outer) vertices have indegree 4 (resp. 0)



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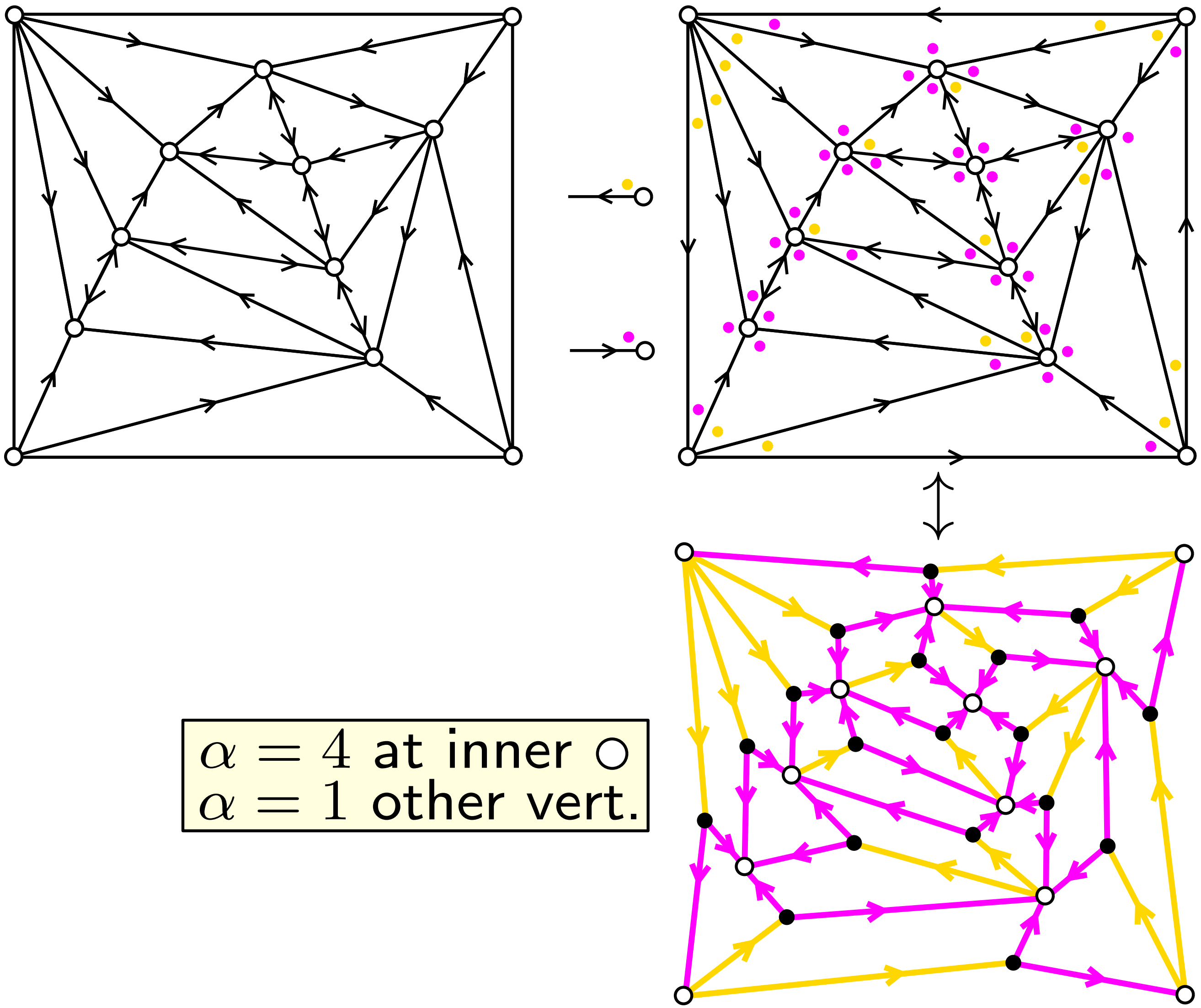
To prove uniqueness, transfer to an α-orientation (on star-graph)



Moreover, the transfer rules preserve minimality

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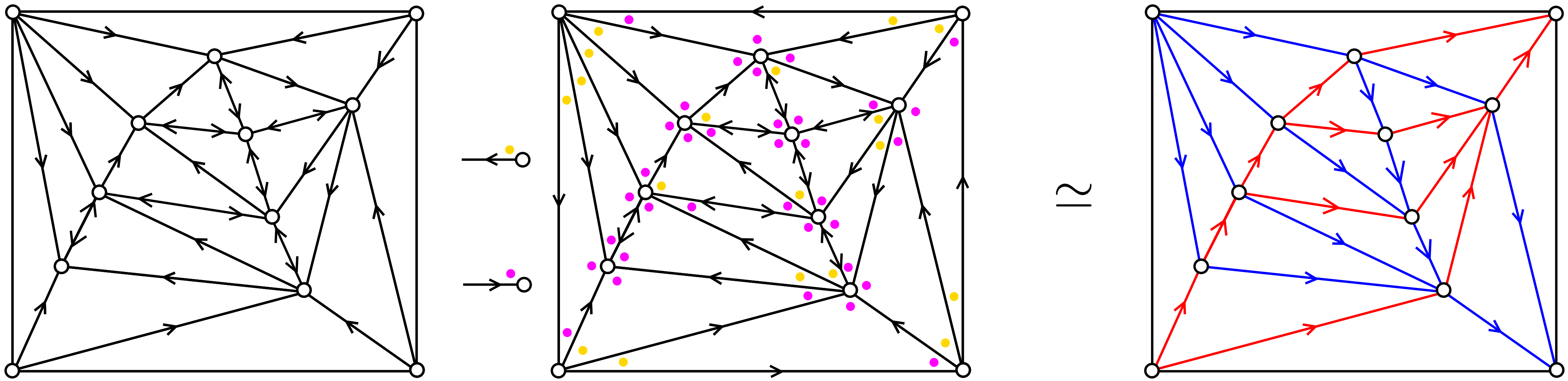
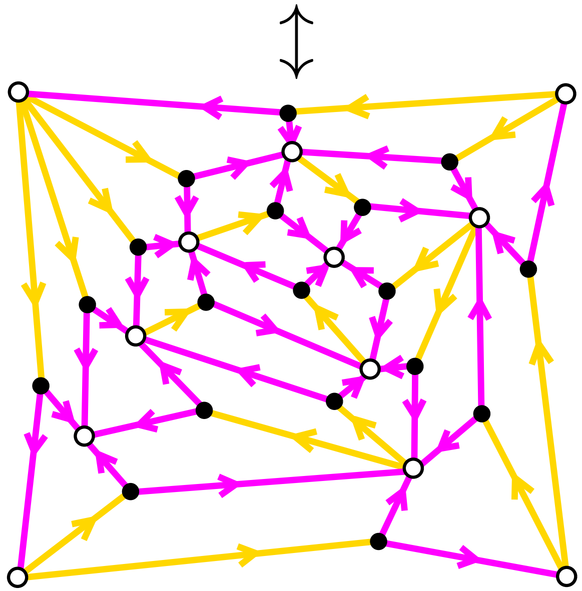




$\alpha = 4$ at inner ○
$\alpha = 1$ other vert.

Byproduct: new algo to find the minimal transversal structure

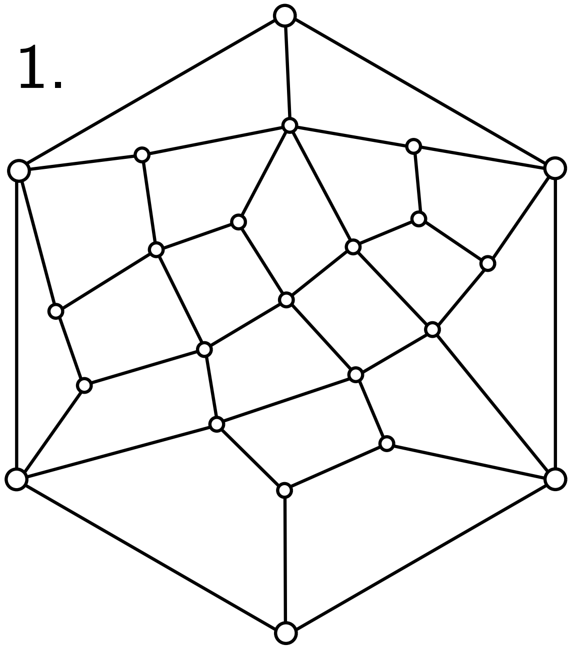
from min. 3-ori

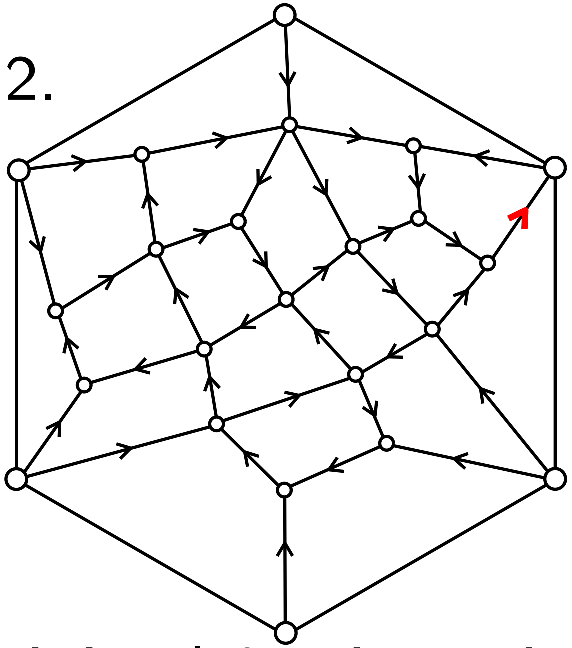
Moreover, the transfer rules preserve minimality

Bijection for irreducible $(6,4)$-dissections (quadrangulated case)

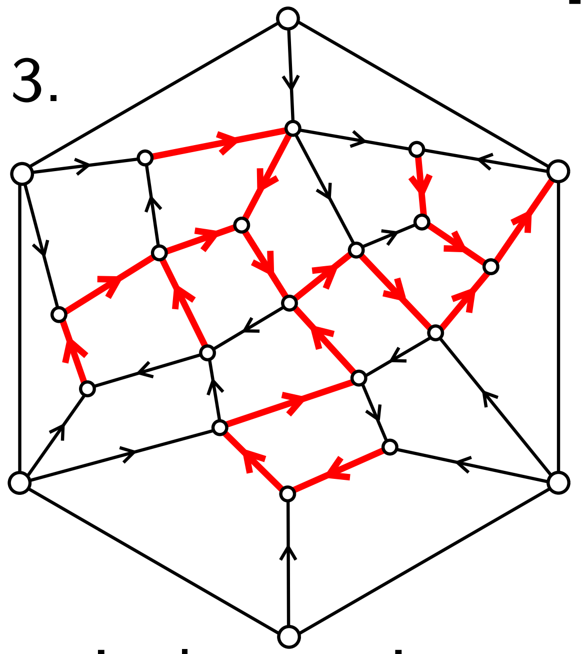
Irreducible $(6,4)$-dissections

recover [F, Poulalhon, Schaeffer'05]

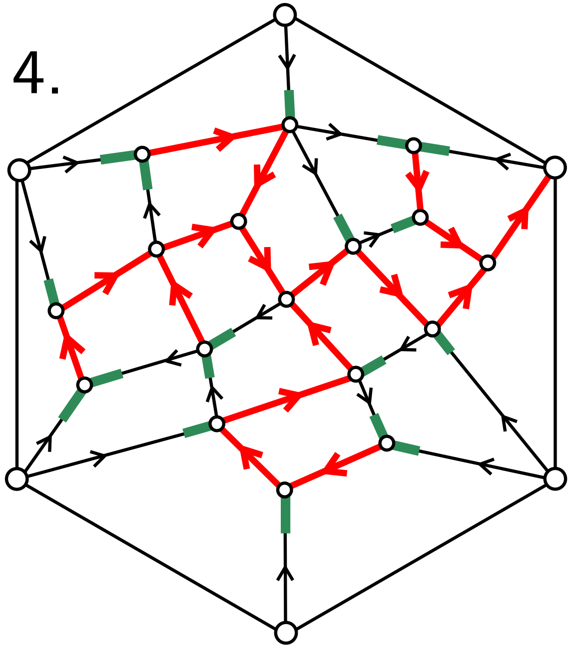




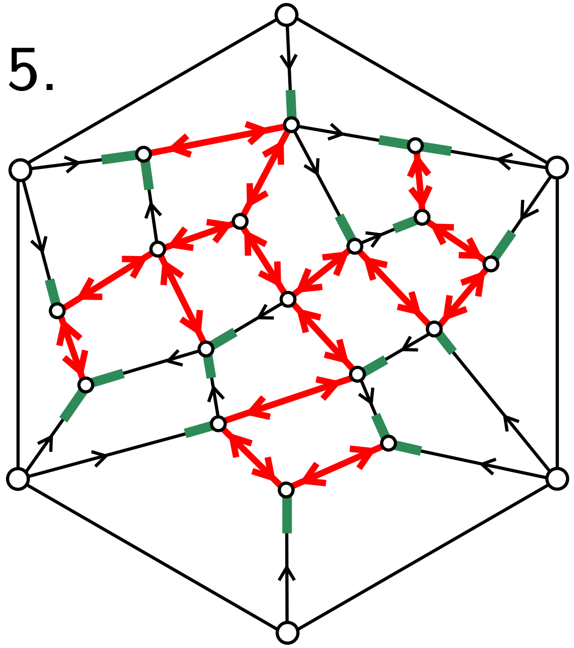
minimal 2-orientation



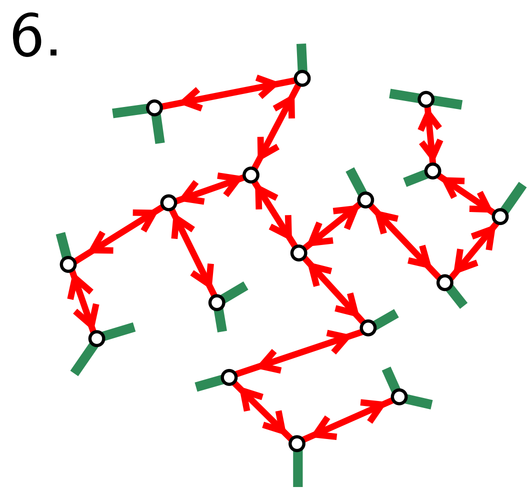
canonical spanning tree



in red-green a rooted binary tree

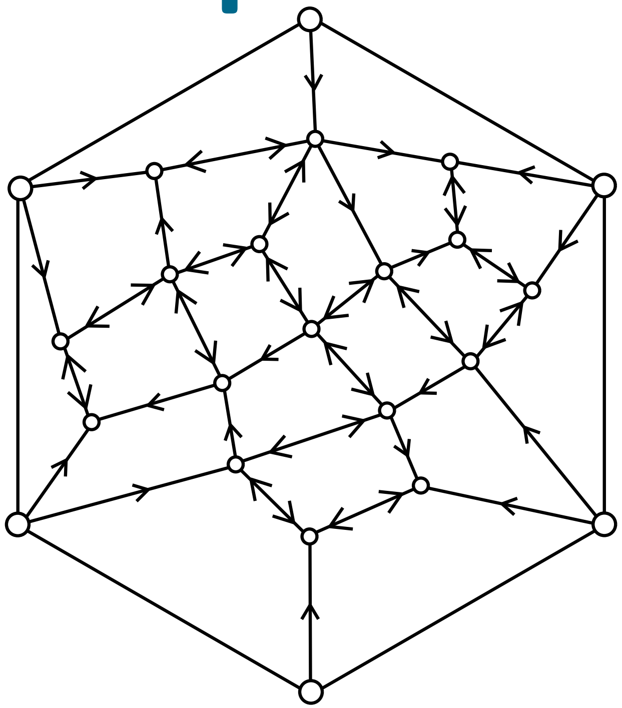


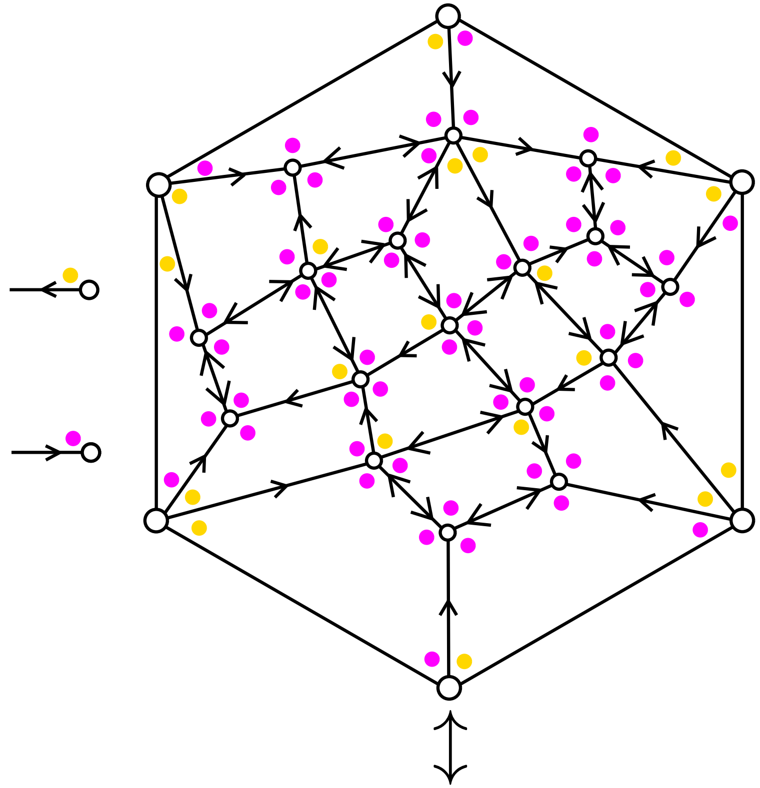
return root-edge
bi-orient tree-edges

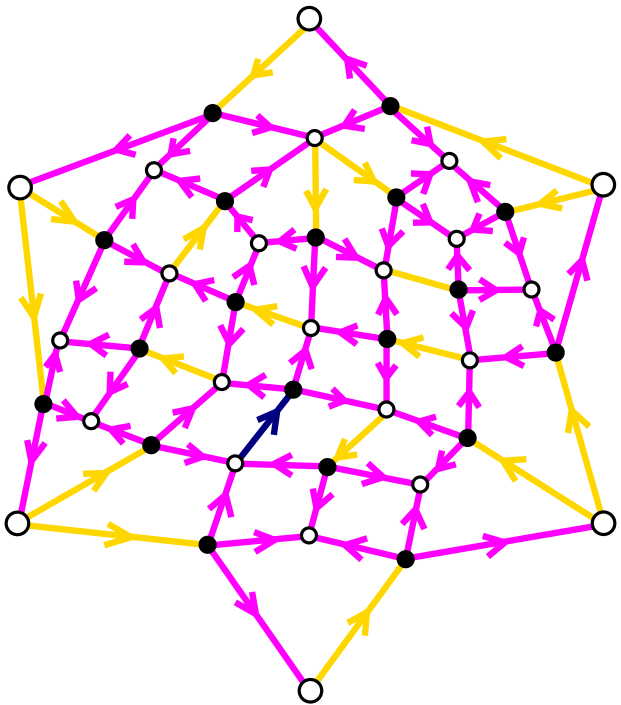


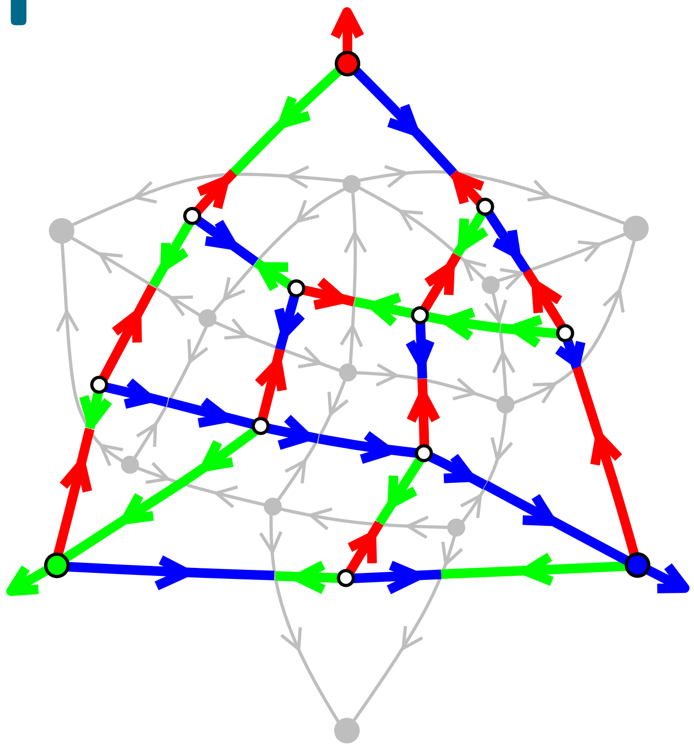
binary tree (unrooted)

Uniqueness of the orientation









$\alpha = 3$ at inner ○
$\alpha = 1$ other vert.

Byproduct: new algo to compute minimal Schnyder wood of a 3-connected map

from min. 2-ori

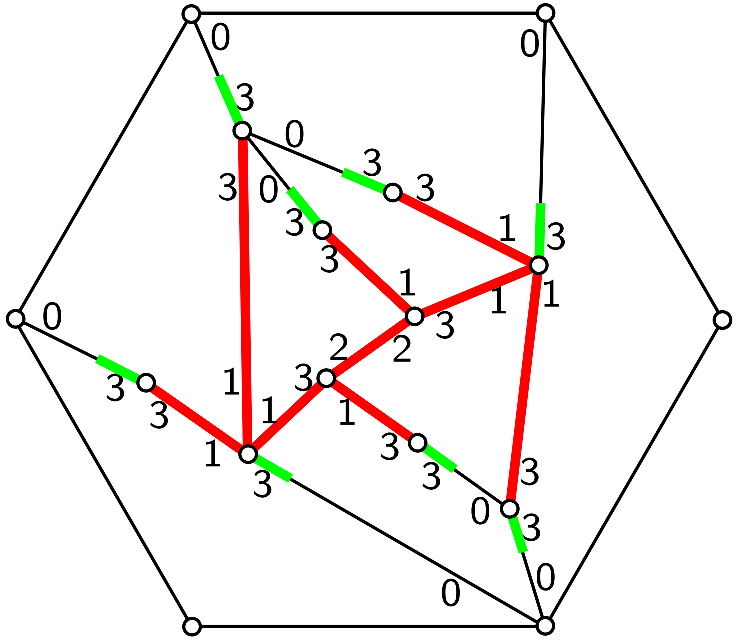
Bijection for irreducible d-angulated dissections

Extension to any d of results for $d \in \{3,4\}$

- Case d odd, irreducible $(d+1,d)$-dissections

Tree-shape

1.	d-2 legs
2.	i j edges i,j positive add up to d-1
3.	Total weight d+1 at each vertex

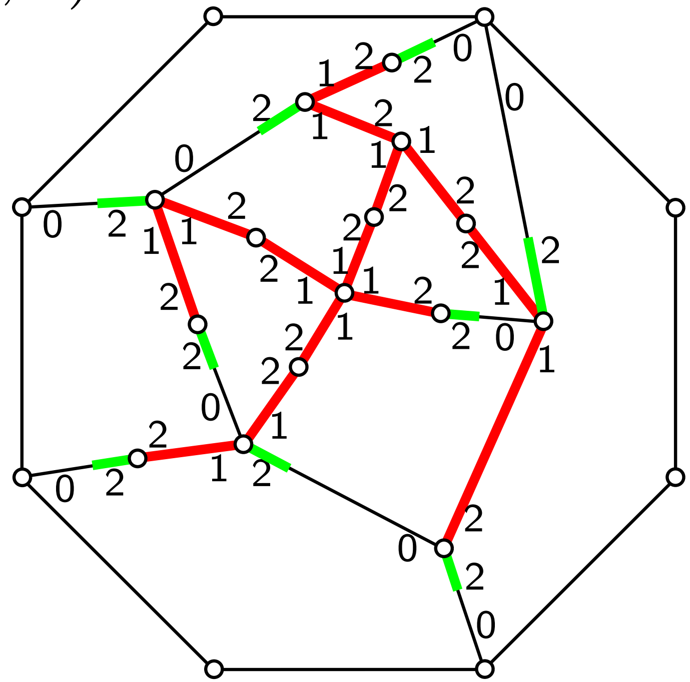


$d=5$

- Case d even, $d=2b$, irreducible $(d+2,d)$-dissections

Tree-shape

1.	b-1 legs
2.	i j edges i,j positive add up to b
3.	Total weight b+1 at each vertex

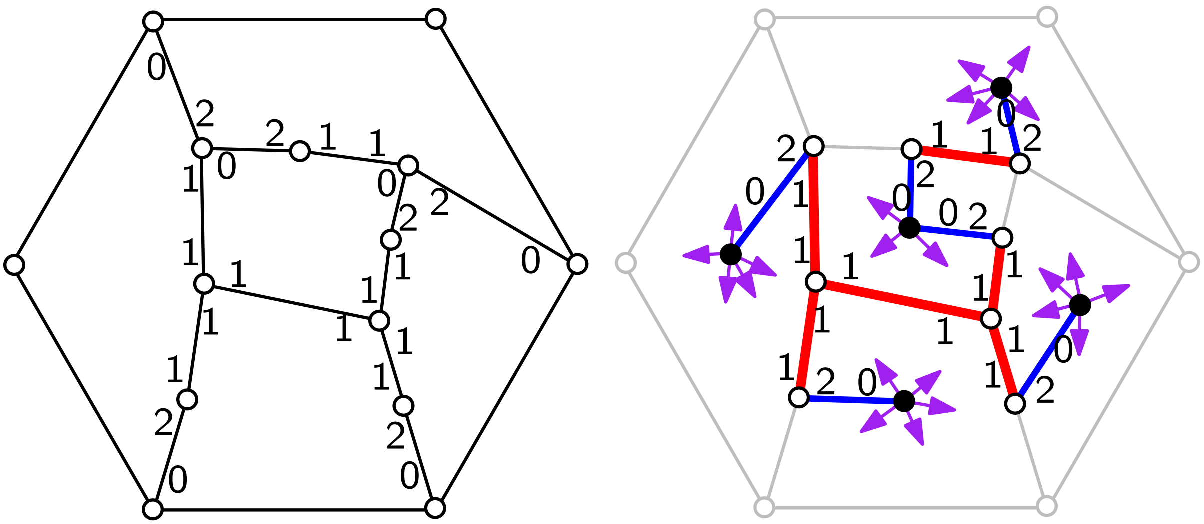


$b=3$

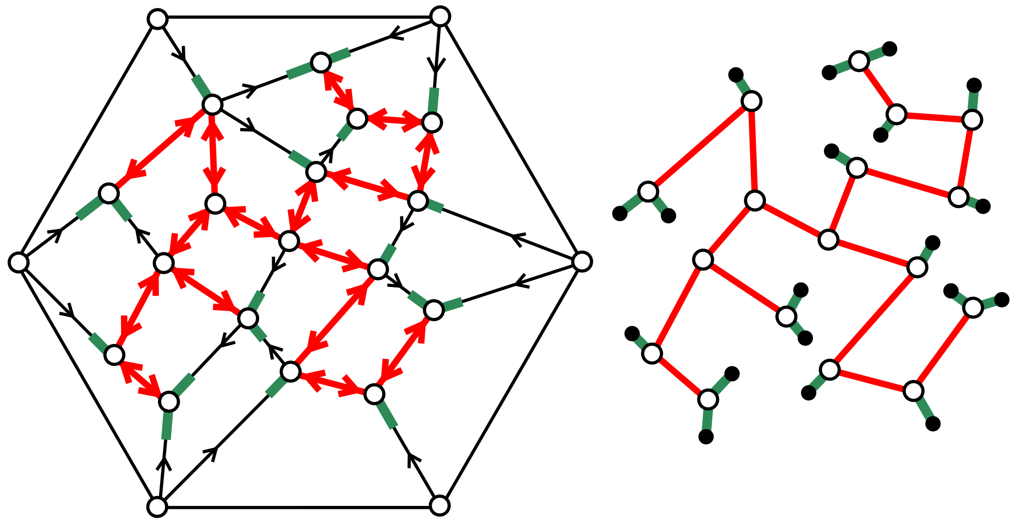
Extensions of the bijections obtained so far

Combining both bijections (bipartite case)

$2b$-angulations girth $2b$



irreducible $(2b, 2b-2)$-dissections



Combining both bijections (bipartite case)

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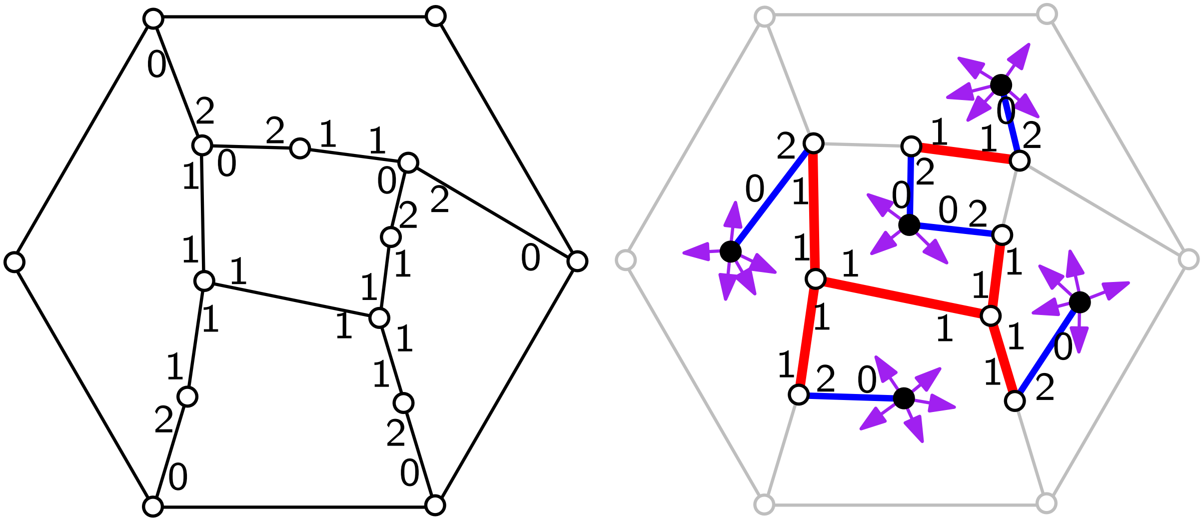
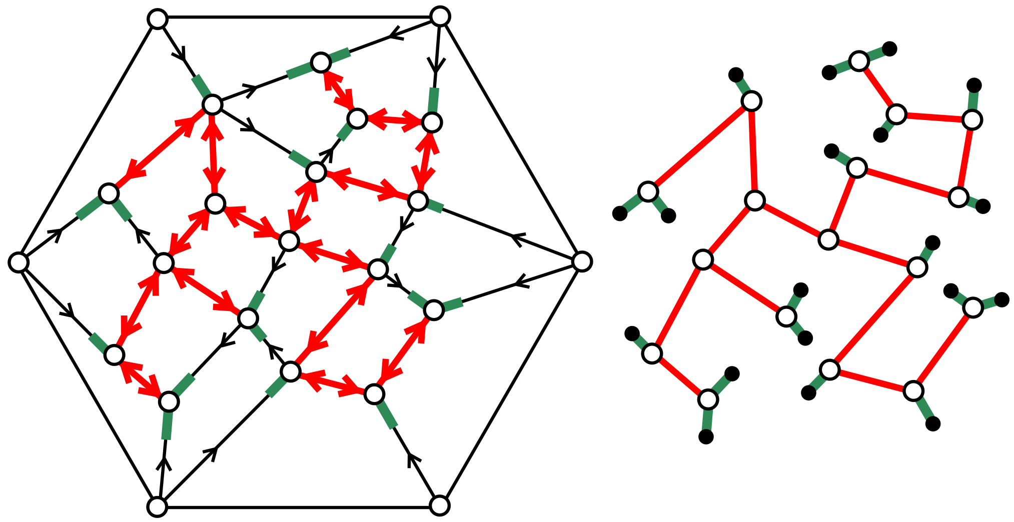
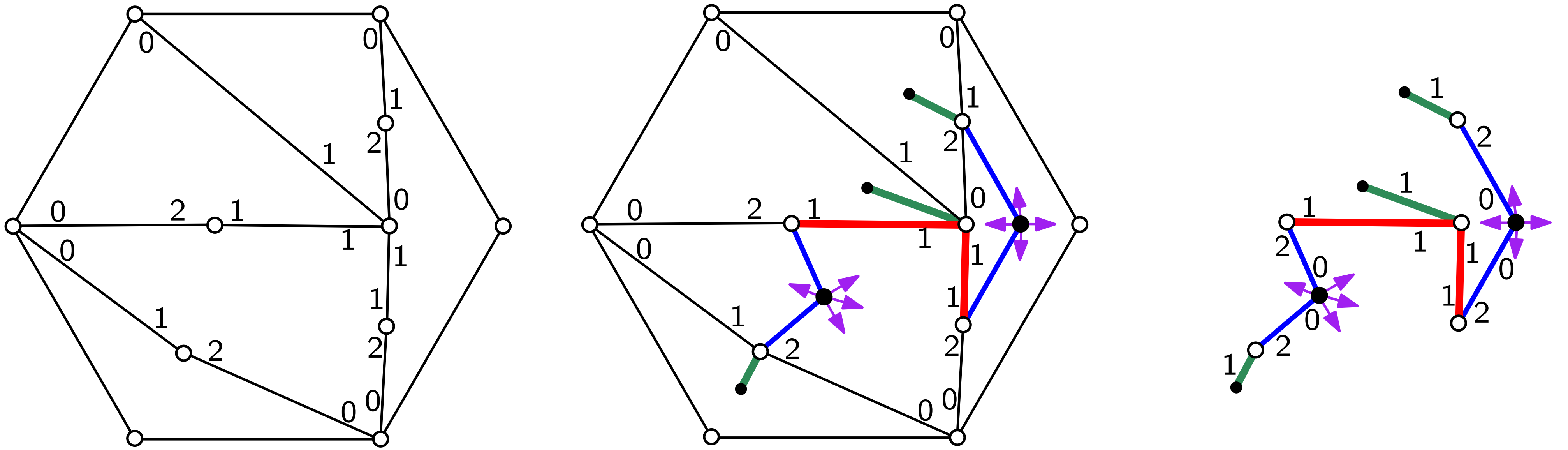


irreducible $(2b, 2b-2)$-dissections

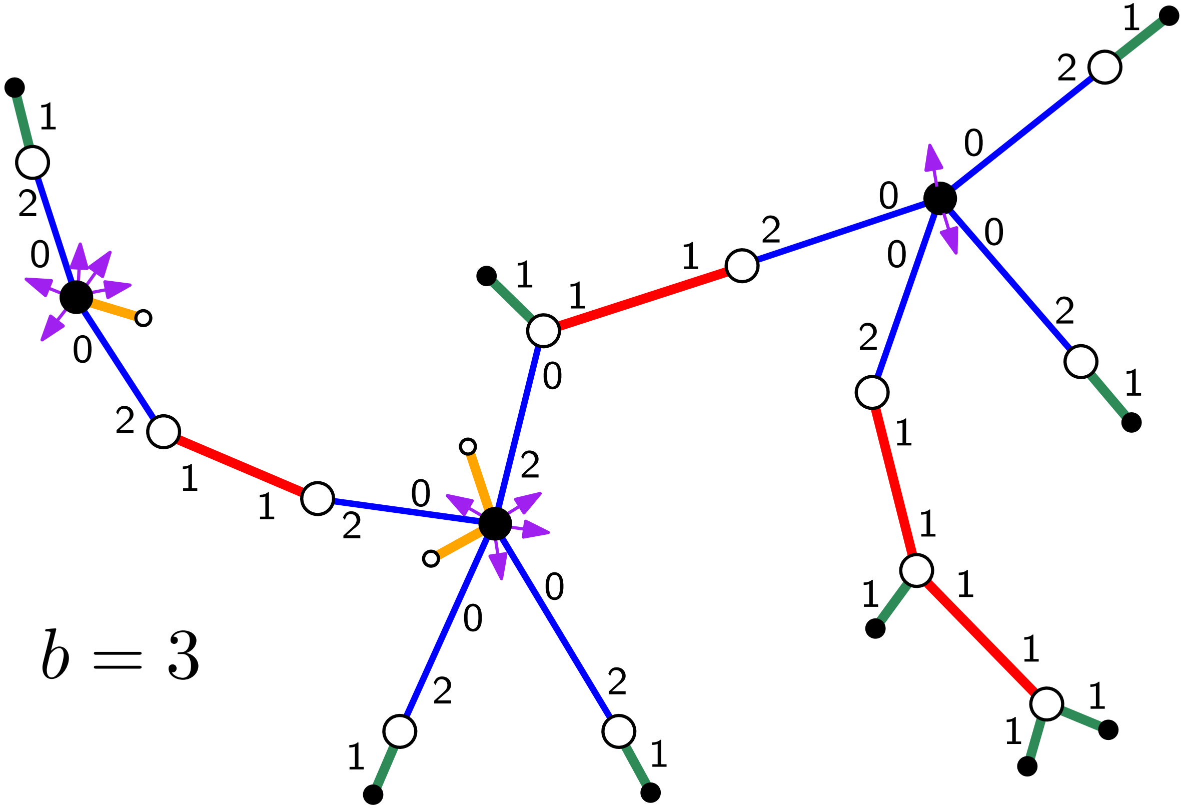


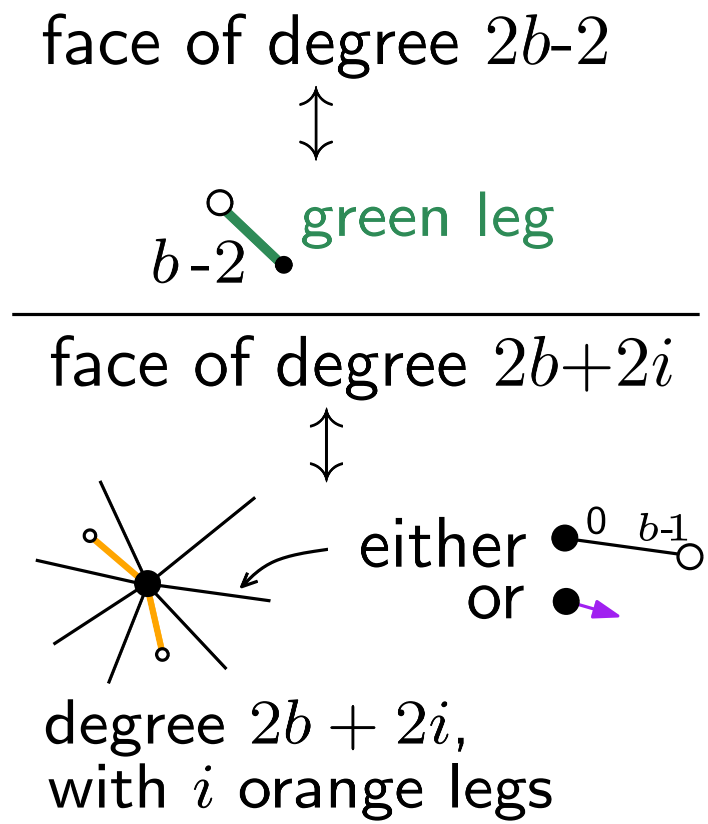
$(2b)$-outer dissections, inner face degrees $\in \{2b-2, 2b\}$
cycles of length $\geq 2b$ except contours of $(2b-2)$-faces



Allowing for higher face-degrees (bipartite case)

$(2b)$-outer dissections, inner face degrees $\in \{2b-2, 2b, 2b+2, 2b+4, \ldots\}$
cycles of length $\geq 2b$ except contours of $(2b-2)$-faces





Allowing for higher face-degrees (bipartite case)

$(2b)$-outer dissections, inner face degrees $\in \{2b-2, 2b, 2b+2, 2b+4, \ldots\}$
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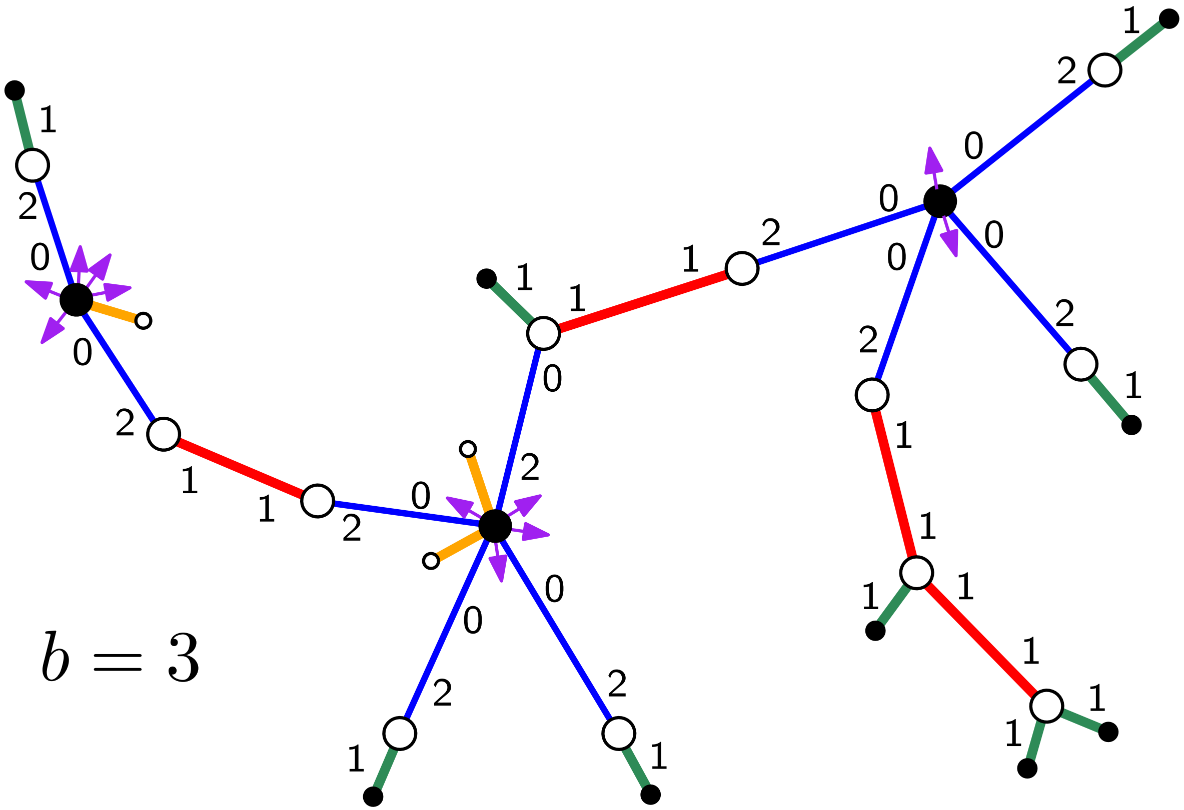
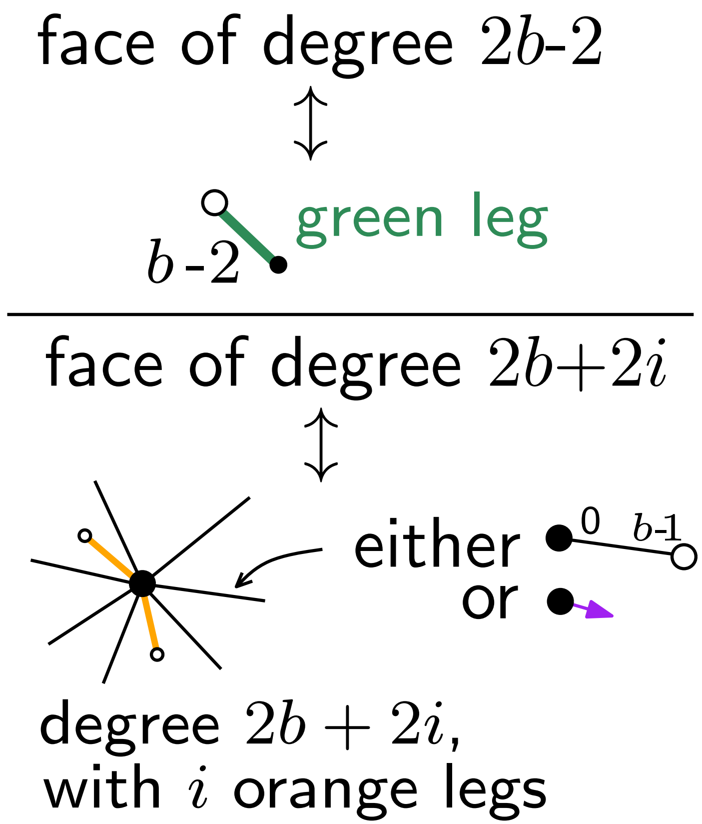


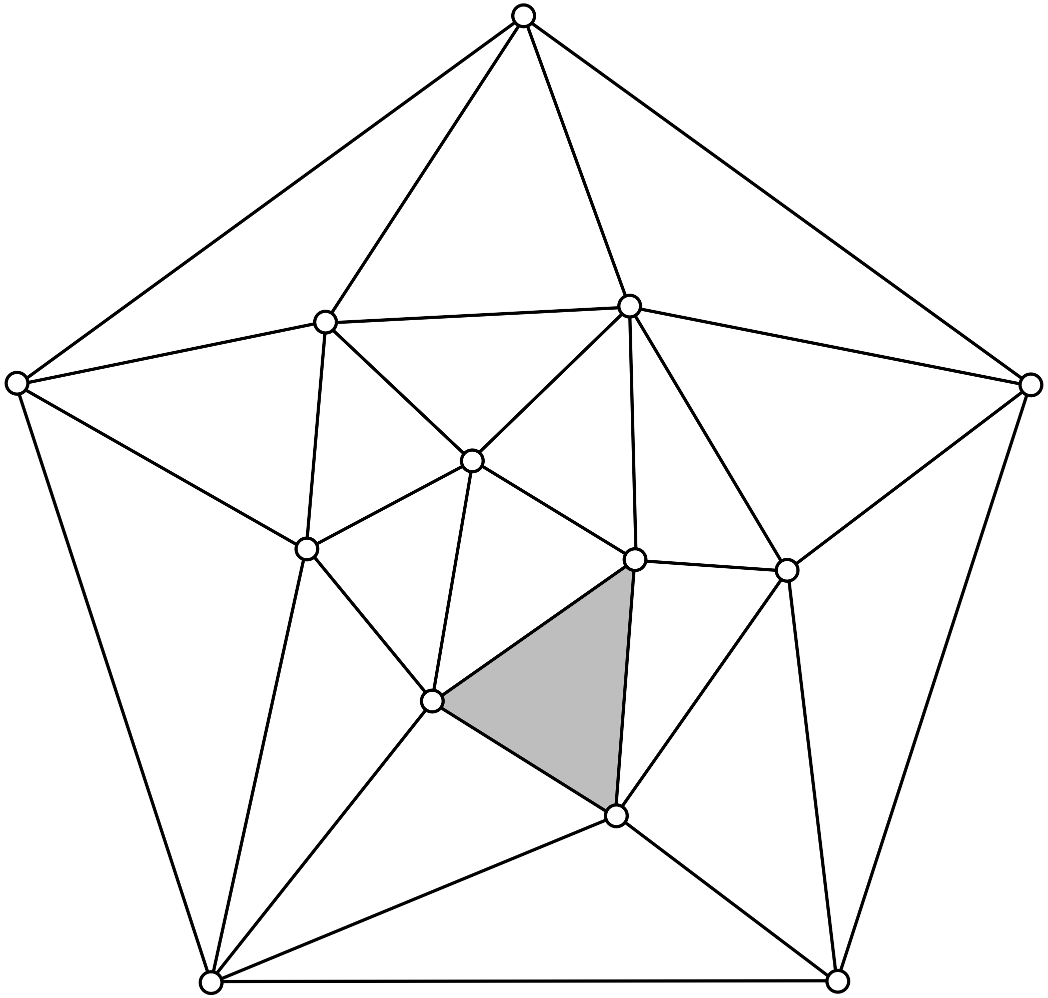
$b = 3$



- can be done also in the non-bipartite setting
- extends GF expressions for annular maps from [Bernardi,F'11]
  this time allowing for "small faces"
- recover annular maps GF expressions in [Bouttier,Guitter'13]

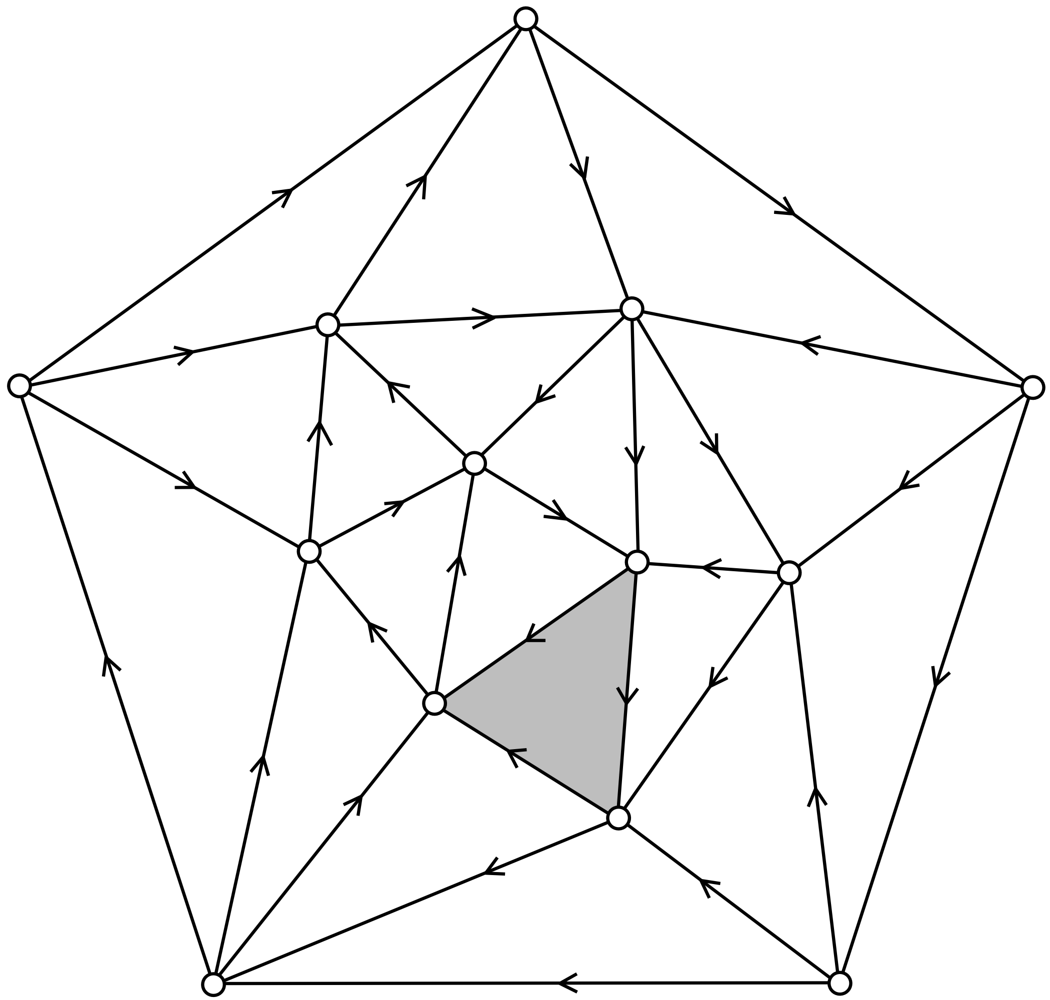
Irreducible (k, d) dissections, for any k

Case $d = 3$, take an irreducible $(k, 3)$-dissection, mark an inner face



Irreducible (k, d) dissections, for any k

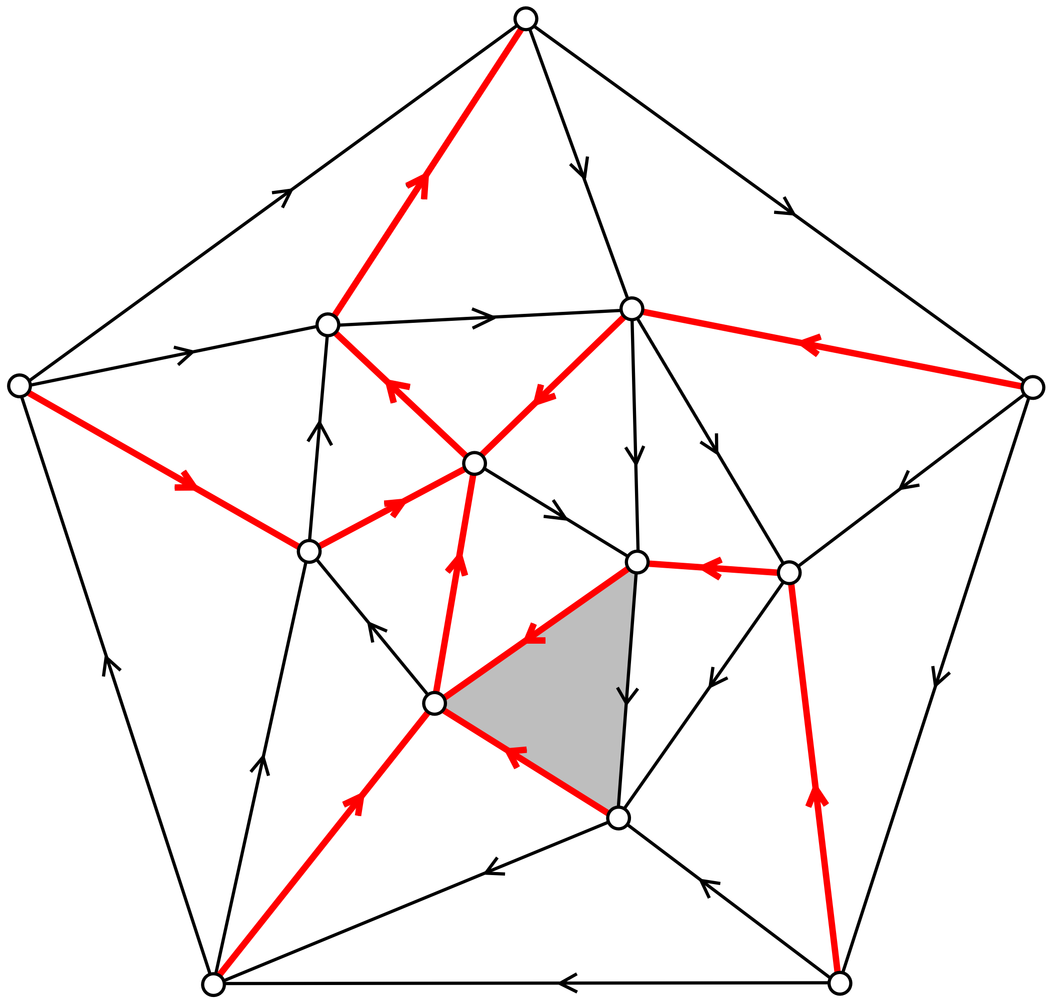
Case $d = 3$, take an irreducible $(k, 3)$-dissection, mark an inner face



endow it with a minimal 3-orientation

Irreducible (k, d) dissections, for any k

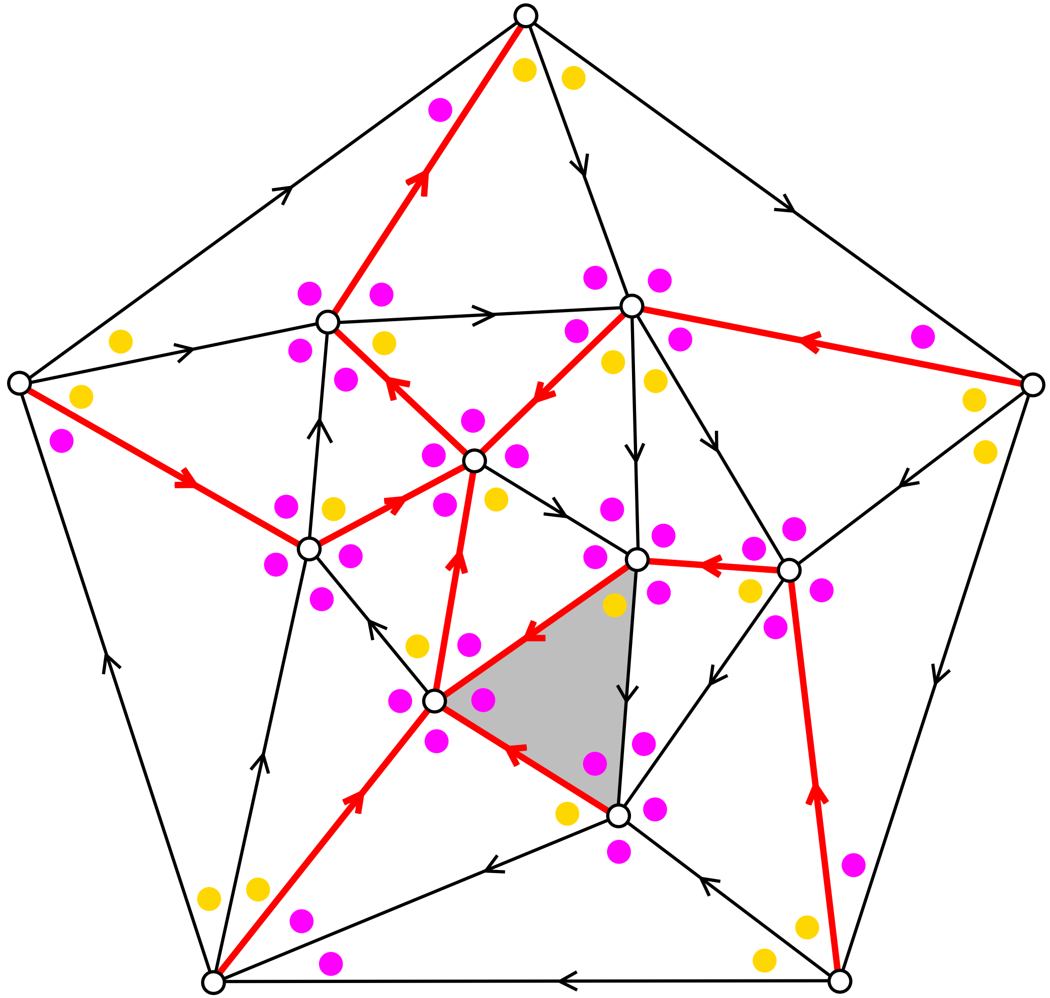
Case $d = 3$, take an irreducible $(k, 3)$-dissection, mark an inner face



endow it with a minimal 3-orientation
compute the canonical spanning tree

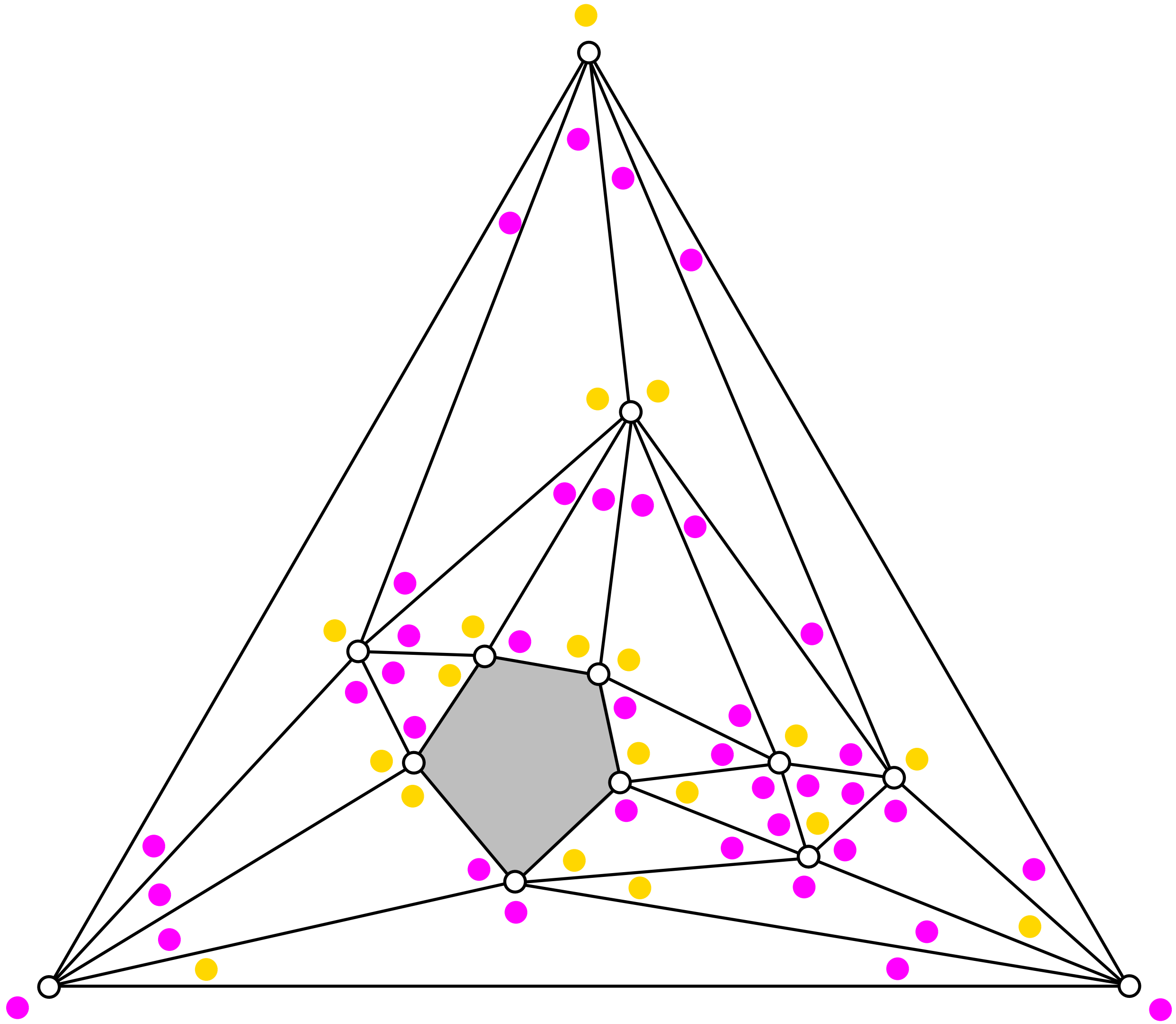
Irreducible (k,d) dissections, for any k

Case $d=3$, take an irreducible $(k,3)$-dissection, mark an inner face



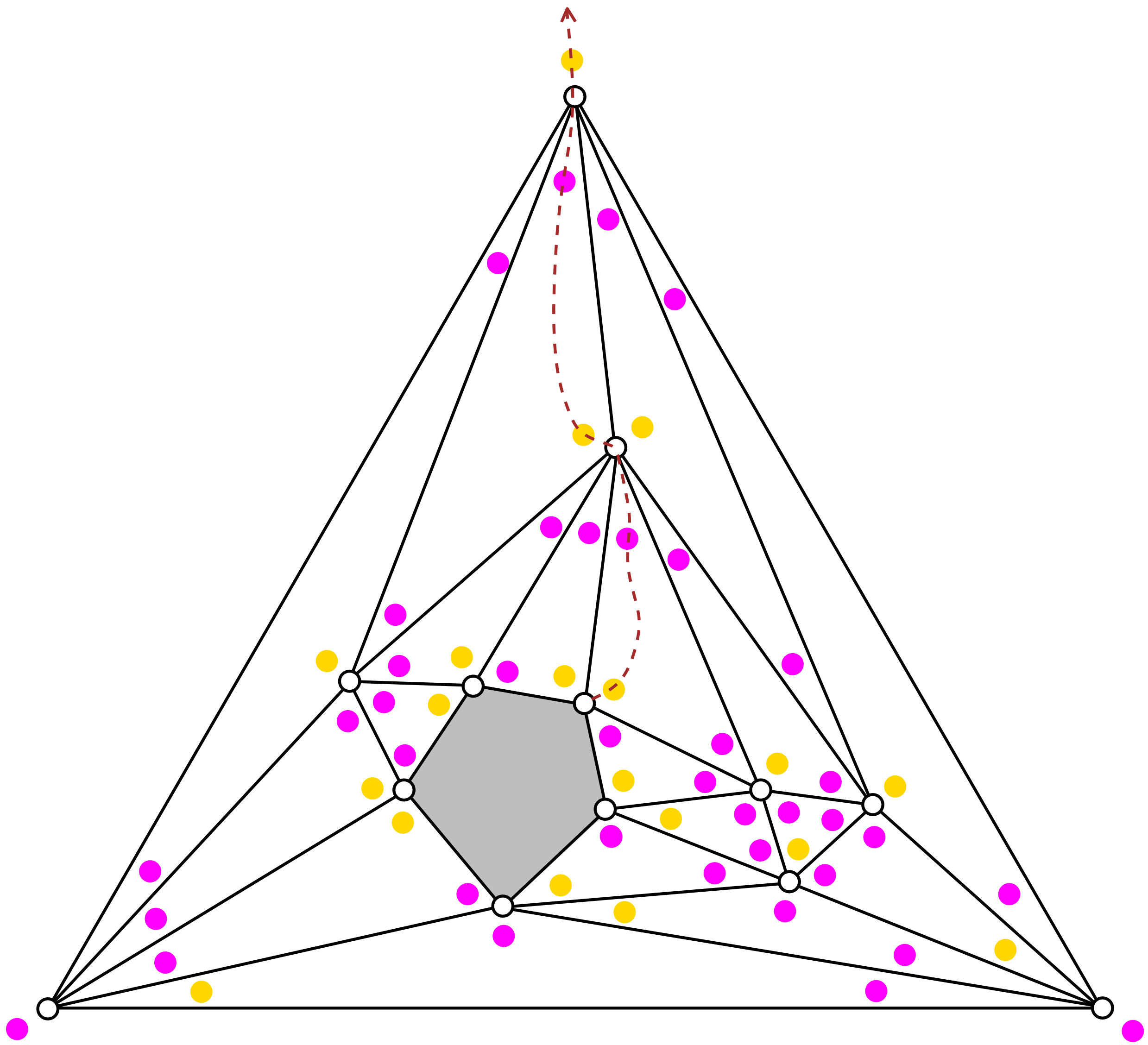
endow it with a minimal 3-orientation
compute the canonical spanning tree
and associated corner bicoloration

Irreducible (k,d) dissections, for any k



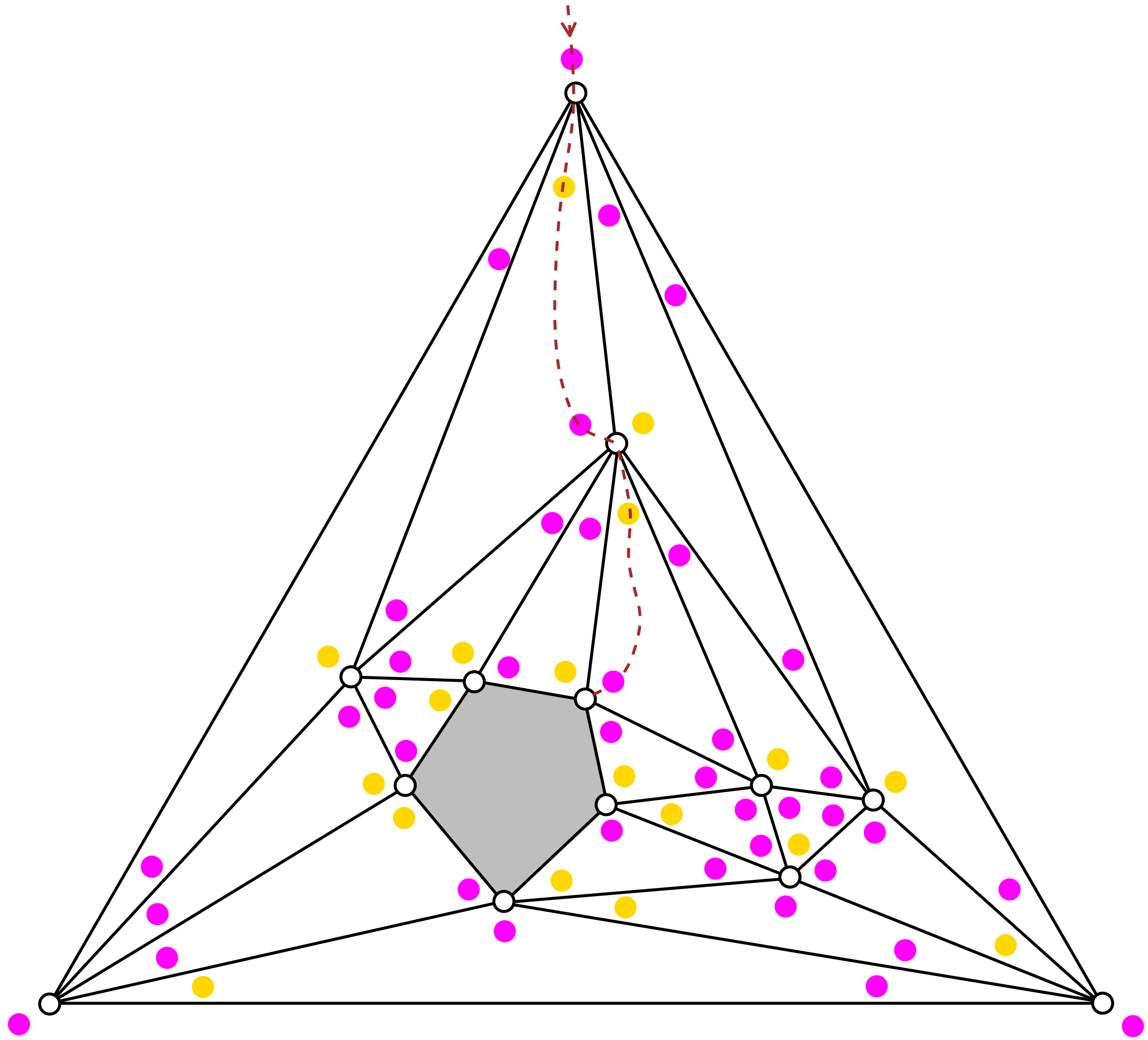
take the marked inner face as outer face

Irreducible (k, d) dissections, for any k



take the marked inner face as outer face
make all corners in the outer face magenta (by returning a path)

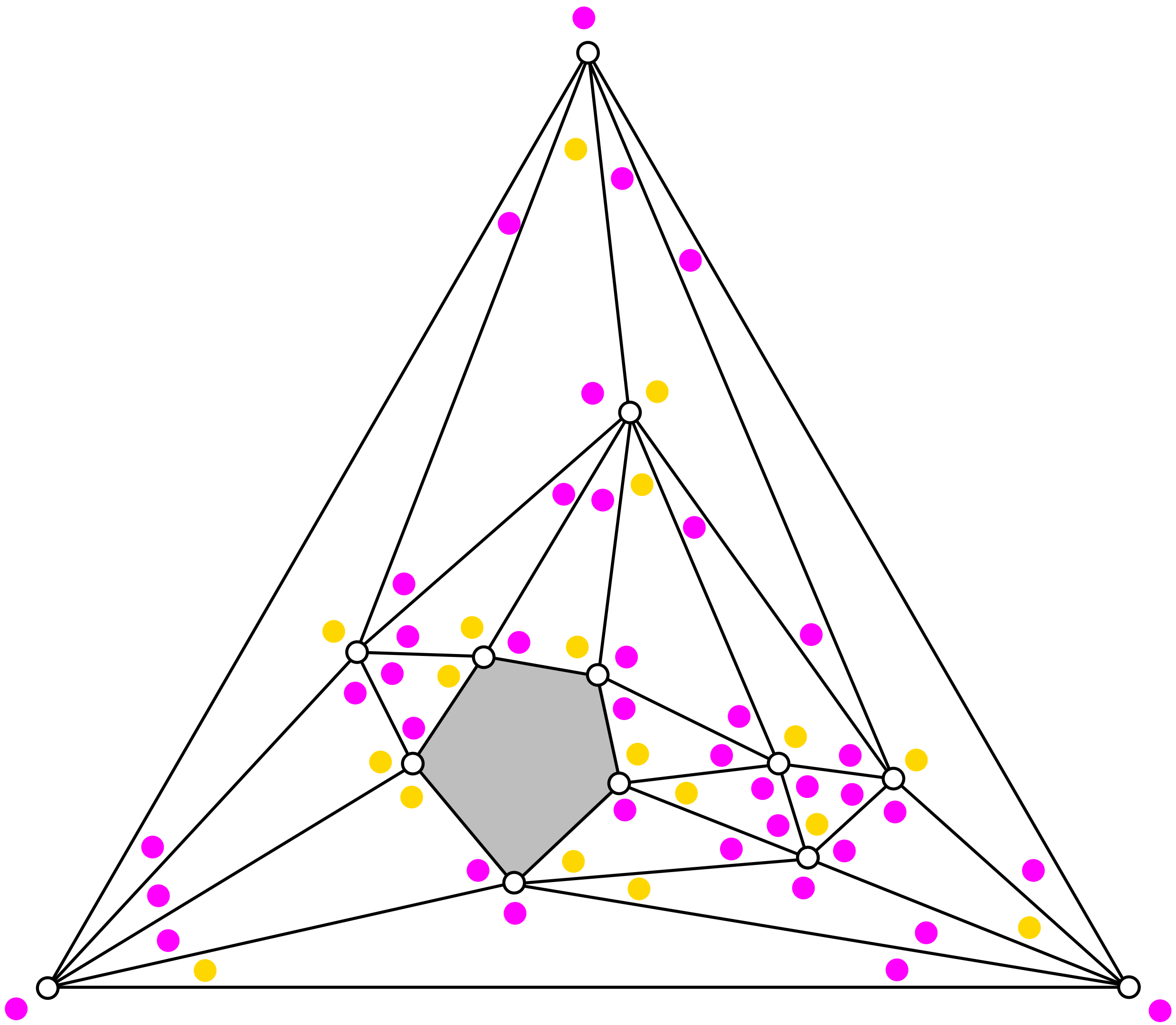
Irreducible (k,d) dissections, for any k



take the marked inner face as outer face

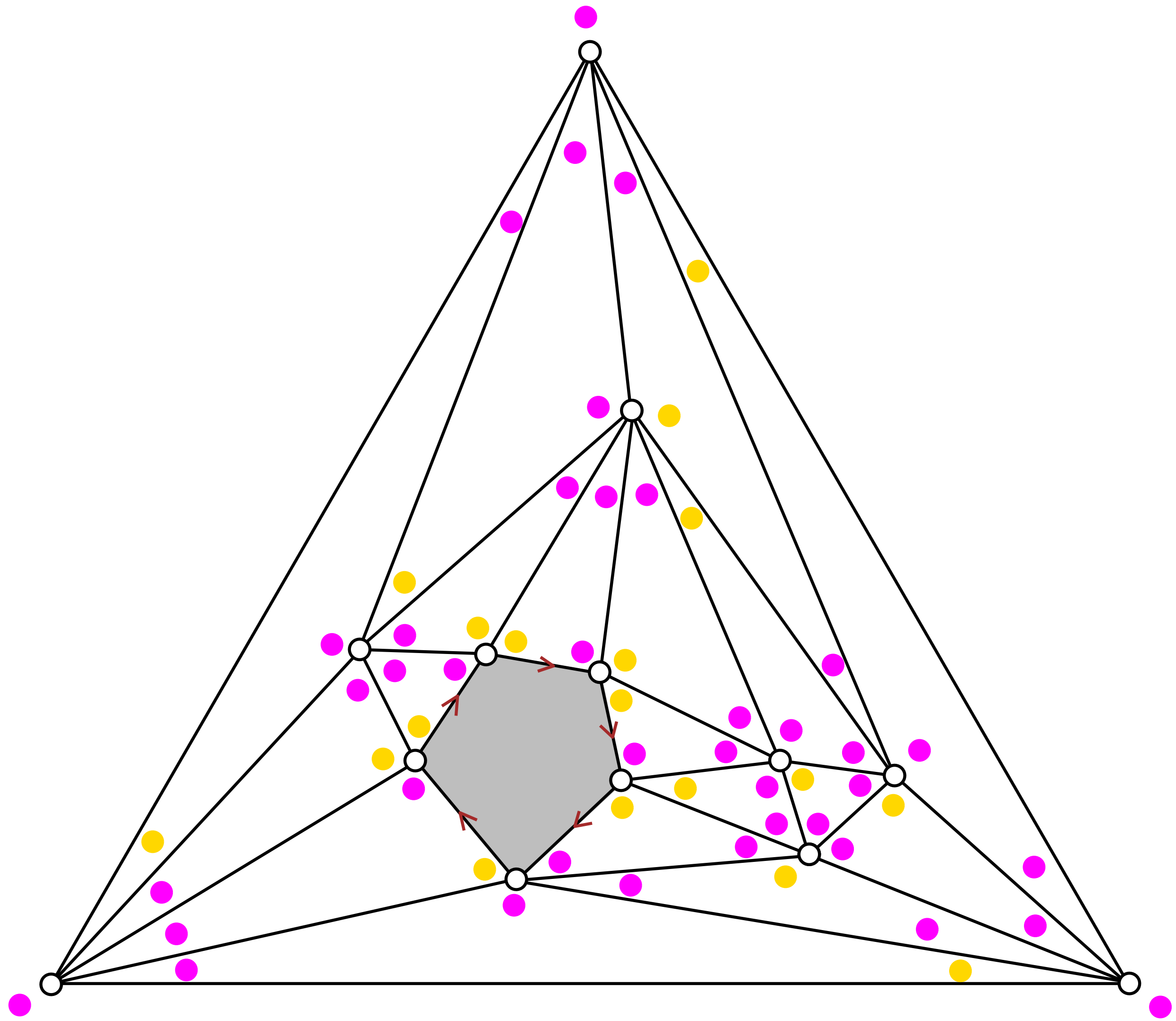
make all corners in the outer face magenta (by returning a path)

Irreducible (k,d) dissections, for any k



then make the underlying α-orientation minimal
(considering the boundary face as a big vertex)

Irreducible (k, d) dissections, for any k



then make the underlying α-orientation minimal
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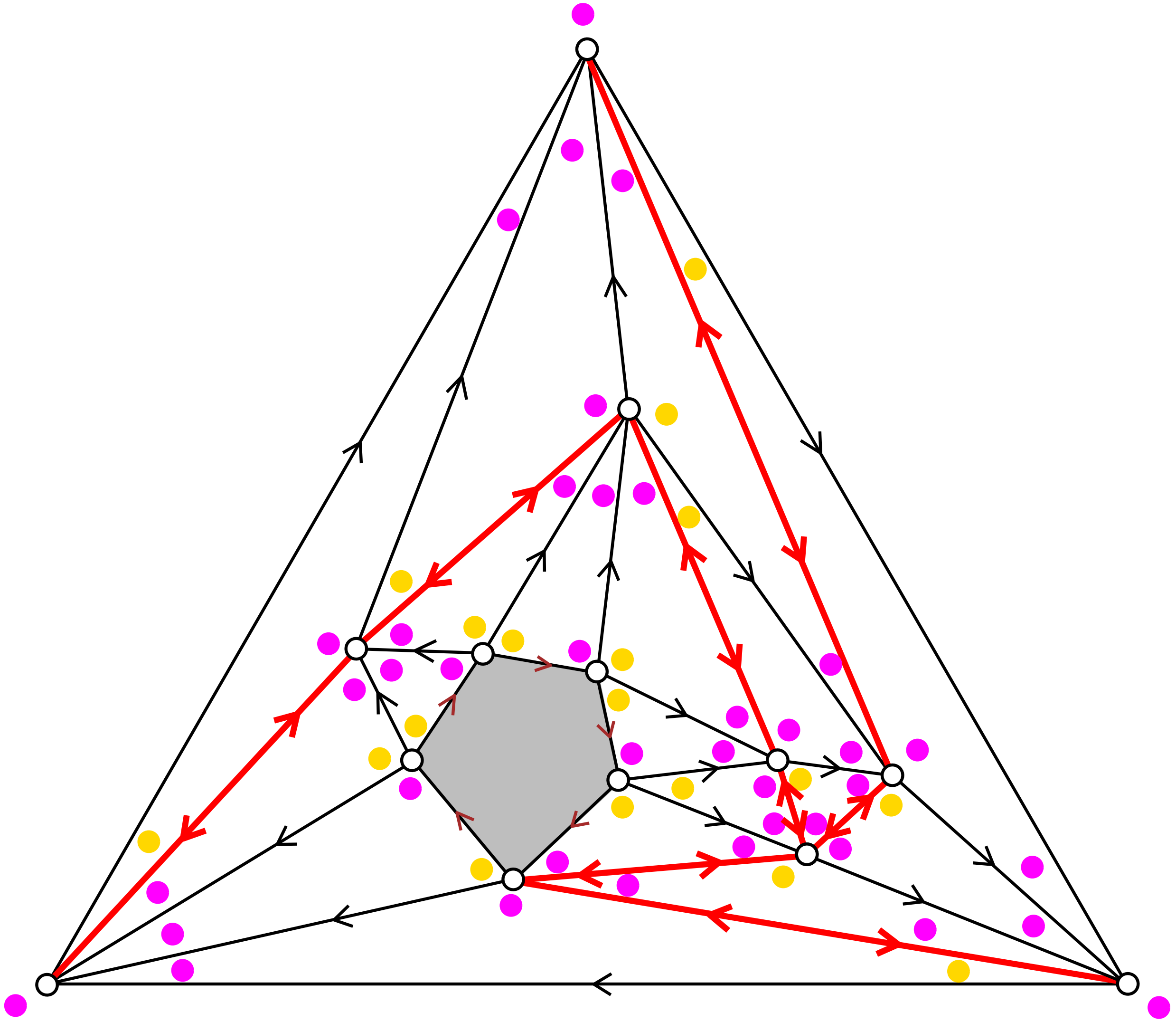
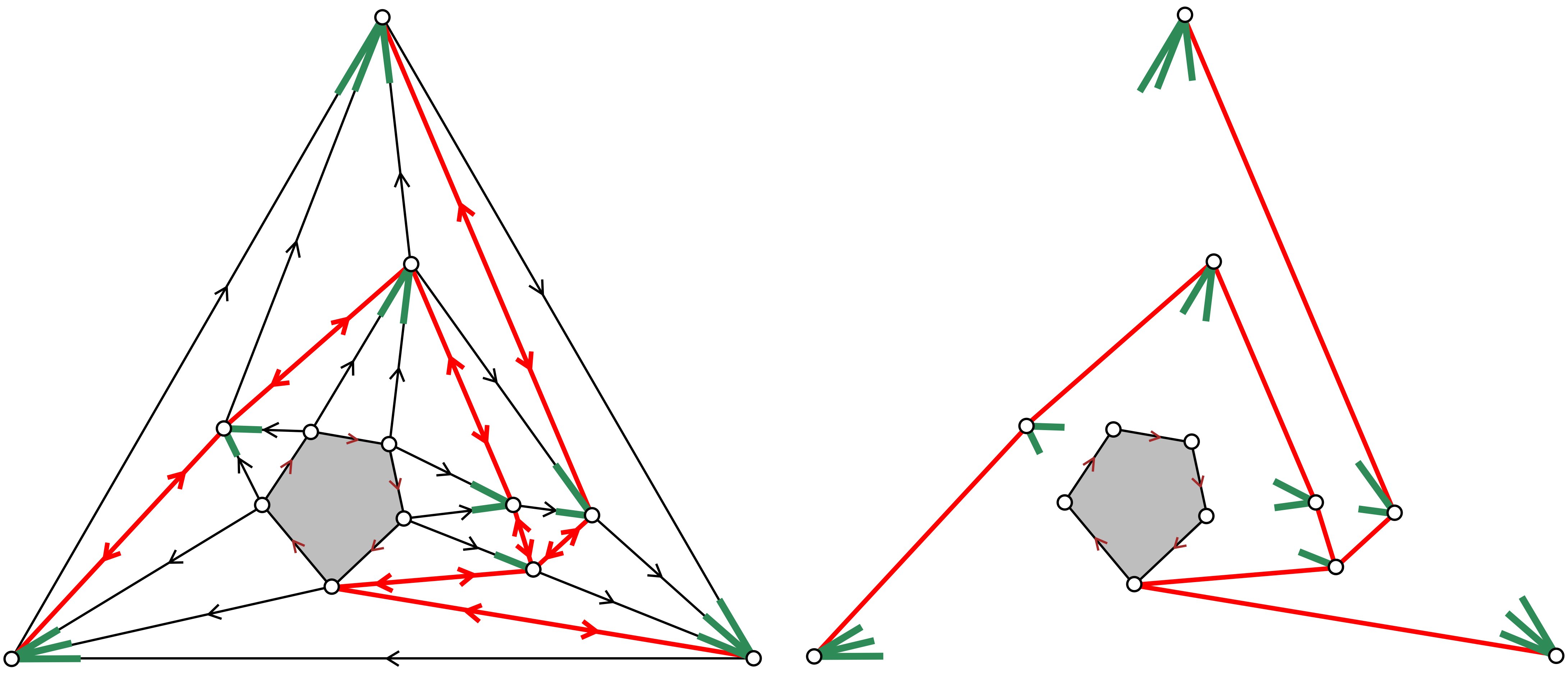
Irreducible (k, d) dissections, for any k



apply the transfer rules in the other direction

Irreducible (k, d) dissections, for any k



forest of $(k-2)$ ternary trees
attached to the boundary

$\Rightarrow$ bijective proof of Tutte's formula $\boxed{\frac{(2k-4)!}{(k-4)!(k-1)!}\frac{(3n+k-4)!}{n!(2n+k-2)!}}$

same approach works for any $d \geq 3$