Enumeration of rectangulations

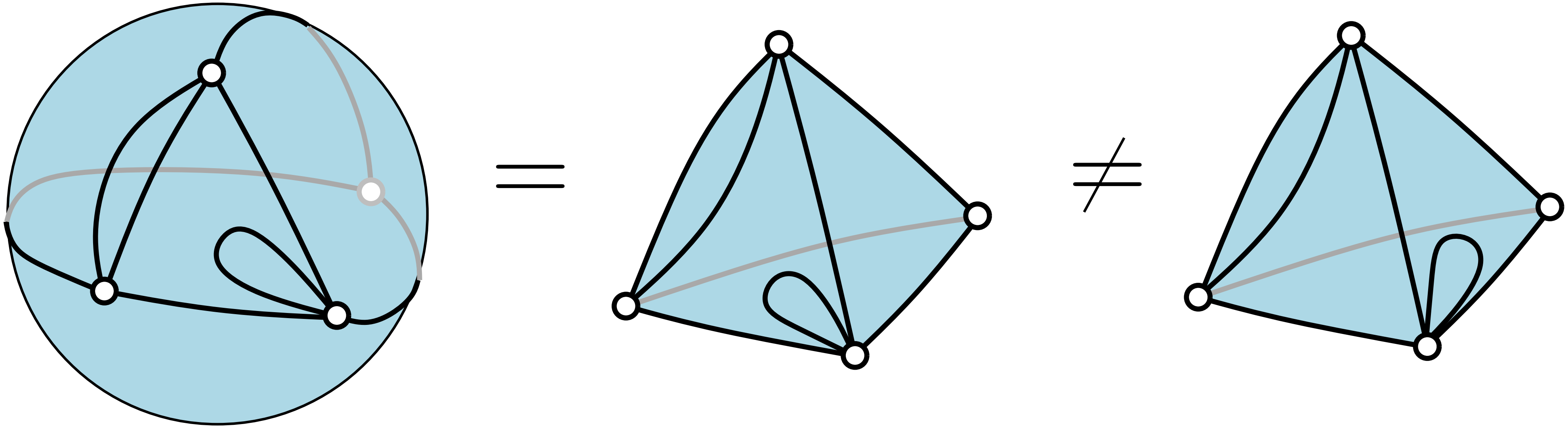
Éric Fusy (LIGM, Univ. Gustave Eiffel)

Joint work with Erkan Narmanli and Gilles Schaeffer

Séminaire combinatoire IRIF

Planar maps

Def. Planar map = connected graph embedded on the sphere



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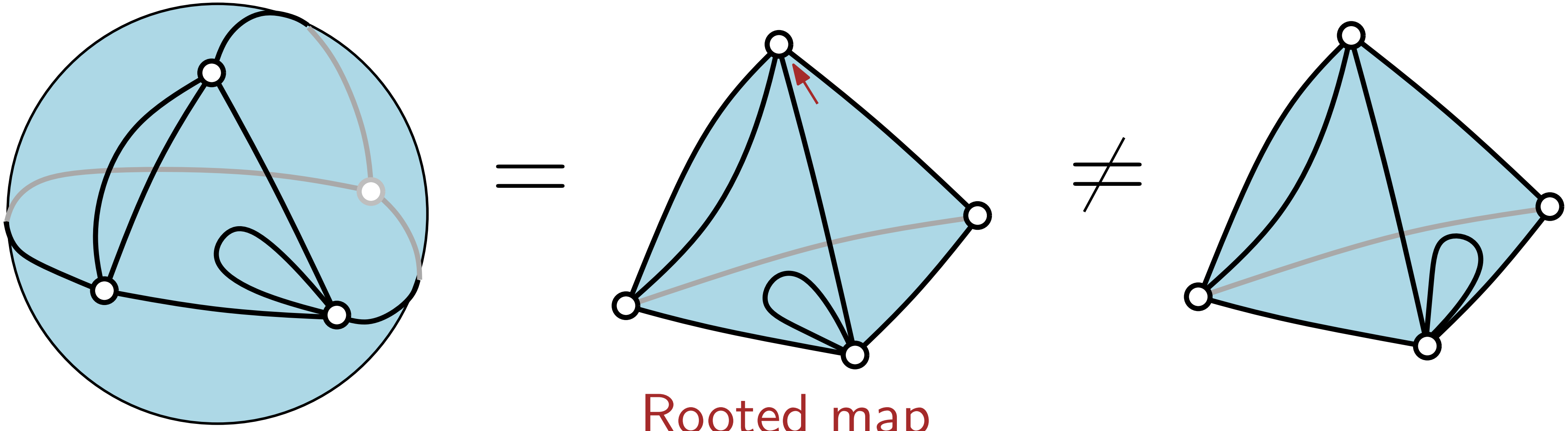

Rooted map
= map with marked corner

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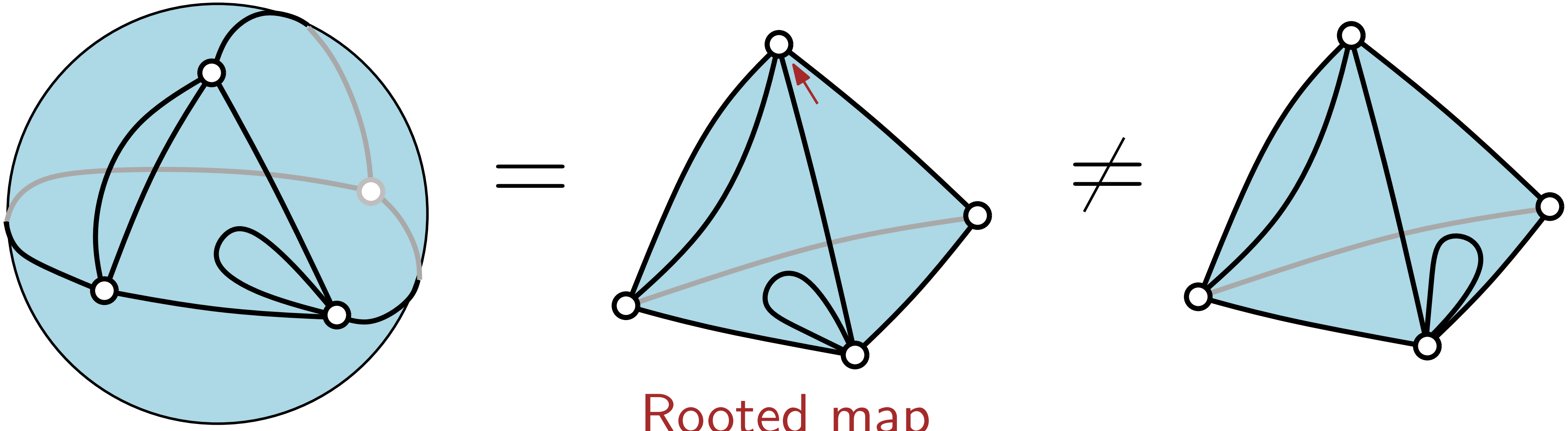

Rooted map
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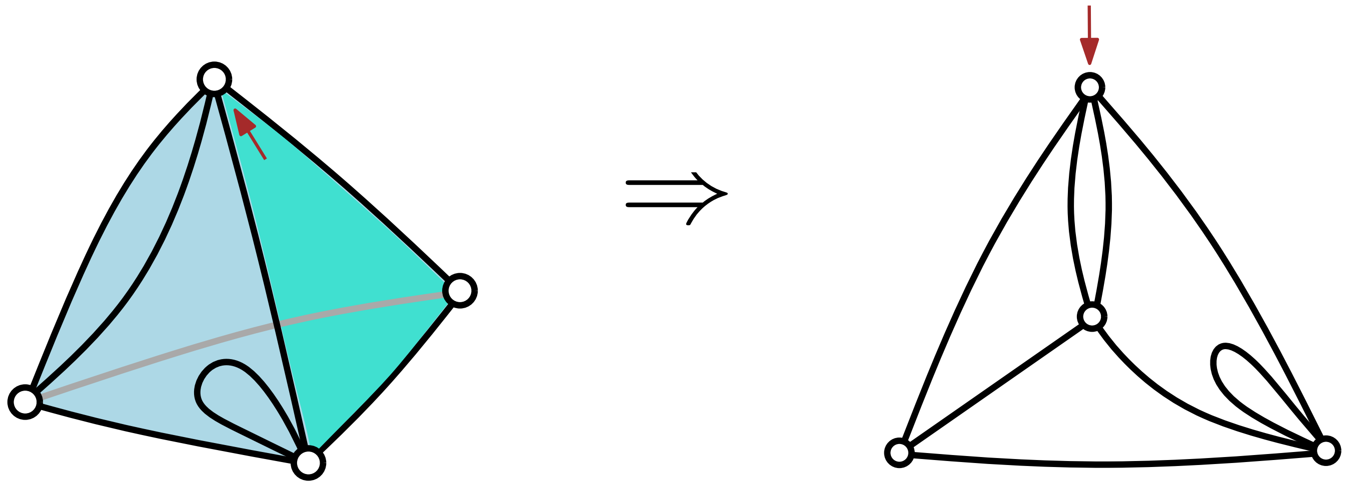
Easier to draw in the plane (choosing root-face to be the outer face)



Universality properties for planar maps

- Nice counting formulas for many natural families [Tutte'60s]

  e.g. $\#$ rooted maps n edges $= \dfrac{2}{n+2} 3^n \mathrm{Cat}_n$

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  $\&$ universal asymptotic behaviour for counting coefficients $c\,\gamma^n n^{-5/2}$

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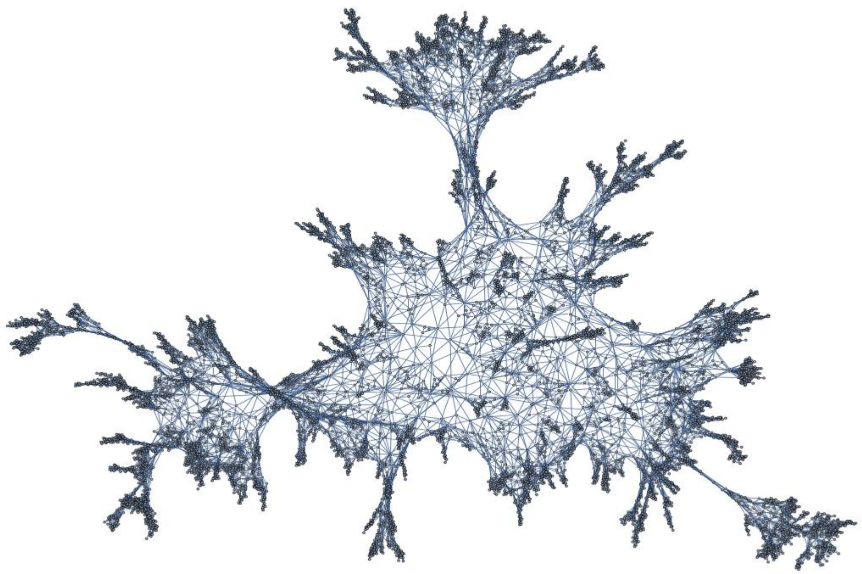
- Universal scaling limit (Brownian sphere) for random planar maps (rescaling distances by $n^{1/4}$)

  [Chassaing,Schaeffer'04]

  [Le Gall'13, Miermont'13]

  Link to Liouville Quantum Gravity [Miller-Sheffield'12]



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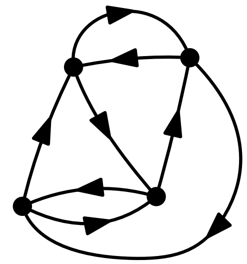
Decorated planar maps

Decorated planar map = planar map + structure

(Ising model, proper coloring, Potts model,
spanning tree, spanning forest, specific orientations,...)



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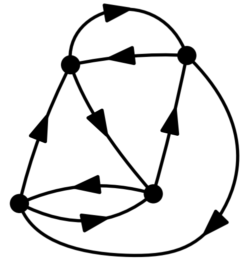

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- new bijections & counting behaviours compared to "pure" planar maps

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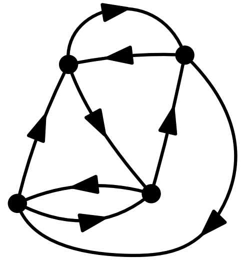

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link to "central charge"

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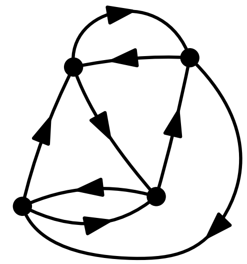

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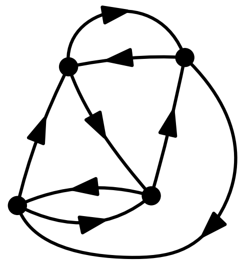

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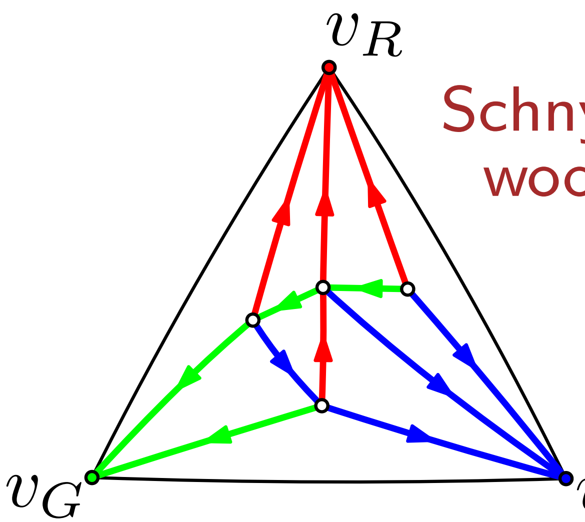
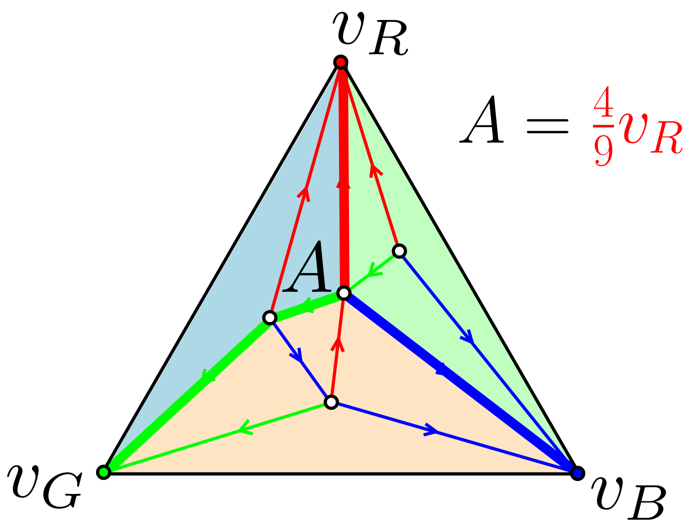
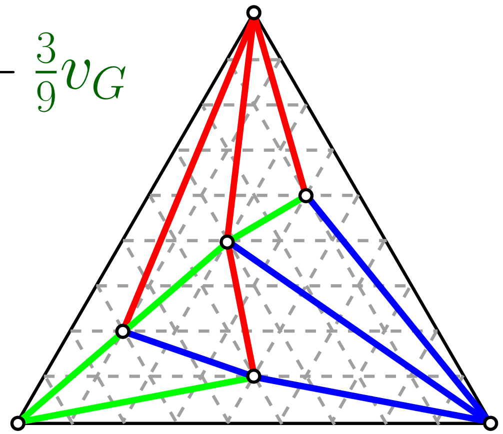
- Some of these structures give nice geometric representations of maps



Schnyder wood

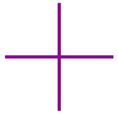


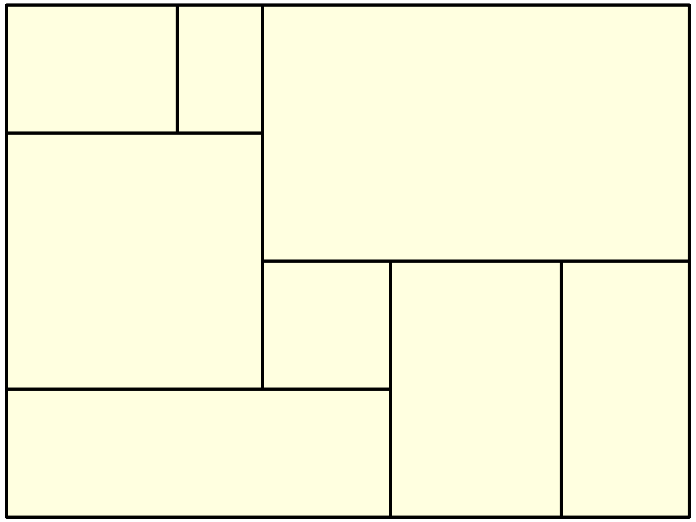
$A = \frac{4}{9}v_R + \frac{2}{9}v_B + \frac{3}{9}v_G$



Rectangulations

Rectangulation = tiling of a rectangle by rectangles

Called “generic” if no 



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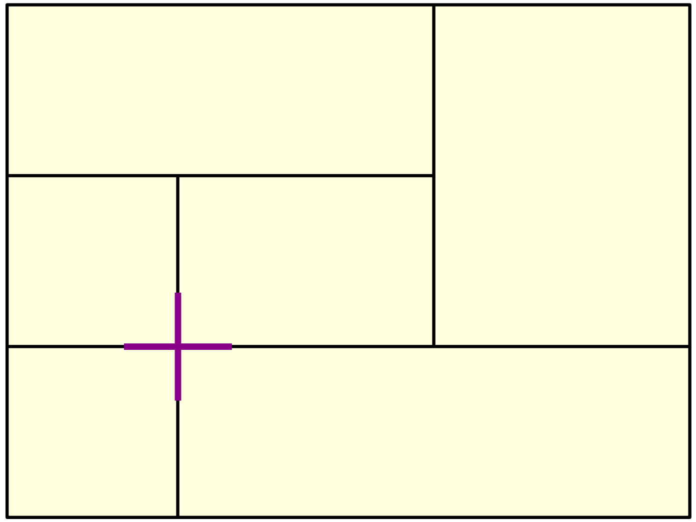

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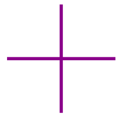
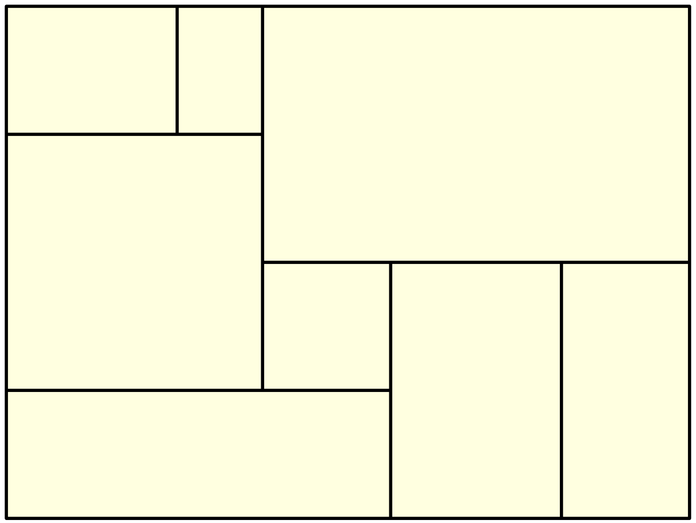
Called "generic" if no 



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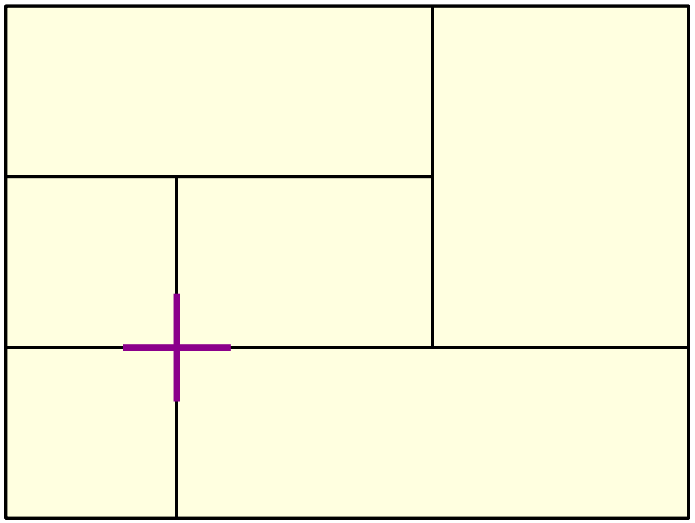
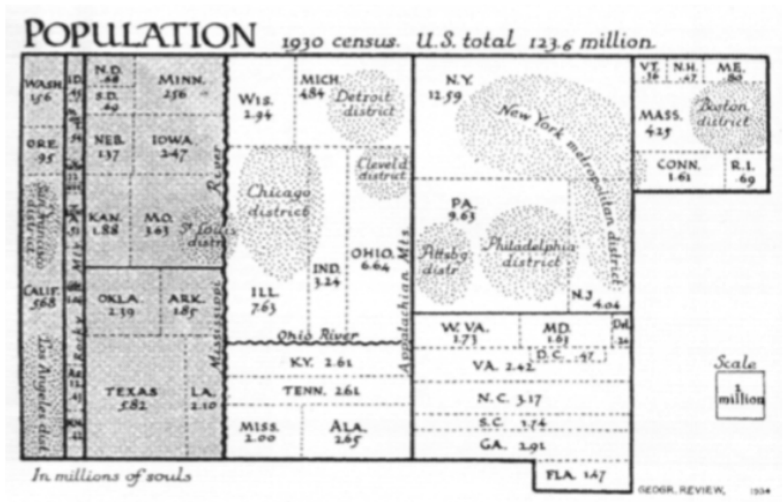


Not generic

used in "cartogram" representations

[van Kreveld-Speckmann'04]

[Eppstein-Mumford-Speckmann-Verbeek'12]



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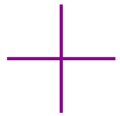
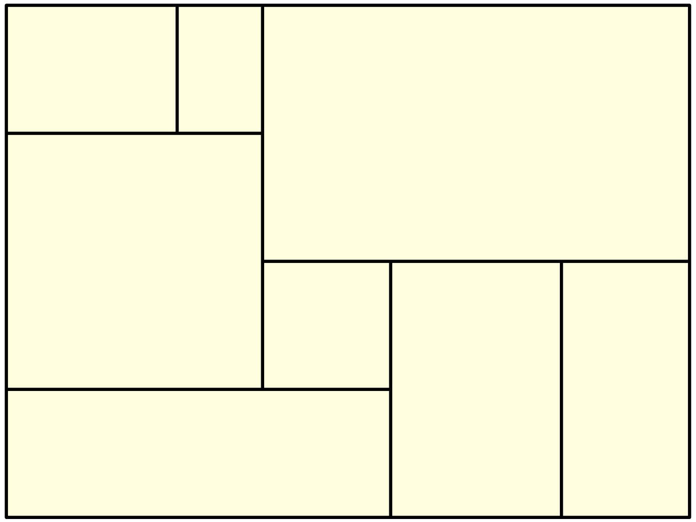
Called "generic" if no 



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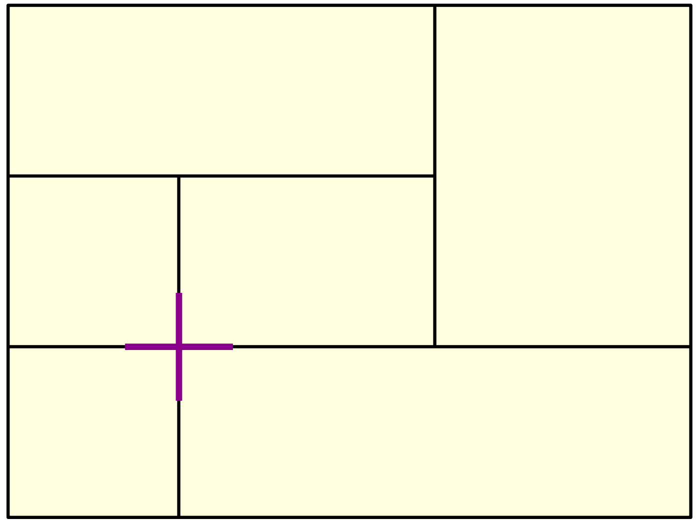


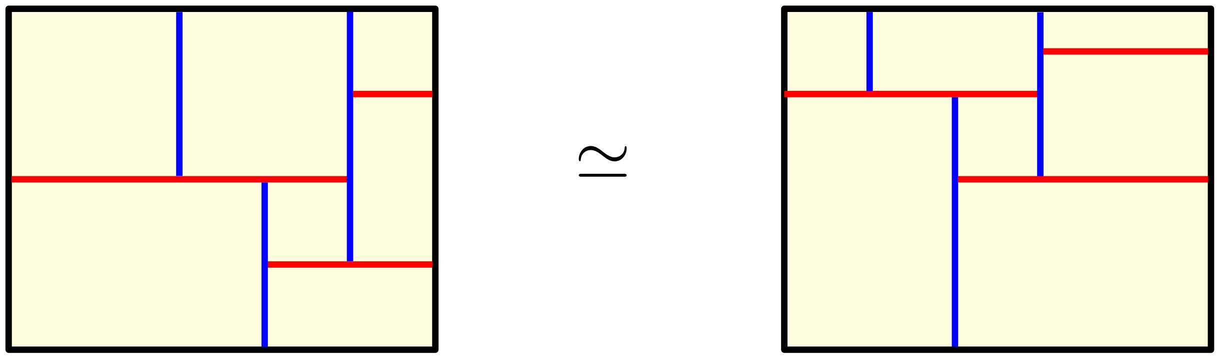
Not generic

This talk:

- Link to decorated planar maps & bijections to walks
- Exact enumeration
- Asymptotic enumeration

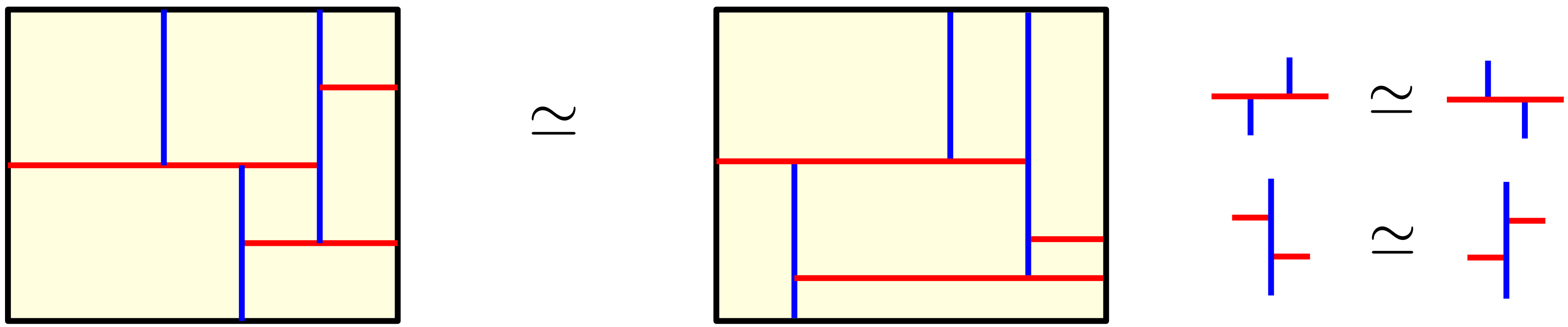
Two types of equivalences for rectangulations

Strong



(order of contacts along each maximal segment is preserved)

Weak



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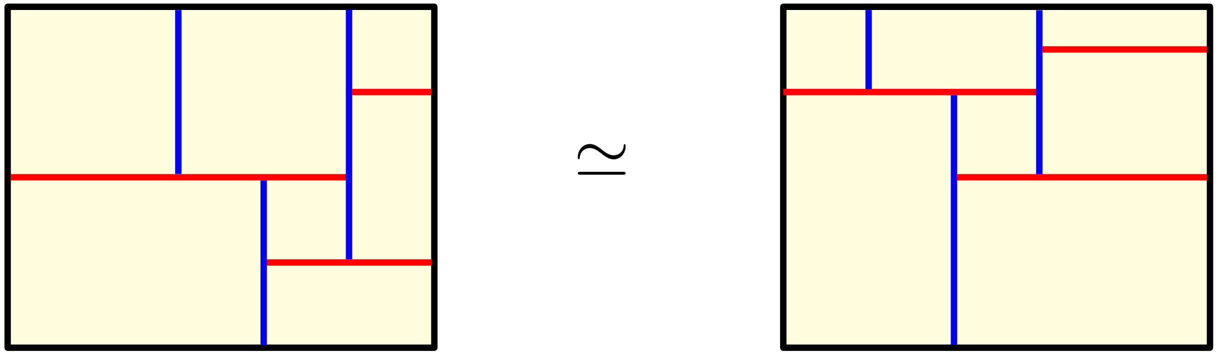

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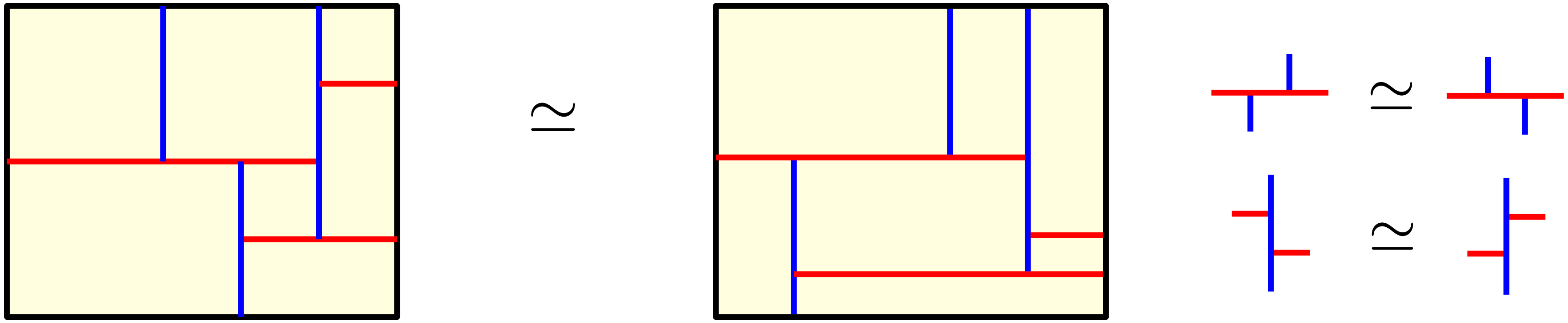


(order of contacts on each side of maximal segments is preserved)

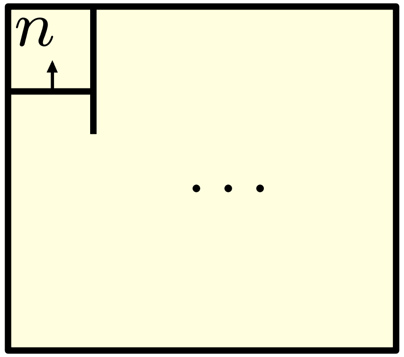
$w_n =$ # weak equivalence classes with n regions

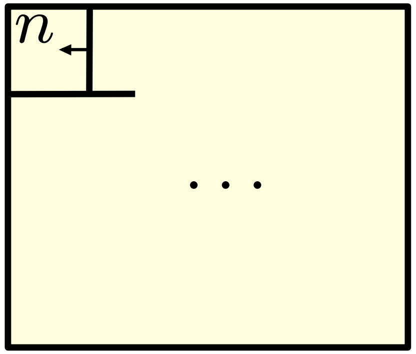
$s_n =$ # strong equivalence classes with n regions

Weak equivalence class: shelling order

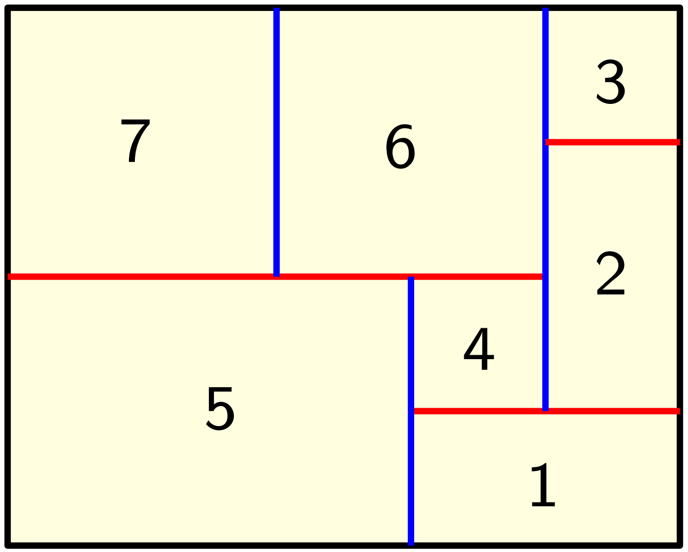
[Ackerman, Barequet, Pinter'06]

Contract top-left region:
two cases

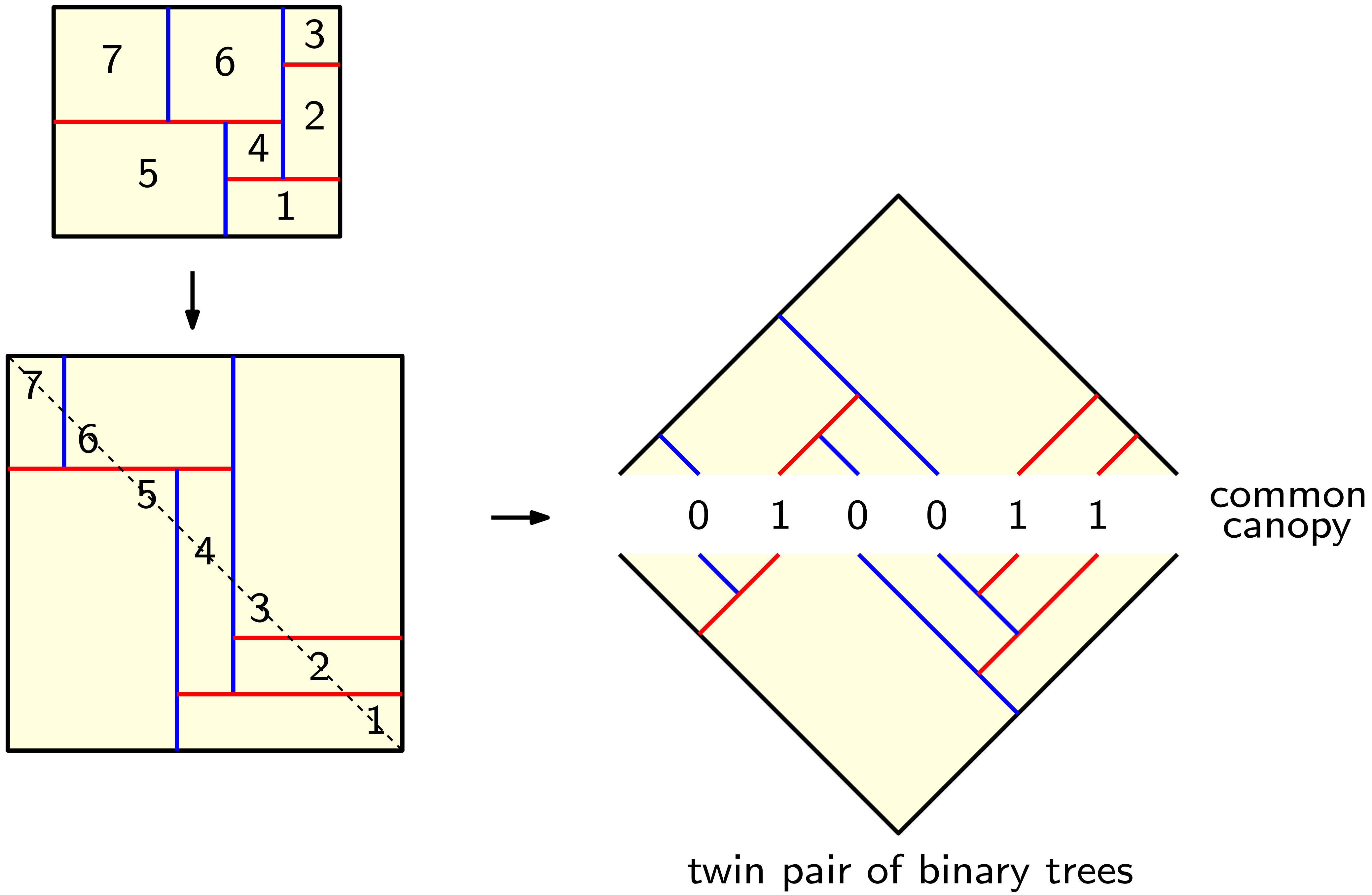




$\Rightarrow$ shelling order on regions

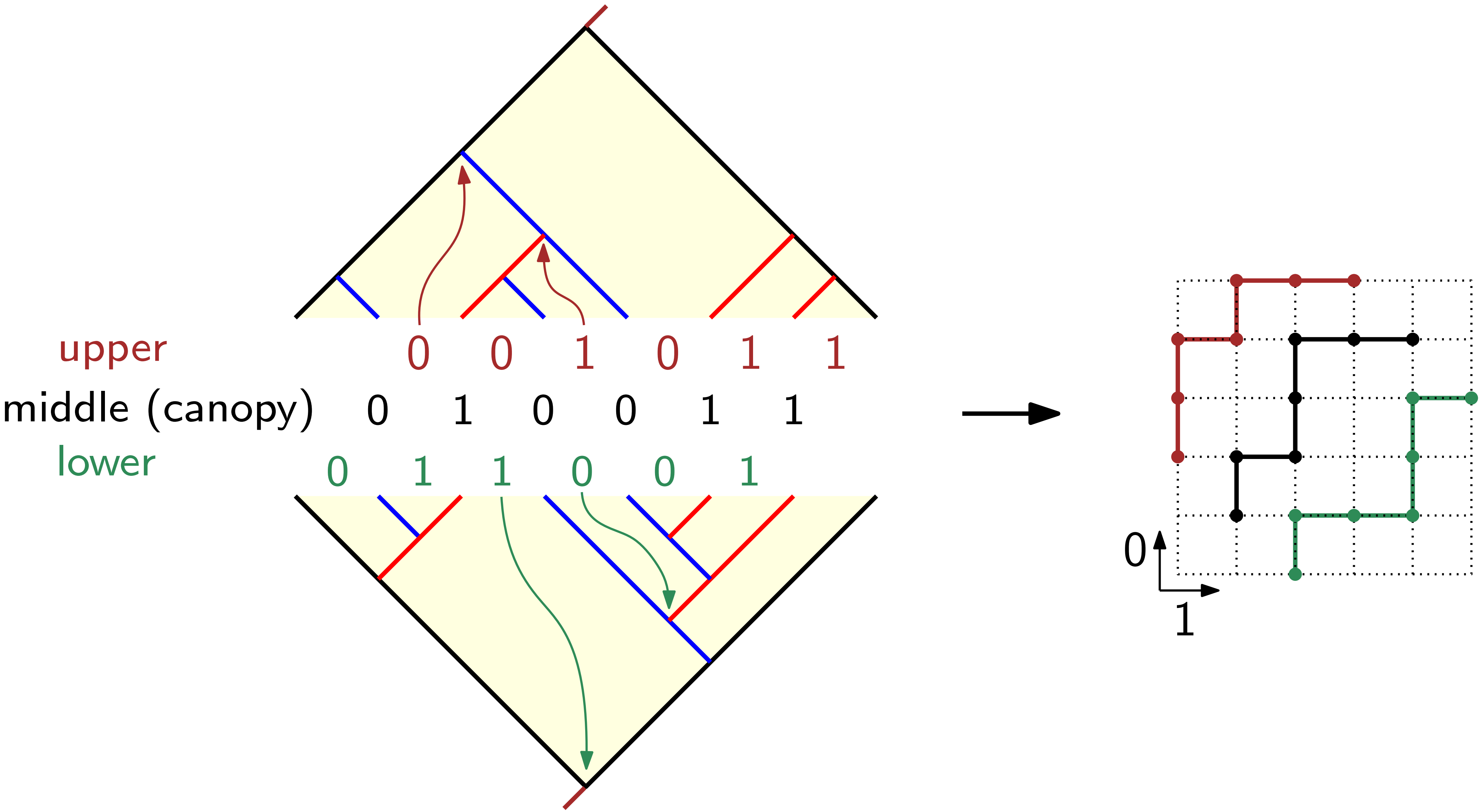


Diagonal representation

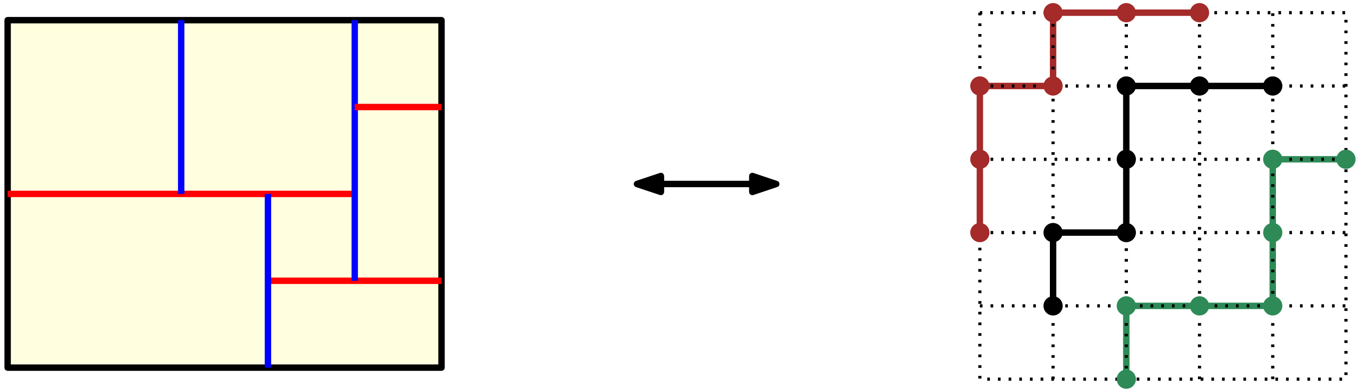


Encoding by a triple of walks

[Dulucq,Guibert'98]



Baxter numbers and Baxter families



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$$w_n \sim \frac{2^5}{\pi\sqrt{3}} 8^n n^{-4}$$

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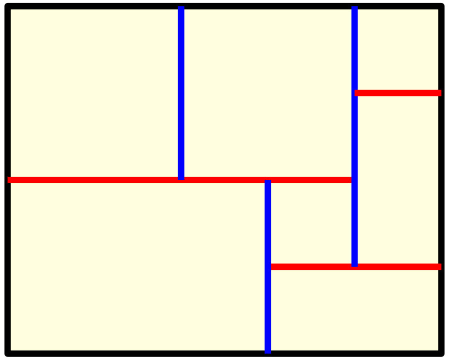

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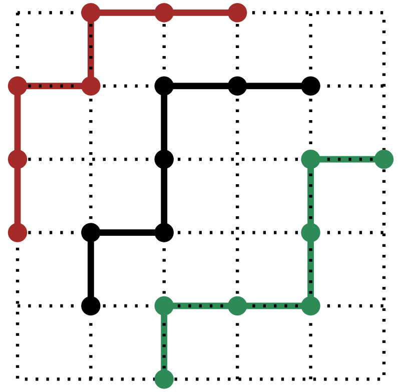

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Baxter families are families counted by Baxter numbers
among which Baxter permutations, plane bipolar orientations, ...

Various bijections relating these families (common generating tree)

[Dulucq-Guibert'98, Chow-Erikkson-Fan'05, Ackerman-Barequet-Pinter'06, F-Poulalhon-Schaeffer'07
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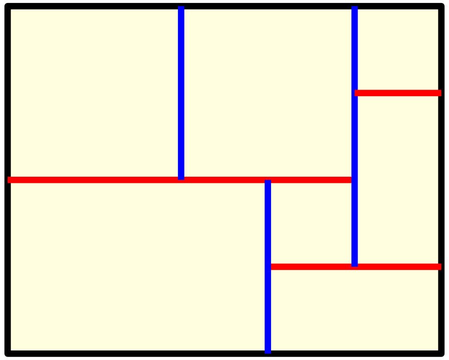
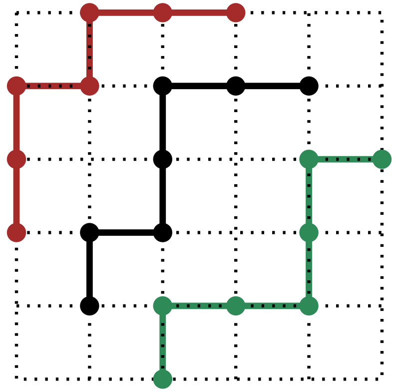


$\longleftrightarrow$



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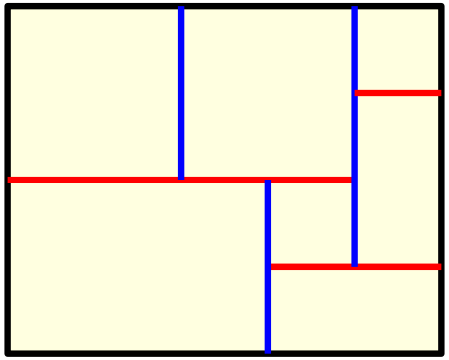

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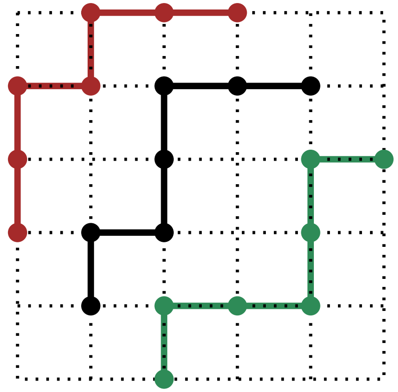

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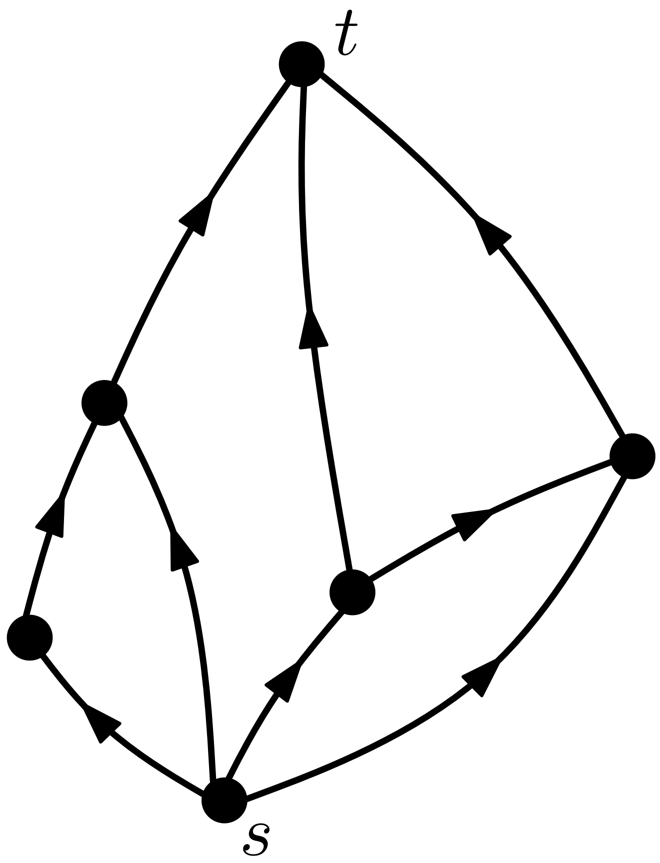
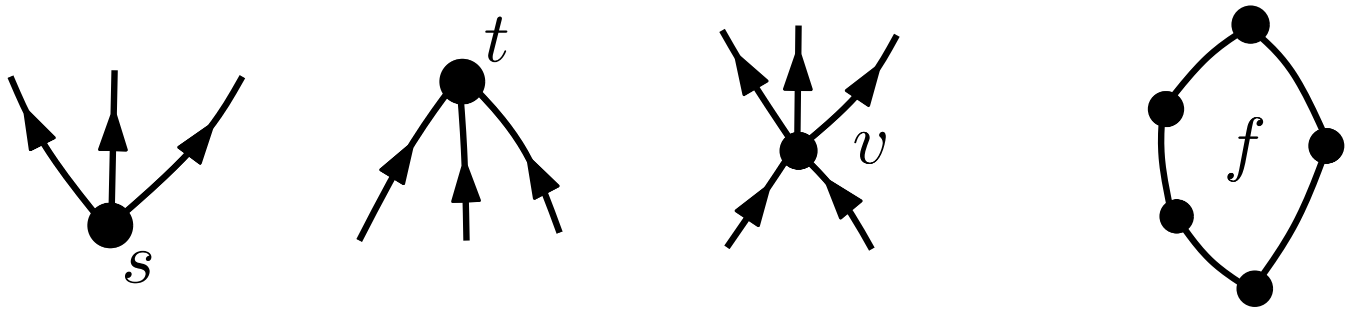
Link to weak order on permutations: [Reading'04,12]
mapping $\mathfrak{S}_n \to \mathcal{R}_n$
grouping permutations by rectangulation gives a lattice congruence

Plane bipolar orientations

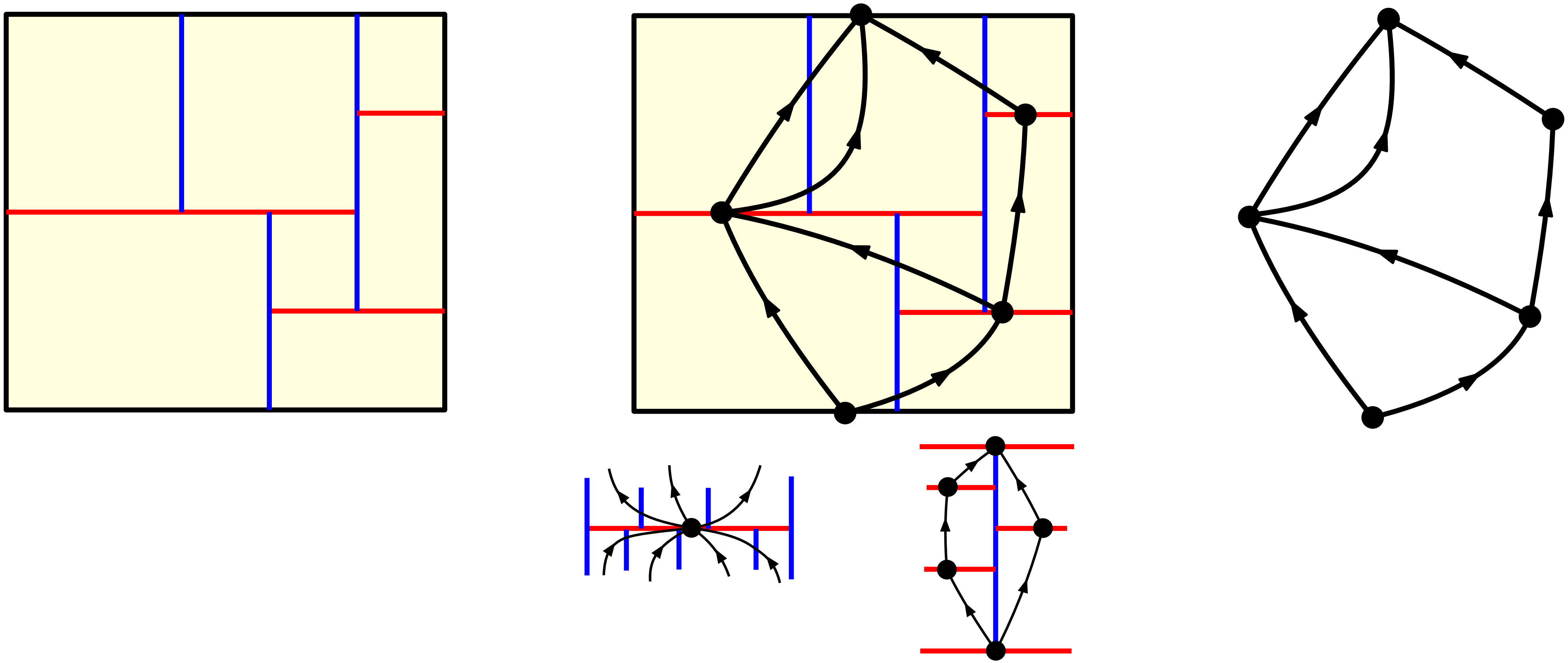


Acyclic orientation on planar map
with single min and single max
both incident to the outer face

Plane bipolar orientations $\Leftrightarrow$ local conditions



Bijective link with weak rectangulations



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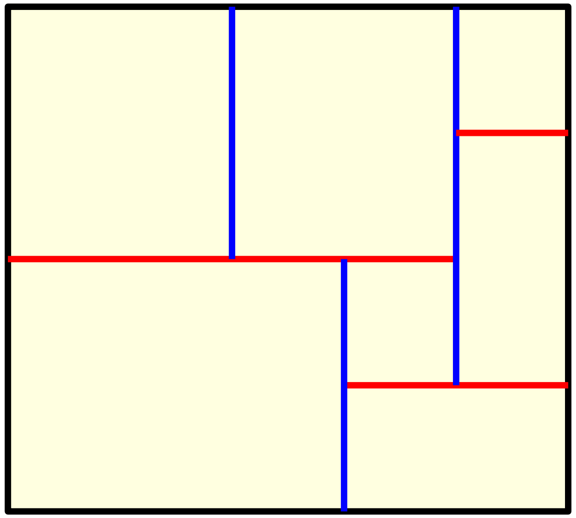
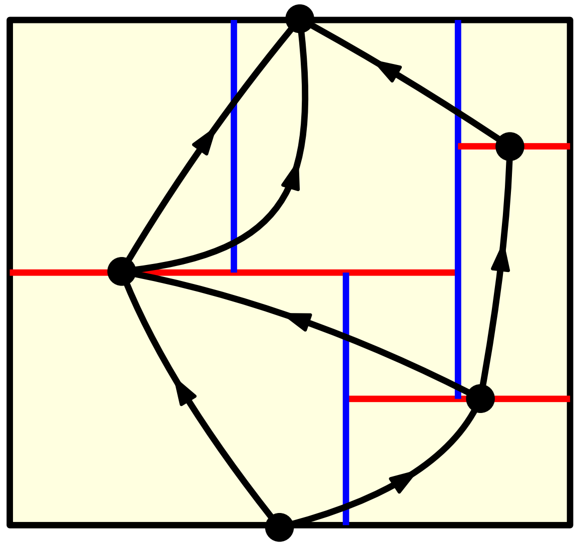
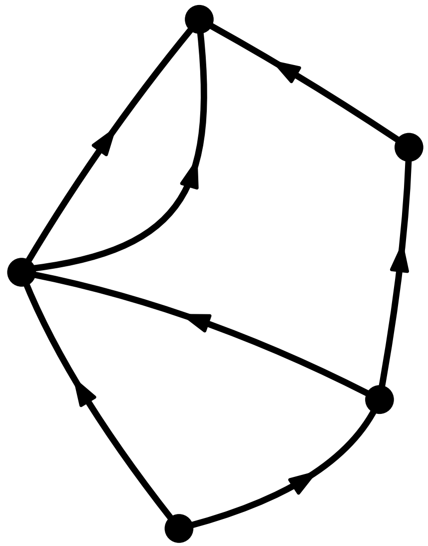
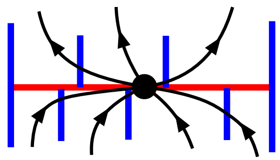
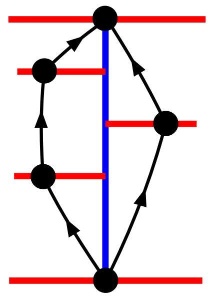




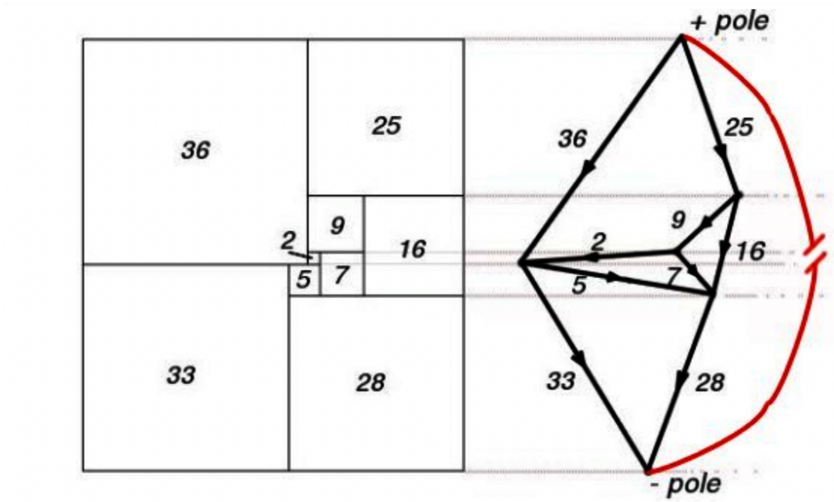






Correspondence used in
problem "squaring the square"

[Brooks, Smith, Stone, Tutte'40]

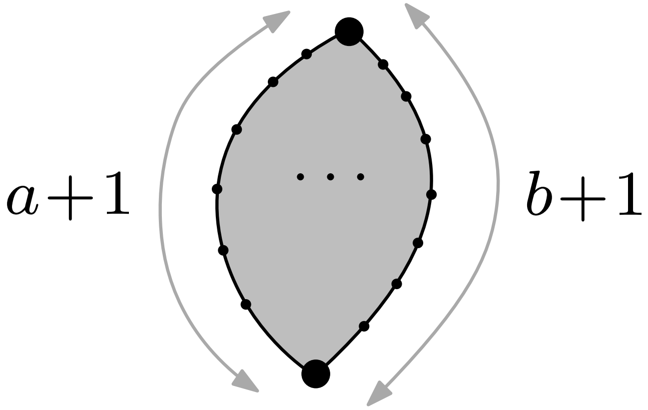


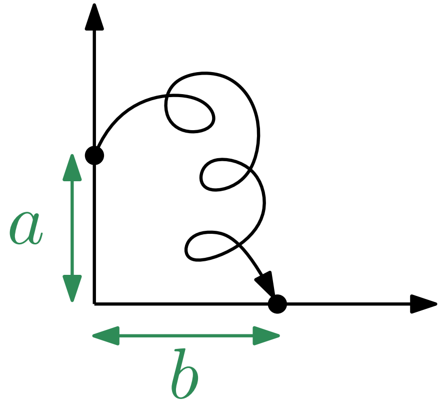
(www.squaring.net/history_theory/gfx/figure73.jpg)

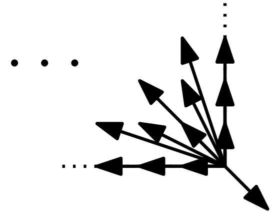
A more precise walk-encoding: the KMSW bijection

[Kenyon, Miller, Sheffield, Wilson'15]

Plane bipolar orientations ⟷ "Tandem walks" in the quadrant



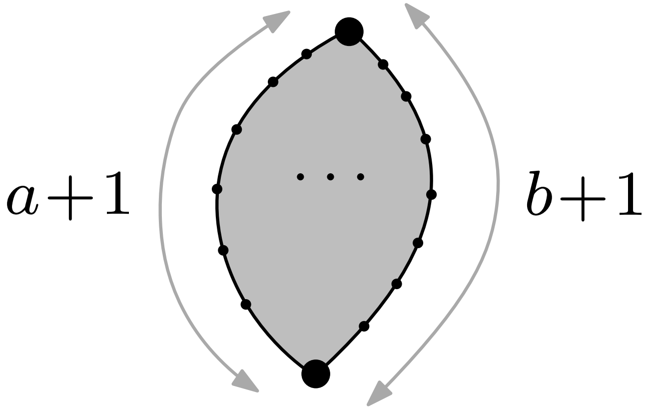


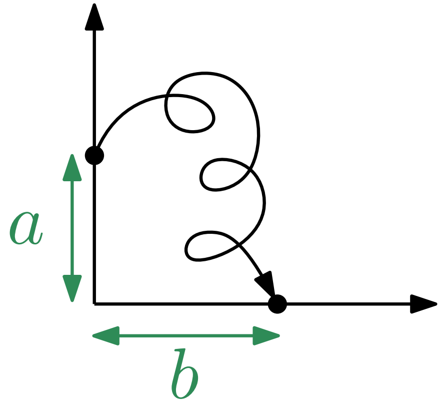


A more precise walk-encoding: the KMSW bijection

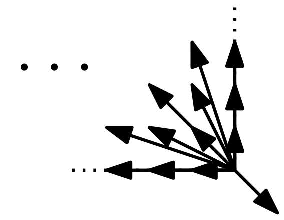
[Kenyon, Miller, Sheffield, Wilson'15]

Plane bipolar orientations ⟷ "Tandem walks" in the quadrant



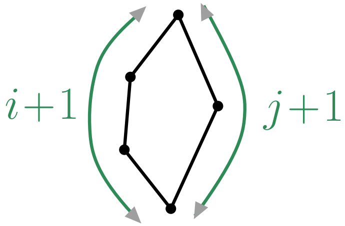


step-set



SE $\cup \{(-i,j), i,j \geq 0\}$

n edges ⟷ length $n-1$

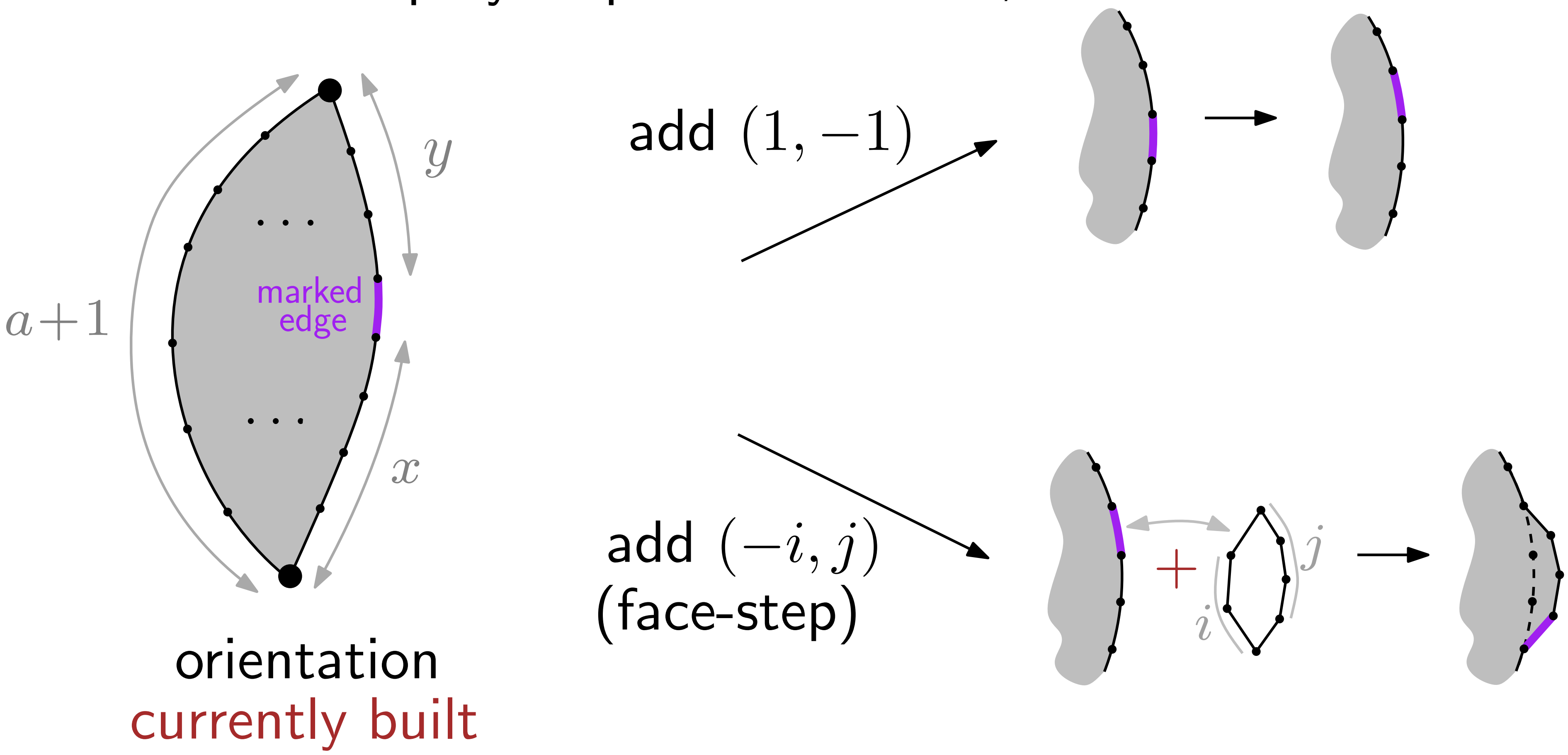
face 

⟷ face-step $(-i,j)$

non-pole vertex ⟷ SE step

The KMSW bijection

Orientation is built step by step from the walk,



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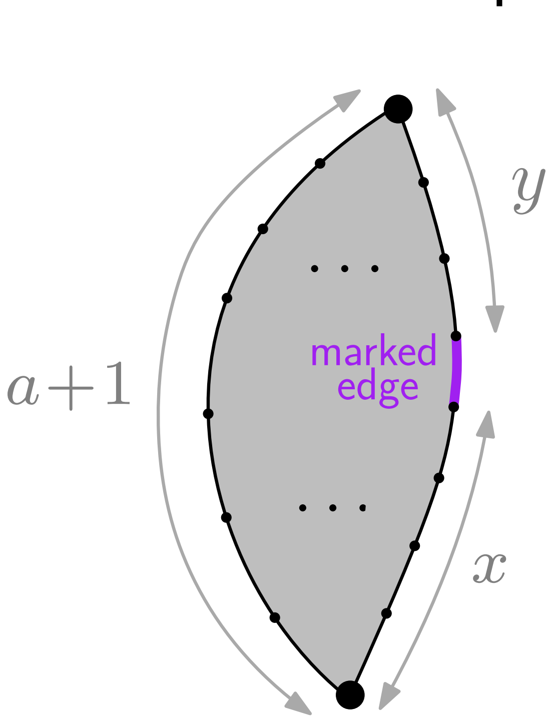
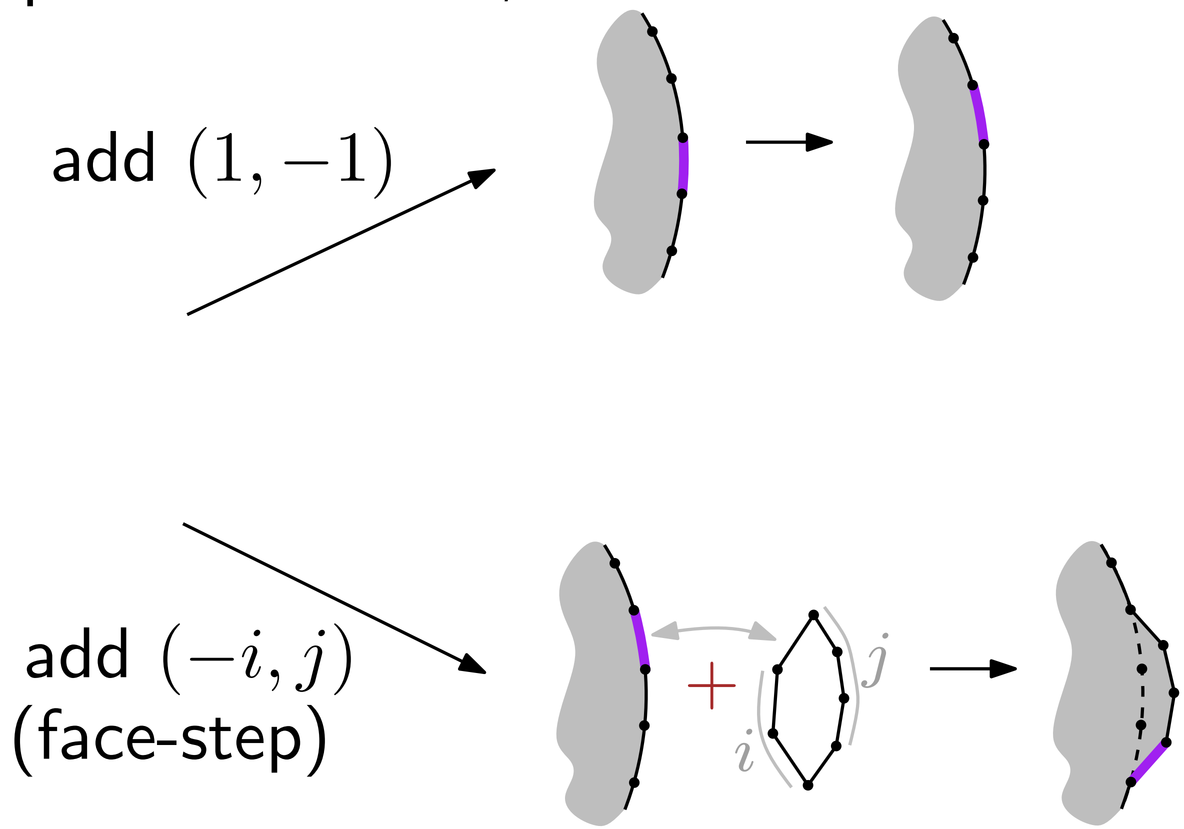
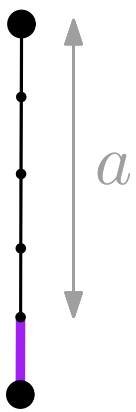


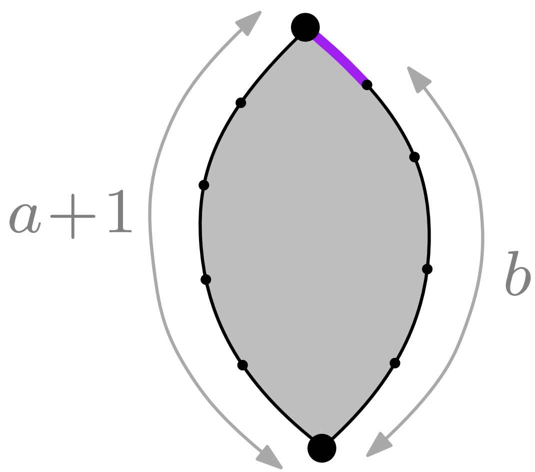
orientation
currently built



Starts with

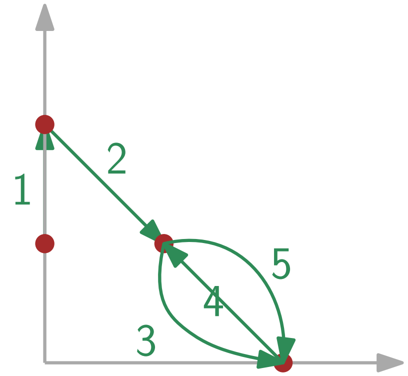


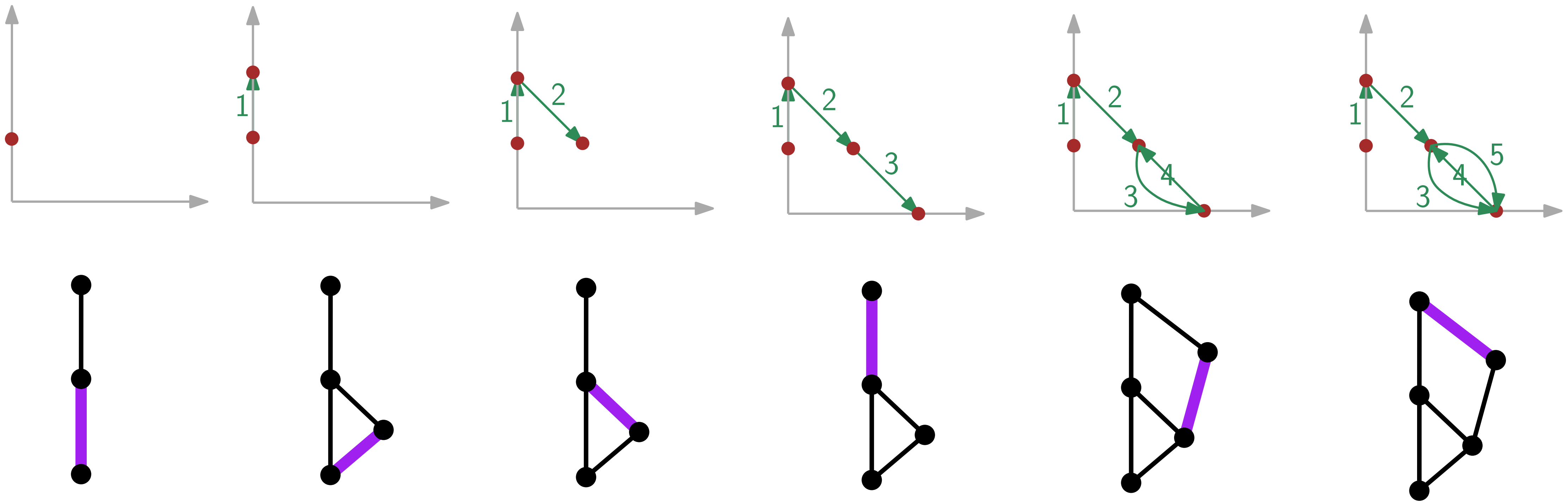
Ends with



The KMSW bijection

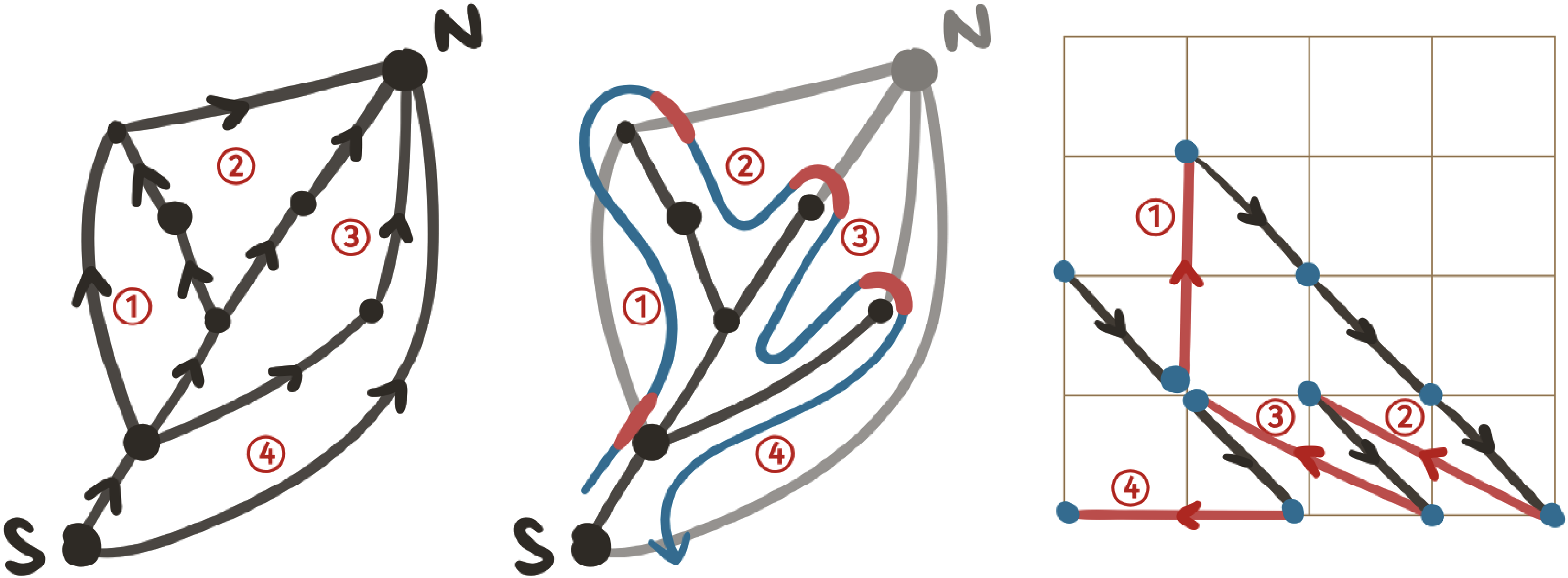
Example: build orientation associated to





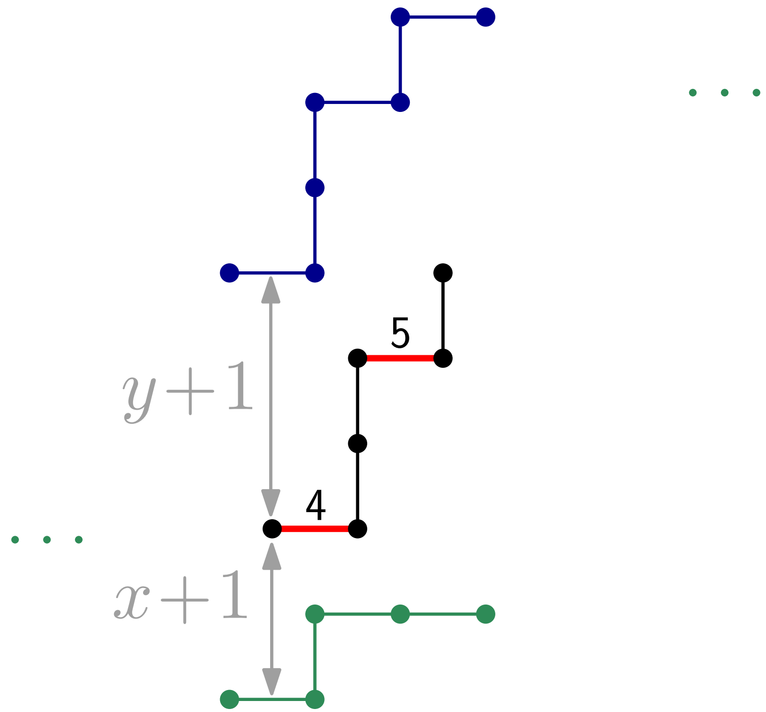
The KMSW bijection

From bipolar orientation to tandem walk

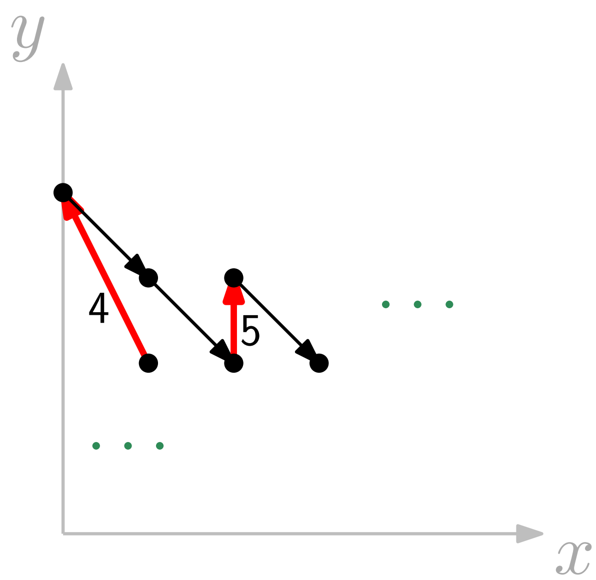


Link with non-intersecting triples of walks

[Bousquet-Mélou,F,Raschel'20]

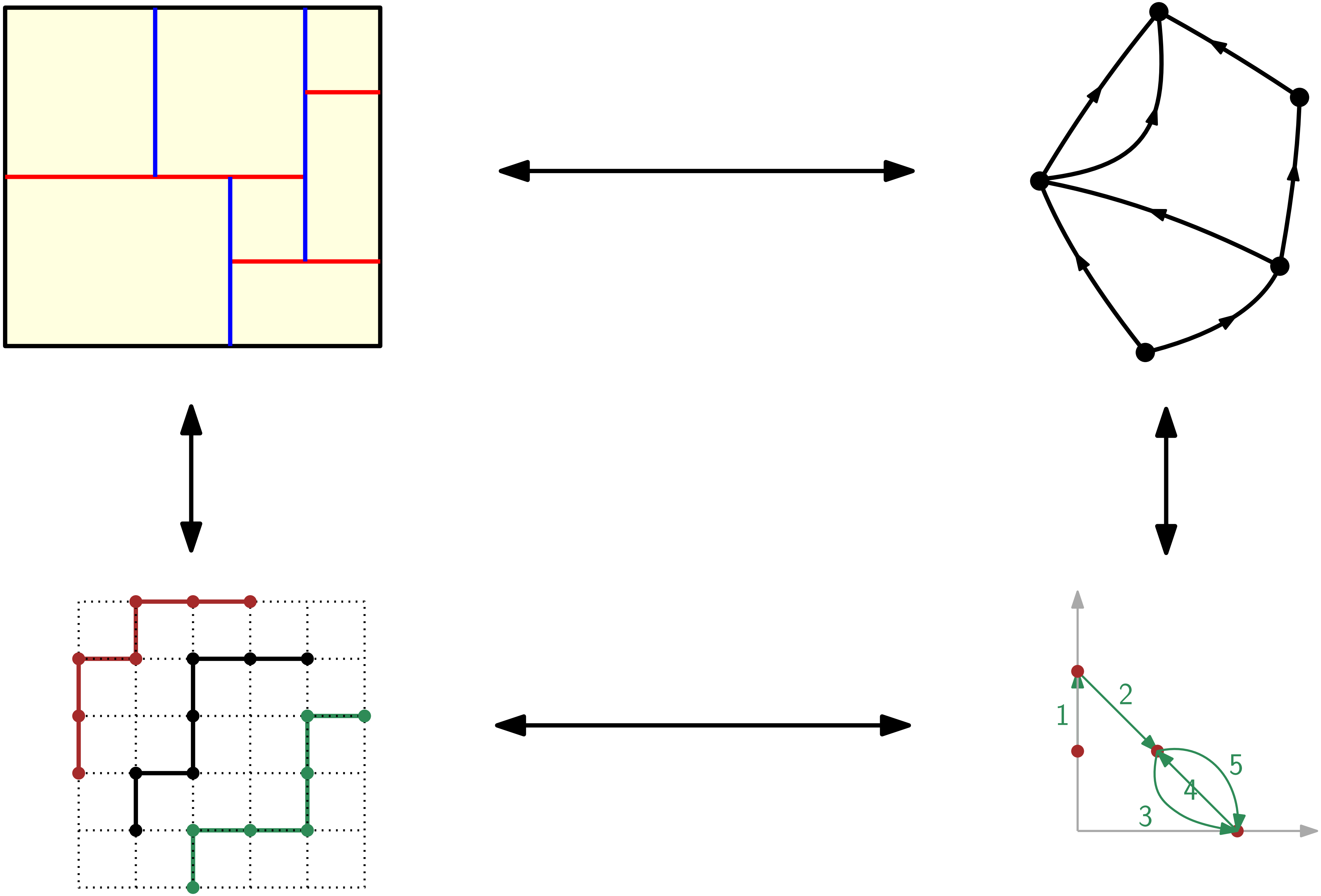


non-intersecting triple



tandem walk

Summary of bijections so far



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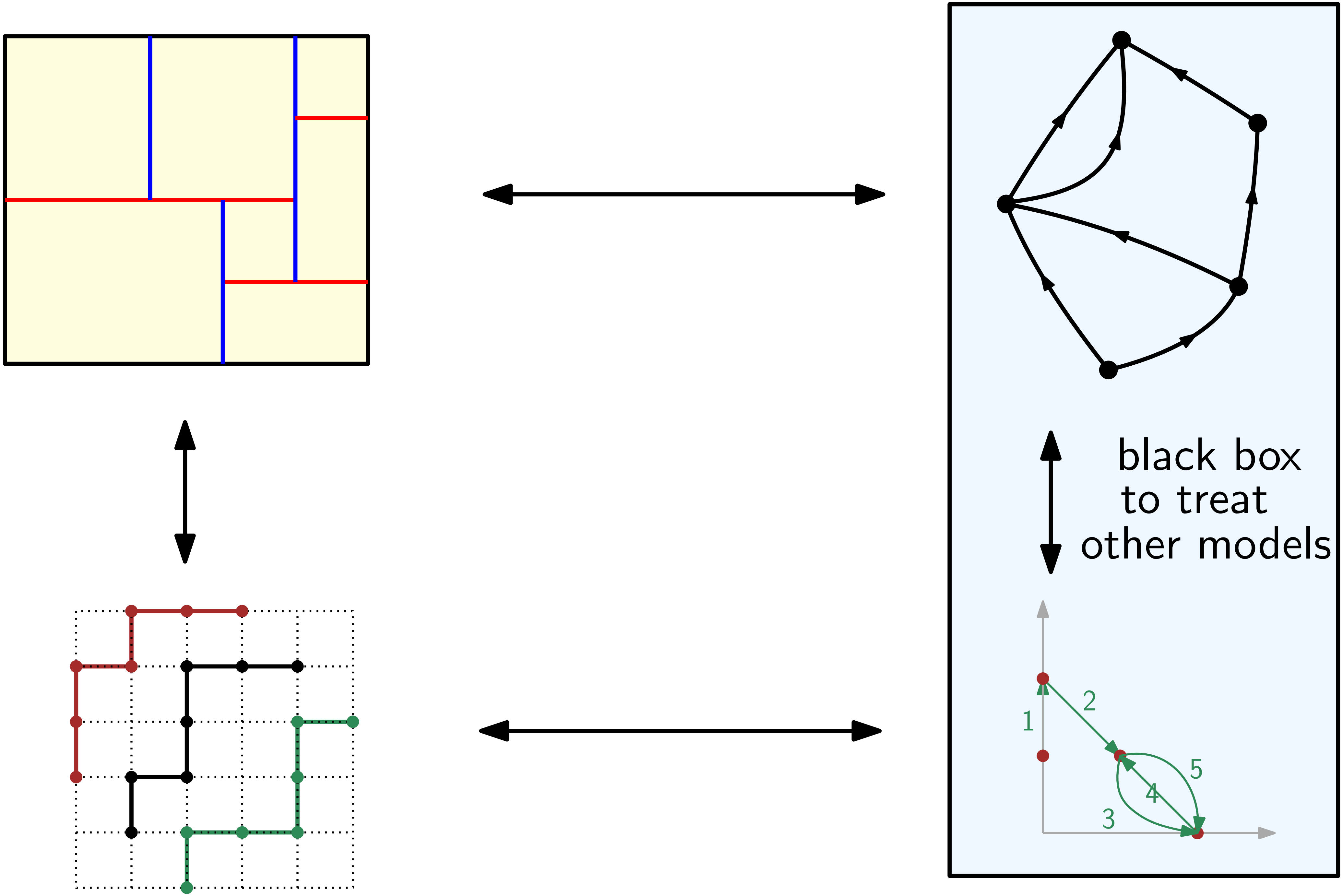
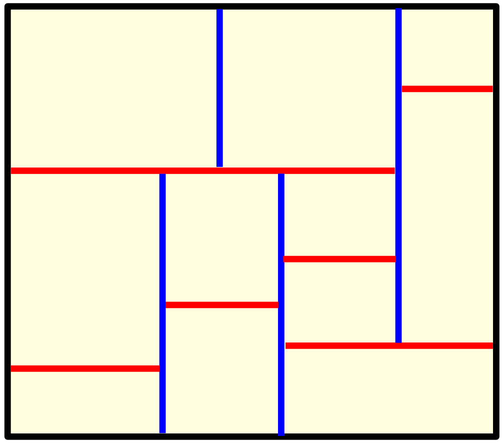
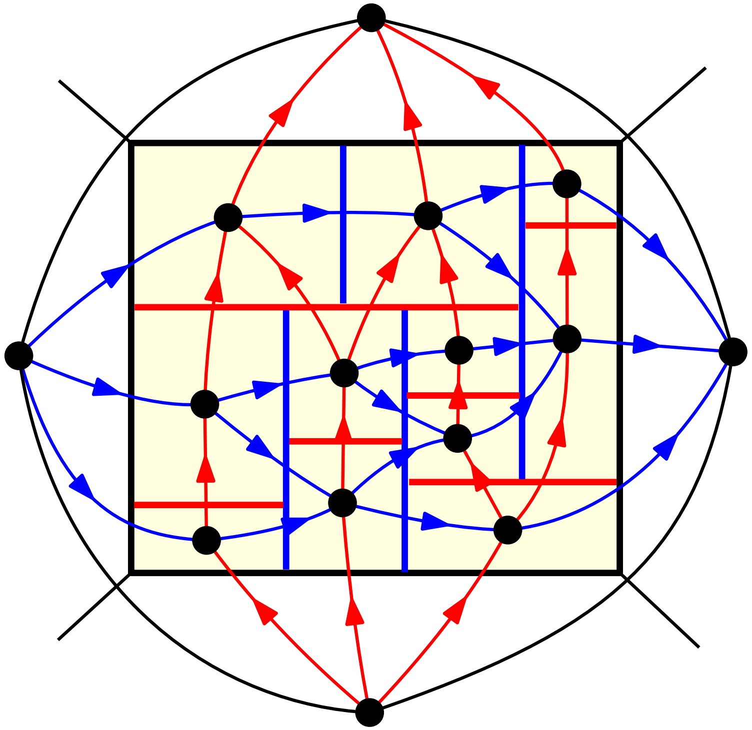
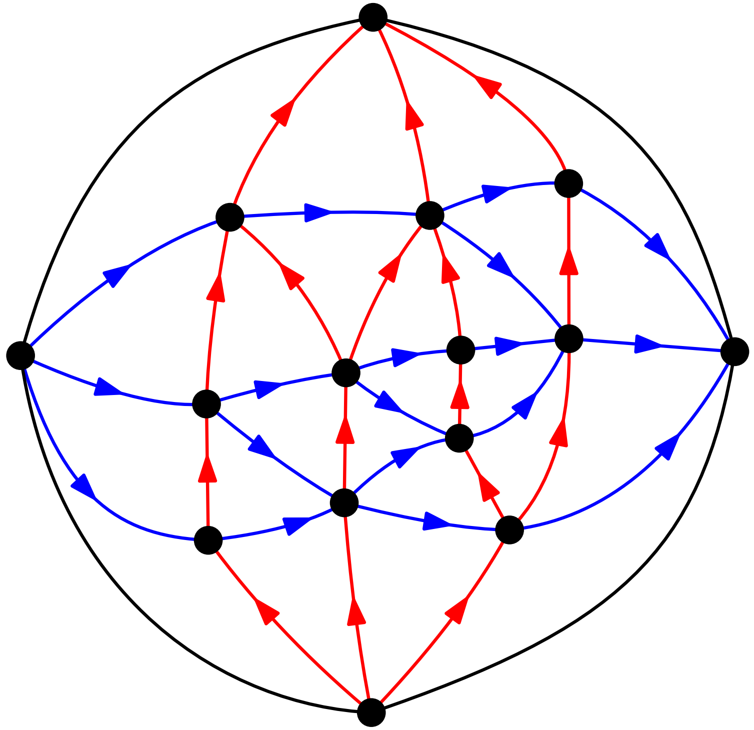


Strong rectangulations

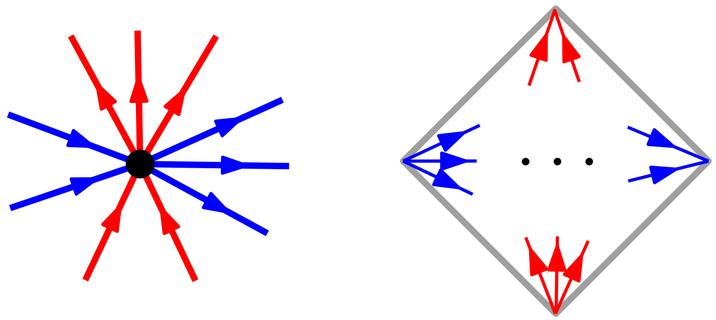
[He'93]







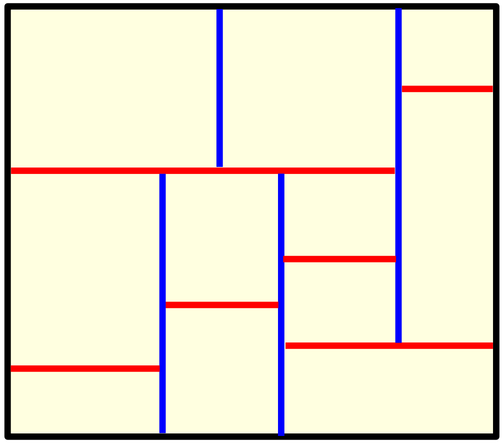
Pair of transversal
plane bipolar orientations

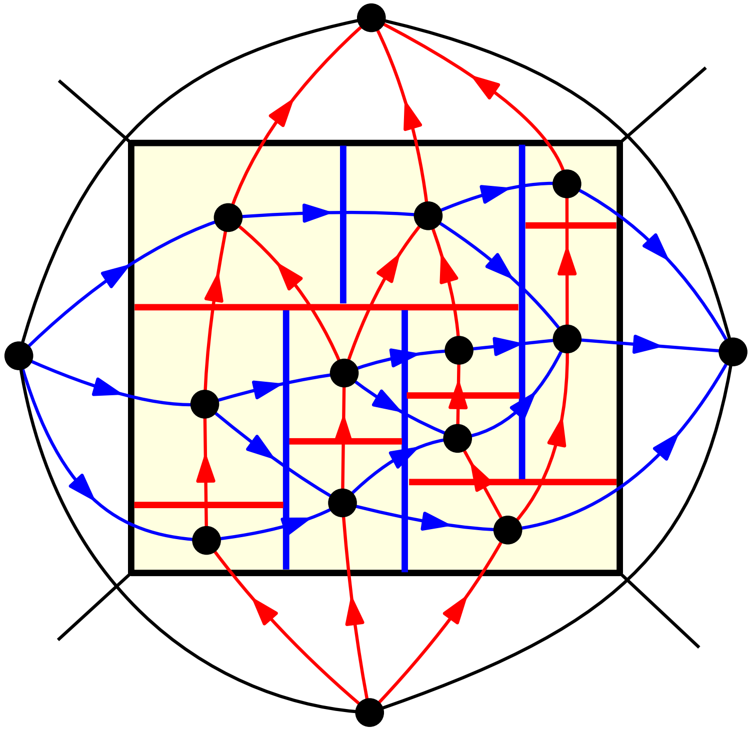


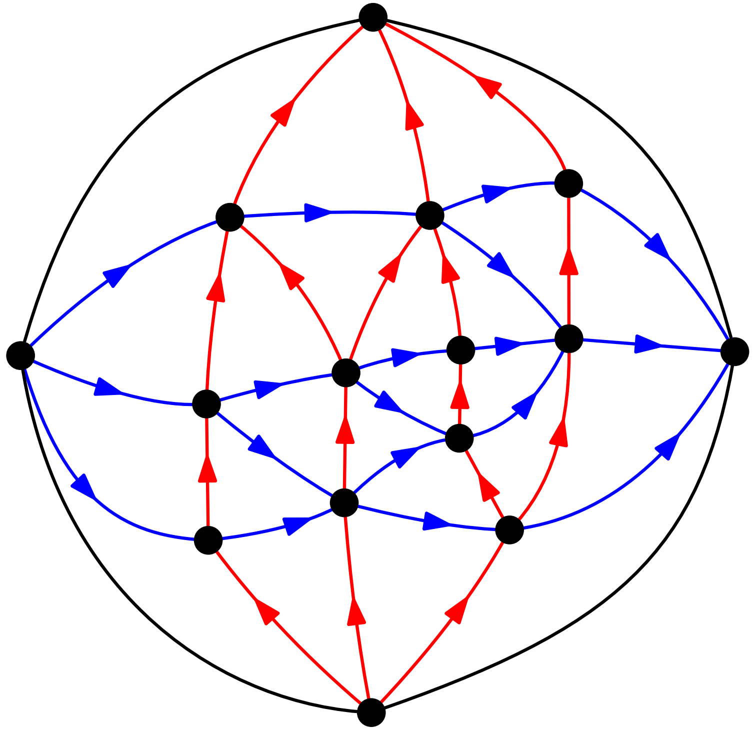
Local conditions

Strong rectangulations

[He'93]



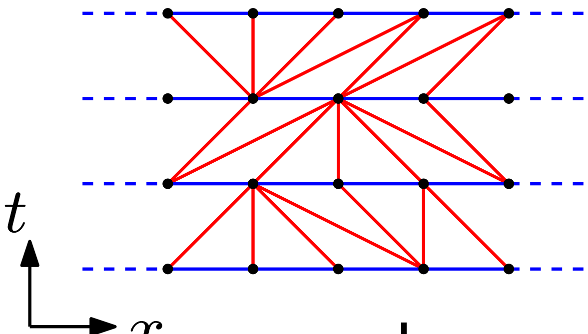




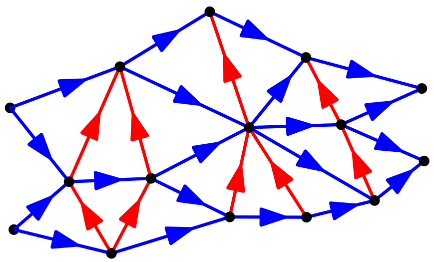
Pair of transversal
plane bipolar orientations

cf locally causal dynamical triangulations

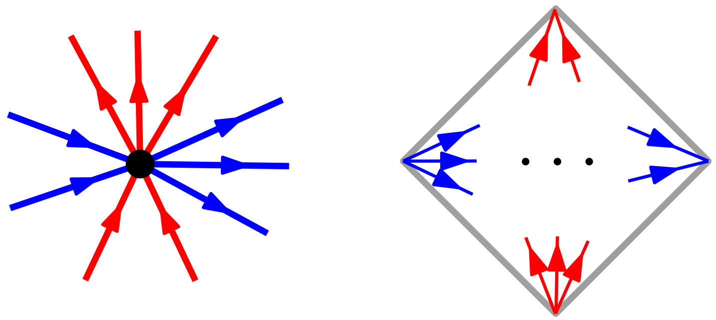
[Loll, Ruijl'15]



causal

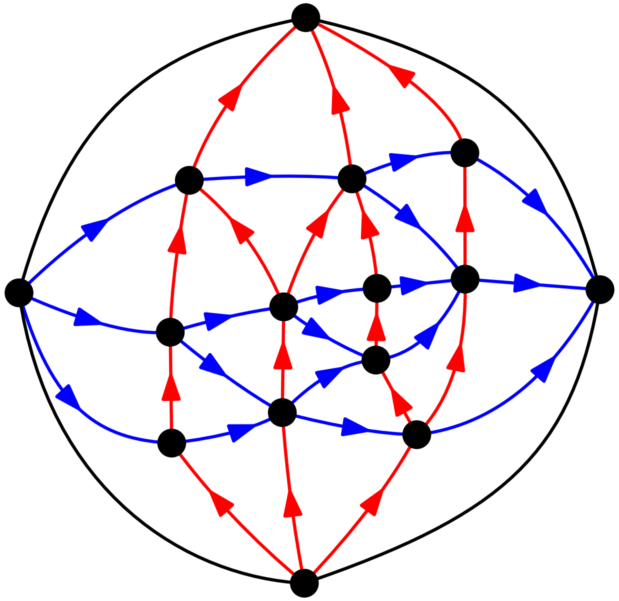


locally causal



Local conditions

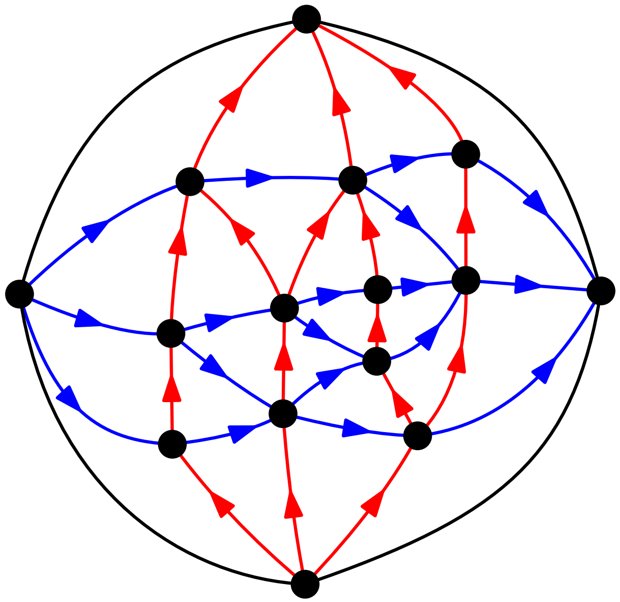
Encoding by (weighted) tandem walks



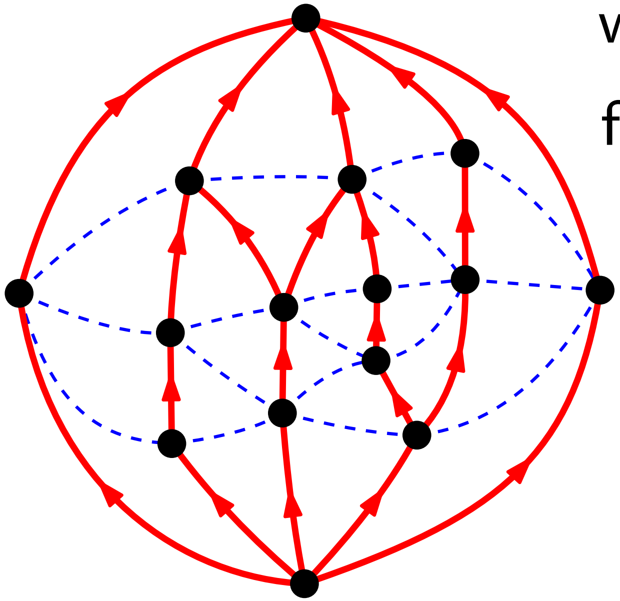
Transversal structure
$n + 4$ vertices

Encoding by (weighted) tandem walks

[F-Narmanli-Schaeffer'21]



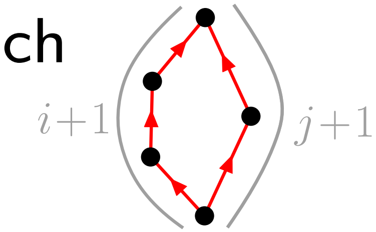
Transversal structure
$n+4$ vertices

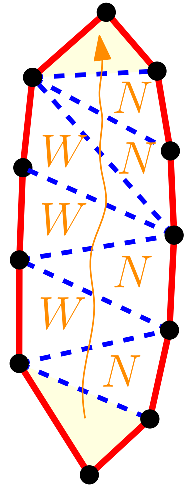


red bipolar poset
$+$ transversal edges

weight $\binom{i+j-2}{i-1}$

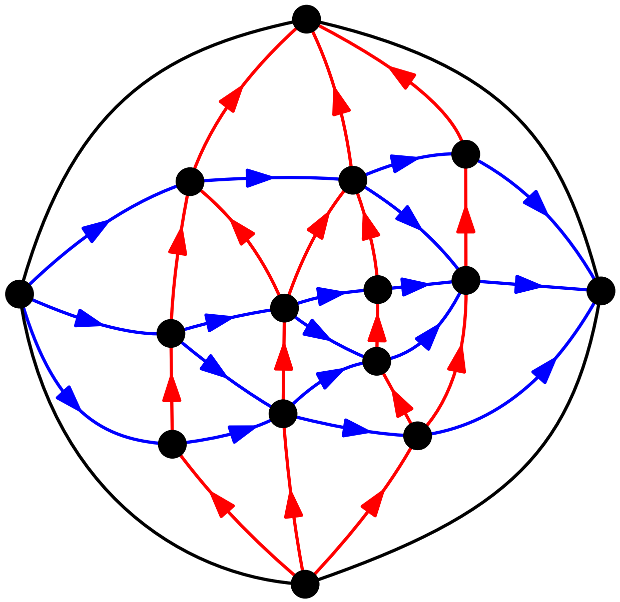
for each



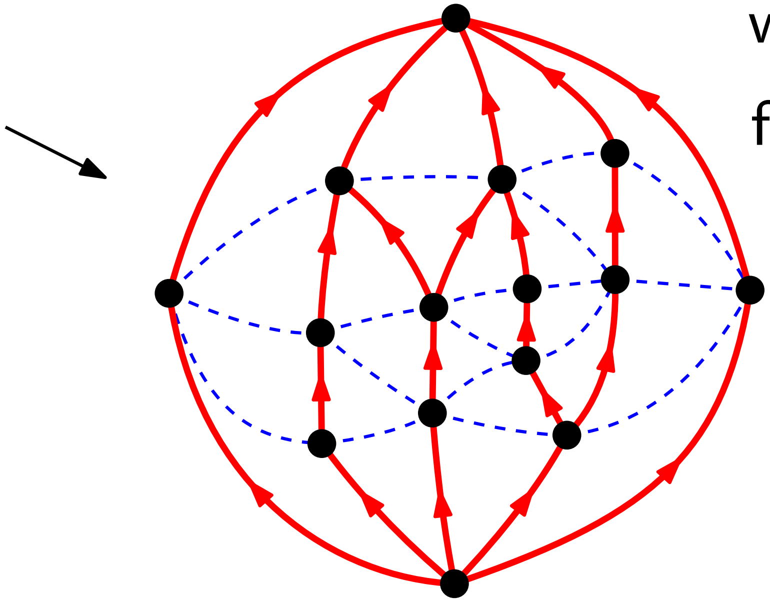


Encoding by (weighted) tandem walks

[F-Narmanli-Schaeffer'21]

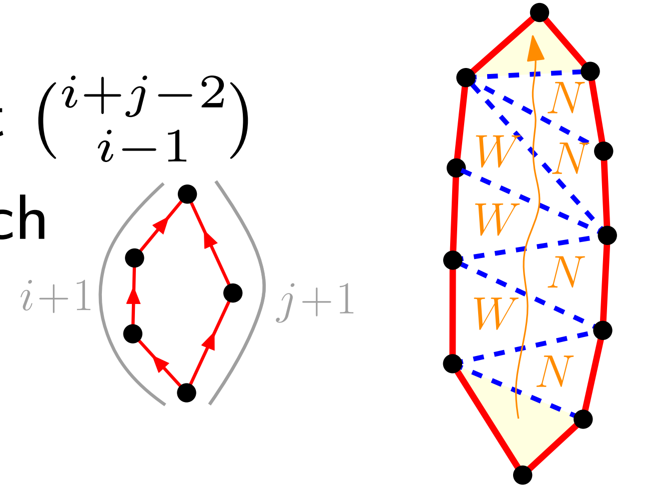


Transversal structure
$n + 4$ vertices

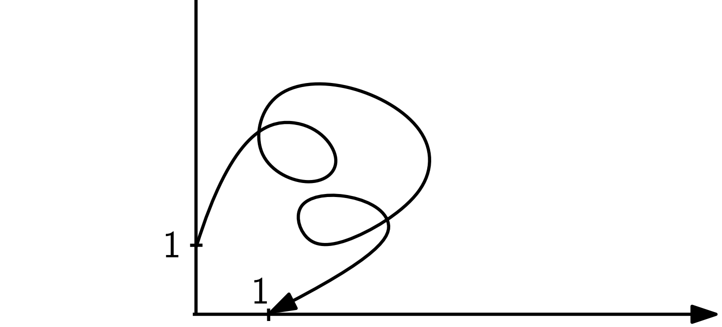


red bipolar poset
$+$ transversal edges

weight $\binom{i+j-2}{i-1}$
for each



weight $\binom{i+j-2}{i-1}$ for
each step $(-i, j)$

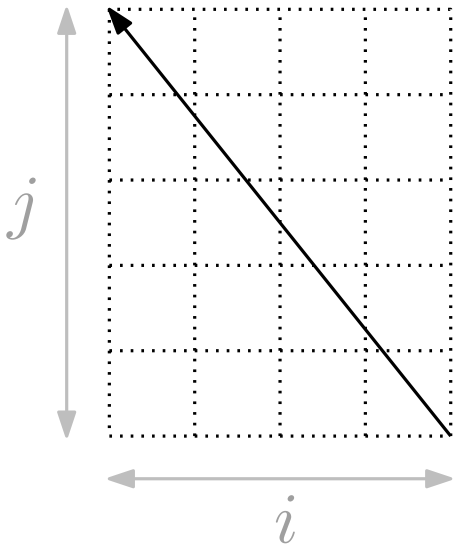


weighted tandem walk
with n SE steps

Encoding by tandem walks with small steps

[F-Narmanli-Schaeffer'21]

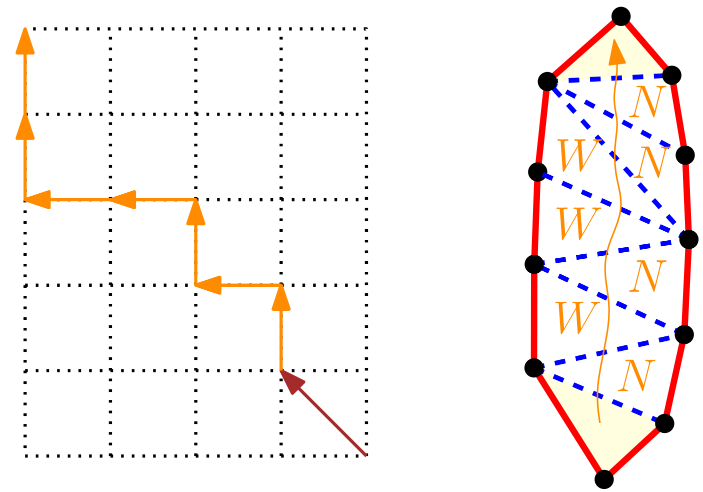
face-step



weight $\binom{i+j-2}{i-1}$

$\Leftrightarrow$

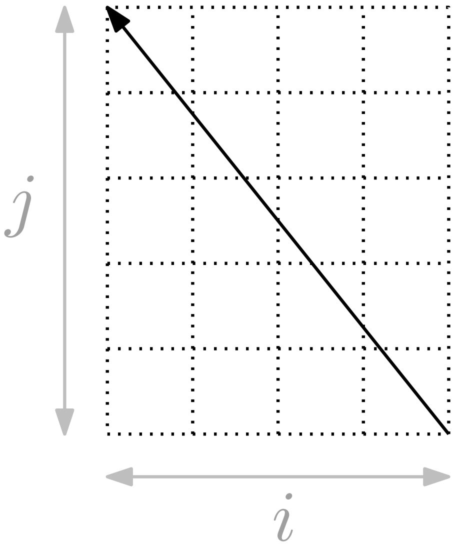
decoration by small-step portion



Encoding by tandem walks with small steps

[F-Narmanli-Schaeffer'21]

face-step



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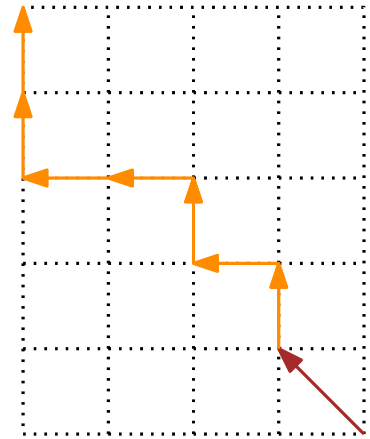
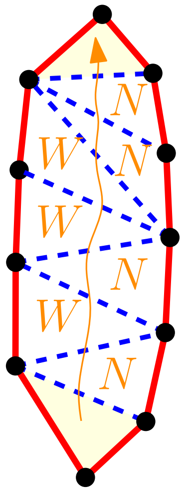
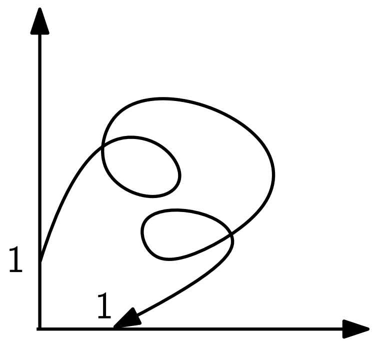
$\Leftrightarrow$

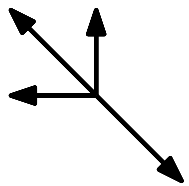
decoration by small-step portion

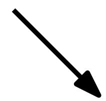




$s_n = \#$ walks



on step-set 

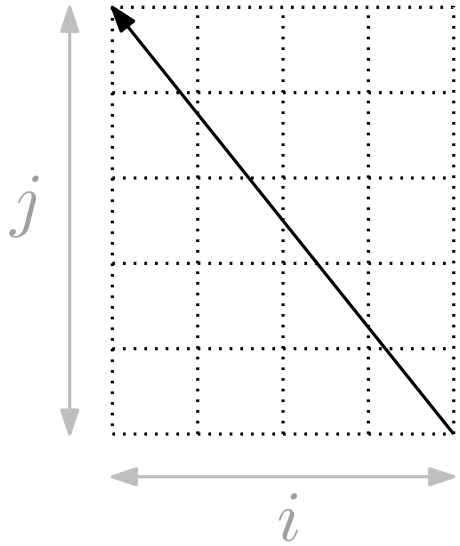
$n-2$ steps 

with no patterns 

Encoding by tandem walks with small steps

[F-Narmanli-Schaeffer'21]

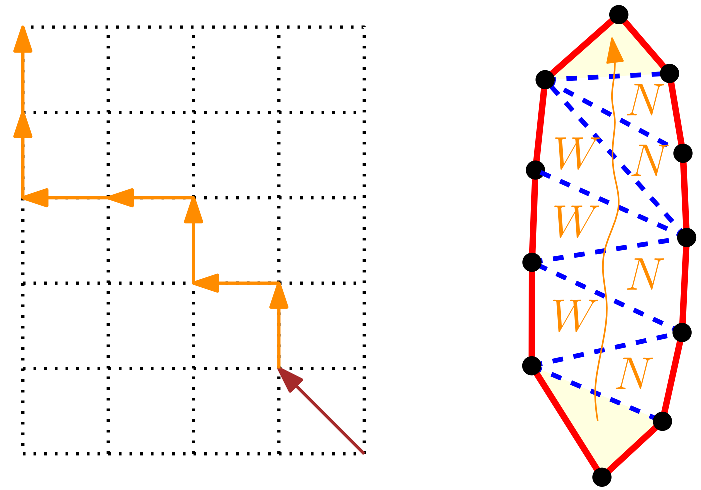
face-step



weight $\binom{i+j-2}{i-1}$

$\Leftrightarrow$

decoration by small-step portion



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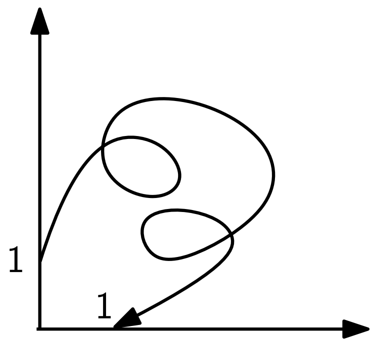
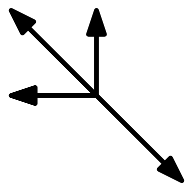


on step-set



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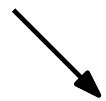


with no patterns

$\Rightarrow$ explicit recurrence

$$\sum_n s_n t^n = t + 2t^2 + 6t^3 + 24t^4 + 116t^5 + 642t^6 + 3938t^7 + \cdots$$

other recurrence (& small step walks) [Inoue, Takahashi, Fujimaki'09]

Asymptotic enumeration

[F-Narmanli-Schaeffer'21]

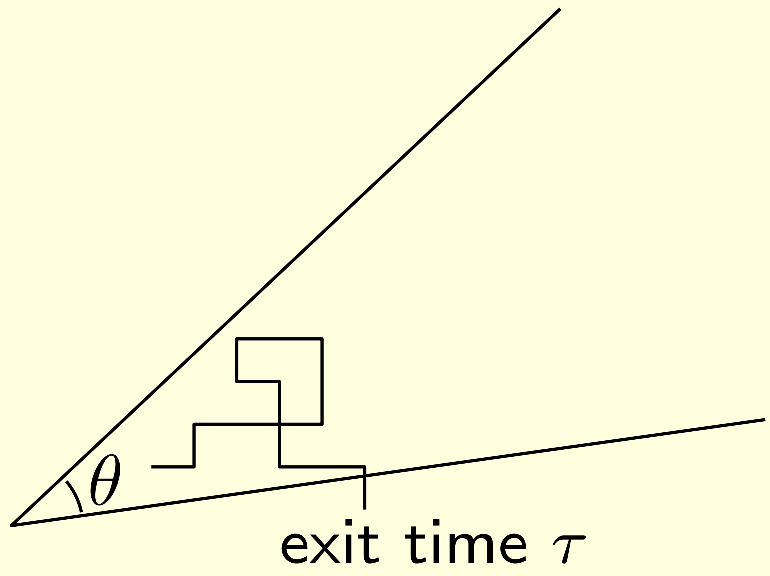
relies on [Denisov-Wachtel'11, Bostan-Raschel-Salvy'14]

Each of the counting sequences w_n, s_n has asymptotics of the form

$$c\ \gamma^n\ n^{-\alpha}$$

$\alpha = 1 + \frac{\pi}{\theta}$

	weak	strong
γ	8	$27/2$
$\cos(\theta)$	$1/2$	$7/8$
α	4	$\approx 7.21 \notin \mathbb{Q}$



$$\mathbb{P}(\tau > n) \sim c\ n^{-\frac{\pi}{2\theta}}$$

$$\mathbb{P}(\tau > n \ \&\ \text{excursion}) \sim c'\ n^{-1-\frac{\pi}{\theta}}$$

Asymptotic enumeration

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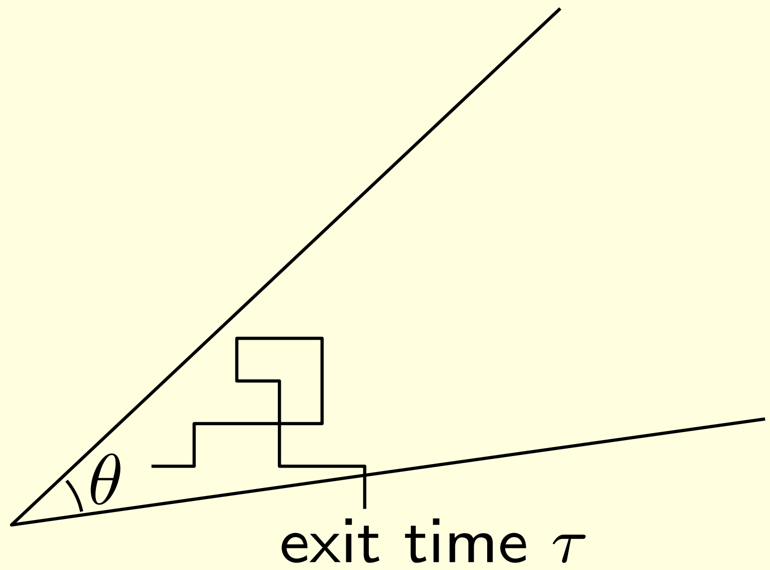
$\notin \mathbb{Q}$ → not D-finite



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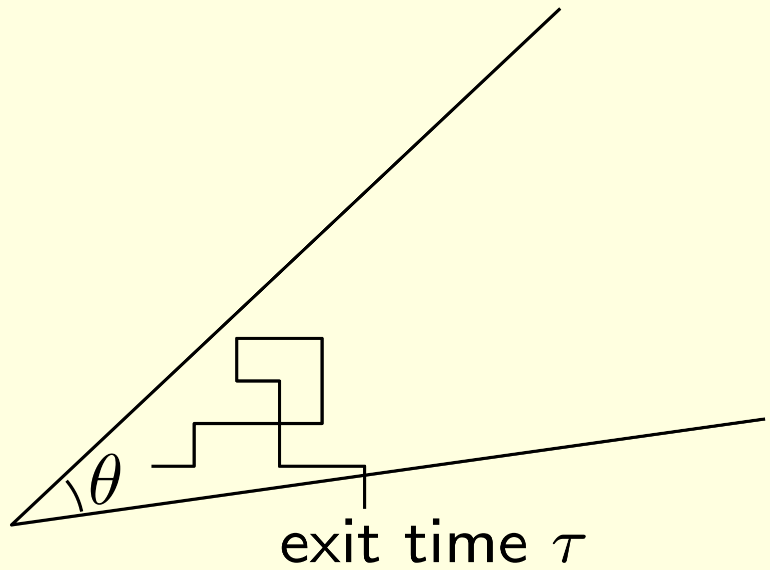

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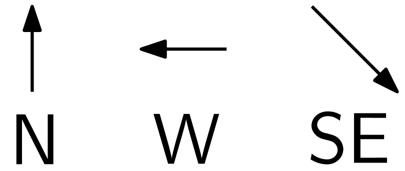
$$\mathbb{P}(\tau > n \ \&\ \text{excursion}) \sim c'\ n^{-1-\frac{\pi}{\theta}}$$

$27/2$ → optimal encoding [Takahashi, Fujimaki, Inoue'09]

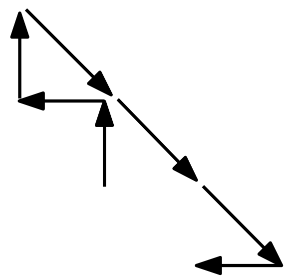
$$s_n \leq \binom{3n}{n} 2^n$$

$\notin \mathbb{Q}$ → not D-finite

Illustration on tandem walks with small steps

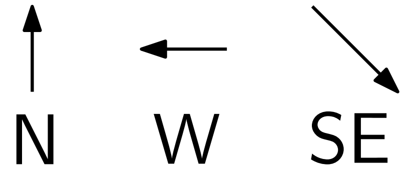
Step-set 

(triangulated bipolar orientations)

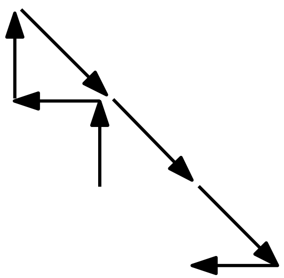
Random walk 

$\mathbb{P}(\text{each step}) = \frac{1}{3}$

Illustration on tandem walks with small steps

Step-set 

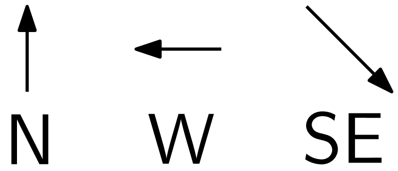
(triangulated bipolar orientations)

Random walk 

$\mathbb{P}(\text{each step}) = \frac{1}{3}$

$$\mathrm{Cov} = \begin{pmatrix} \mathbb{E}(X^2) & \mathbb{E}(XY) \\ \mathbb{E}(XY) & \mathbb{E}(Y^2) \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

Illustration on tandem walks with small steps

Step-set 

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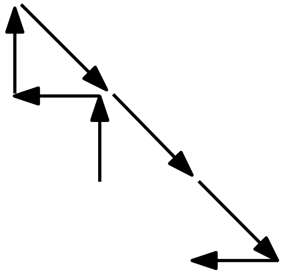
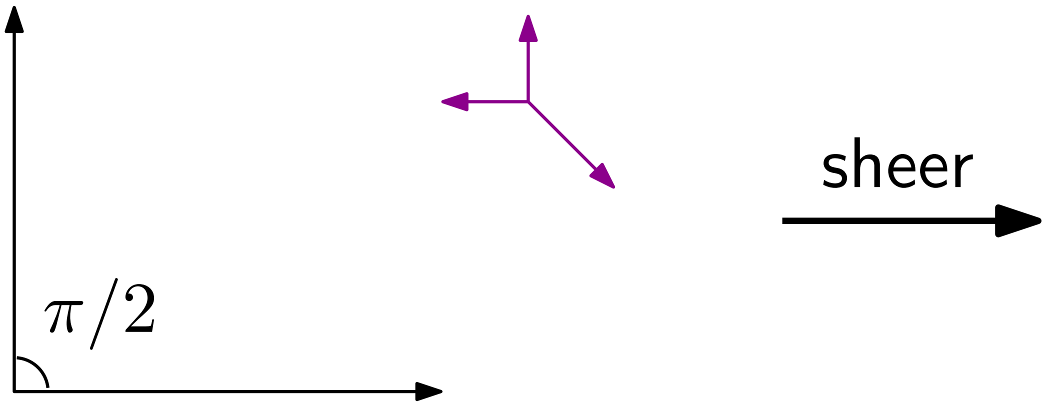
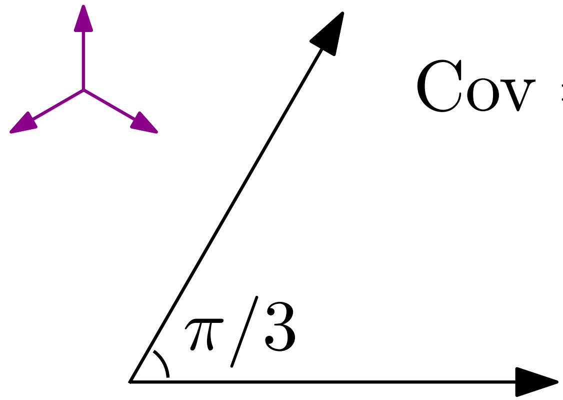
Random walk 

$\mathbb{P}(\text{each step}) = \frac{1}{3}$

$$\mathrm{Cov} = \begin{pmatrix} \mathbb{E}(X^2) & \mathbb{E}(XY) \\ \mathbb{E}(XY) & \mathbb{E}(Y^2) \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$$



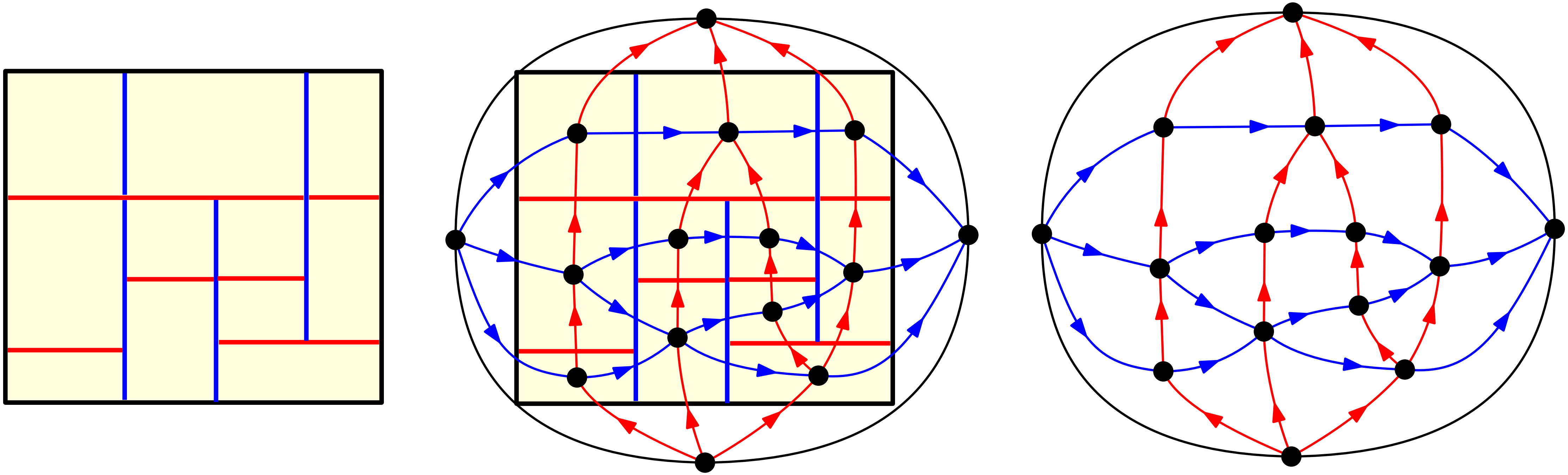


$$\mathrm{Cov} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

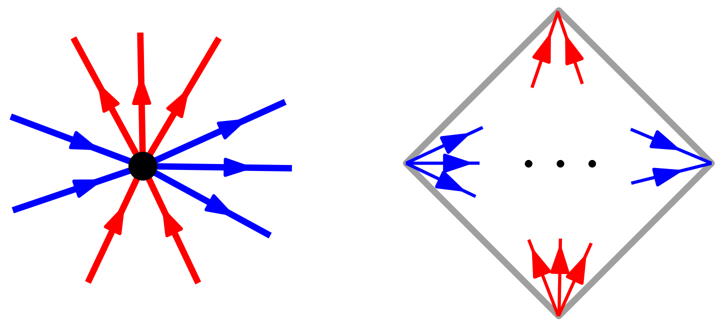
$\Rightarrow$ # quadrant excursions length $3n \sim c \cdot 27^n n^{-4}$

($\alpha = 4$ universal for plane bipolar orientations)

Non-generic rectangulations

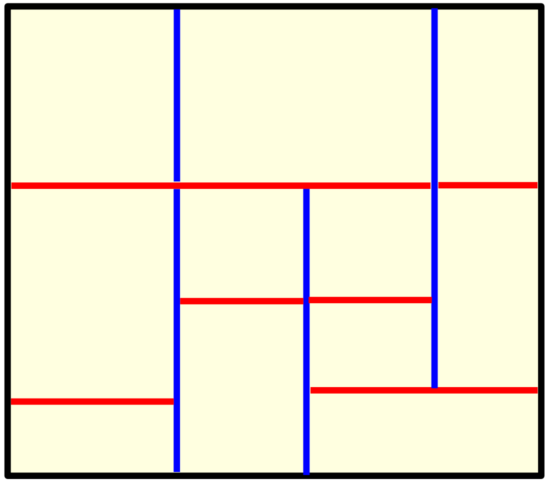


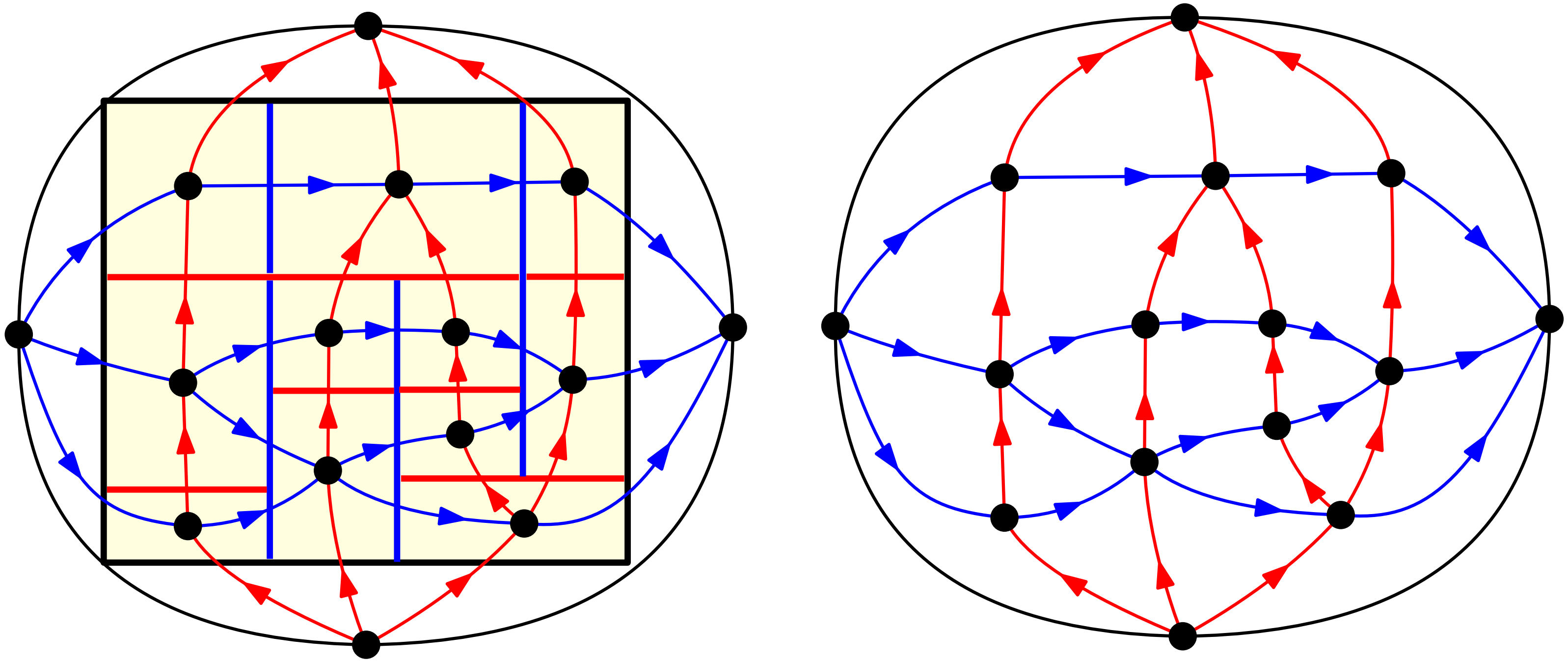
Pair of transversal
plane bipolar orientations



Local conditions

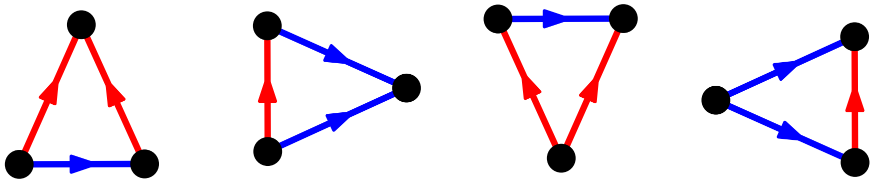
Non-generic rectangulations



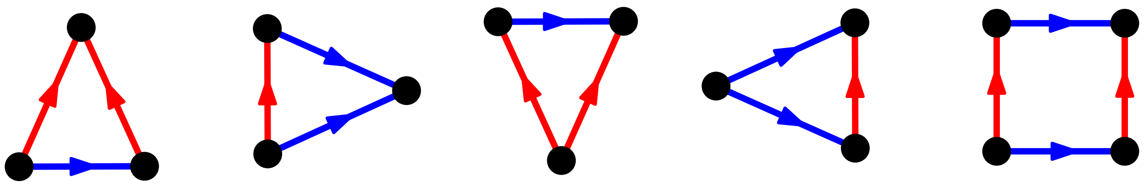


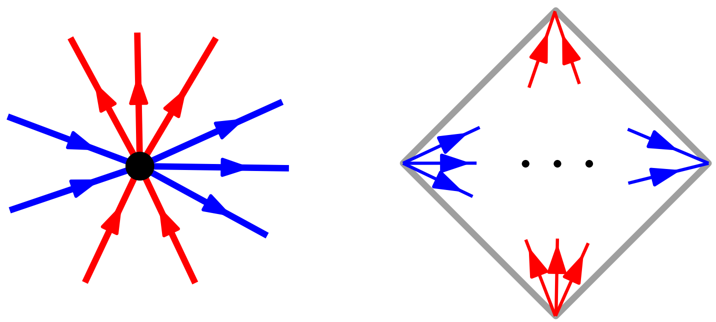
Pair of transversal
plane bipolar orientations

Generic case: inner faces of 4 types



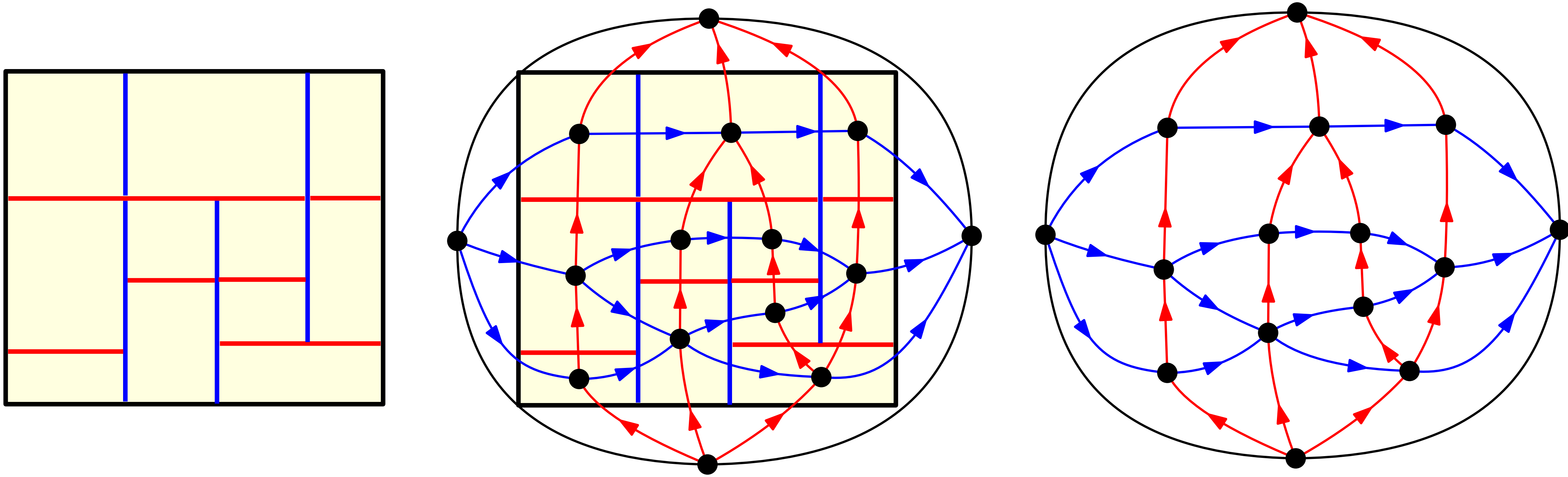
Non-generic case: inner faces of 5 types

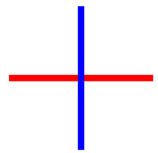


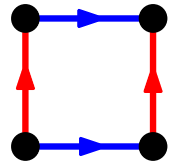


Local conditions

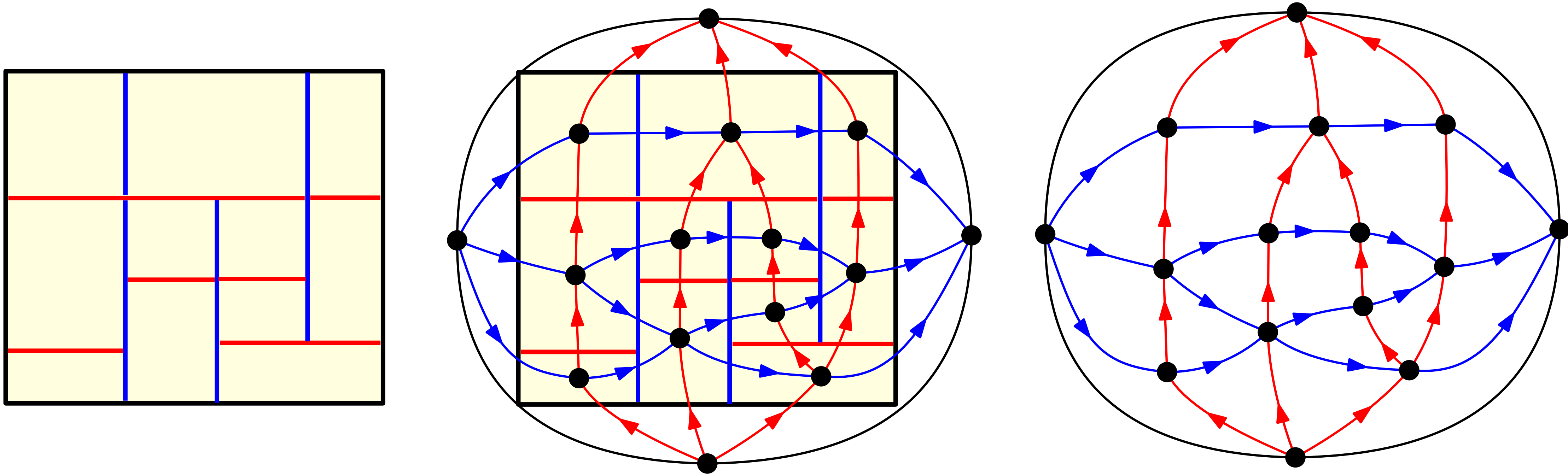
Non-generic rectangulations with weighted quadrangles

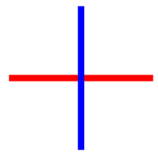


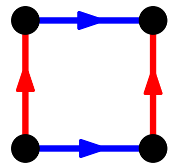
Let $s_n(v) = \#$ rectangulations with n regions, weight v per 

$= \#$ transversal structures with $n+4$ vertices, weight v per 

Non-generic rectangulations with weighted quadrangles

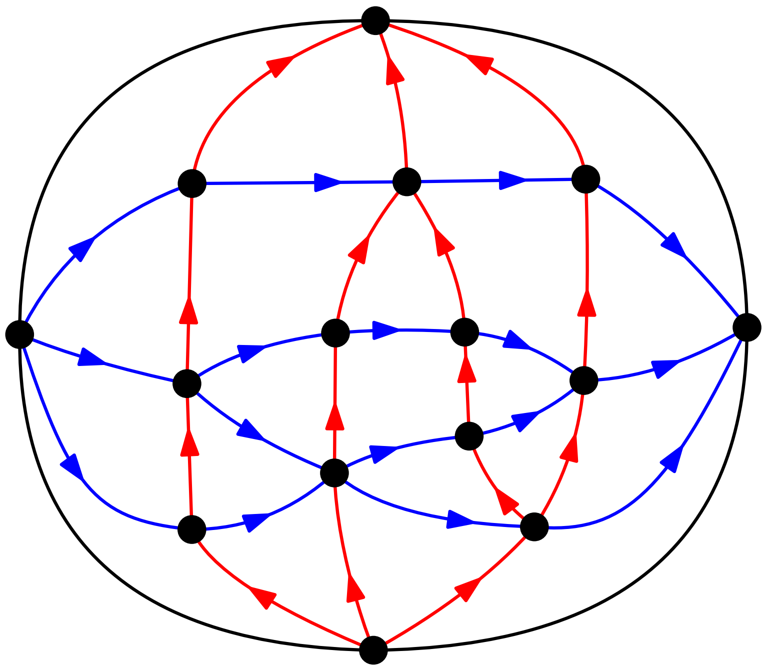


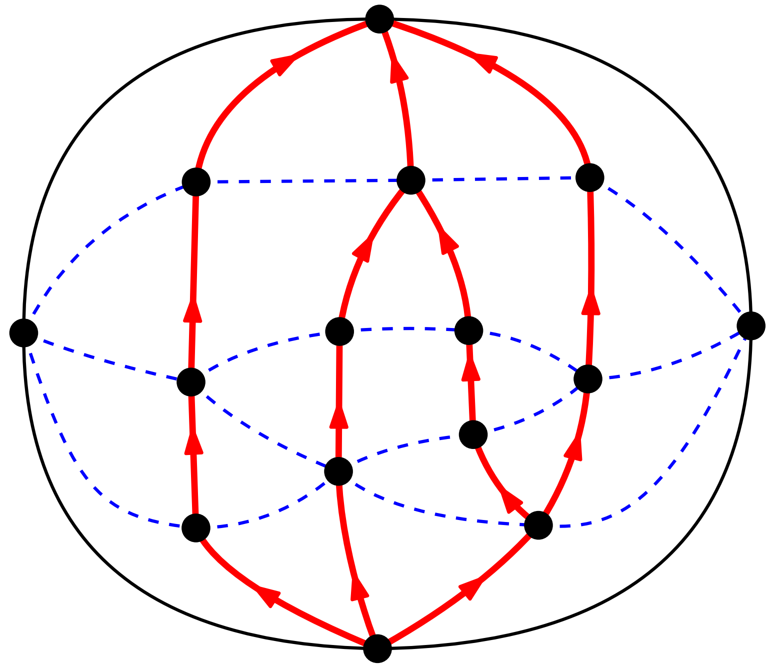
Let $s_n(v) = \#$ rectangulations with n regions, weight v per 

$= \#$ transversal structures with $n+4$ vertices, weight v per 

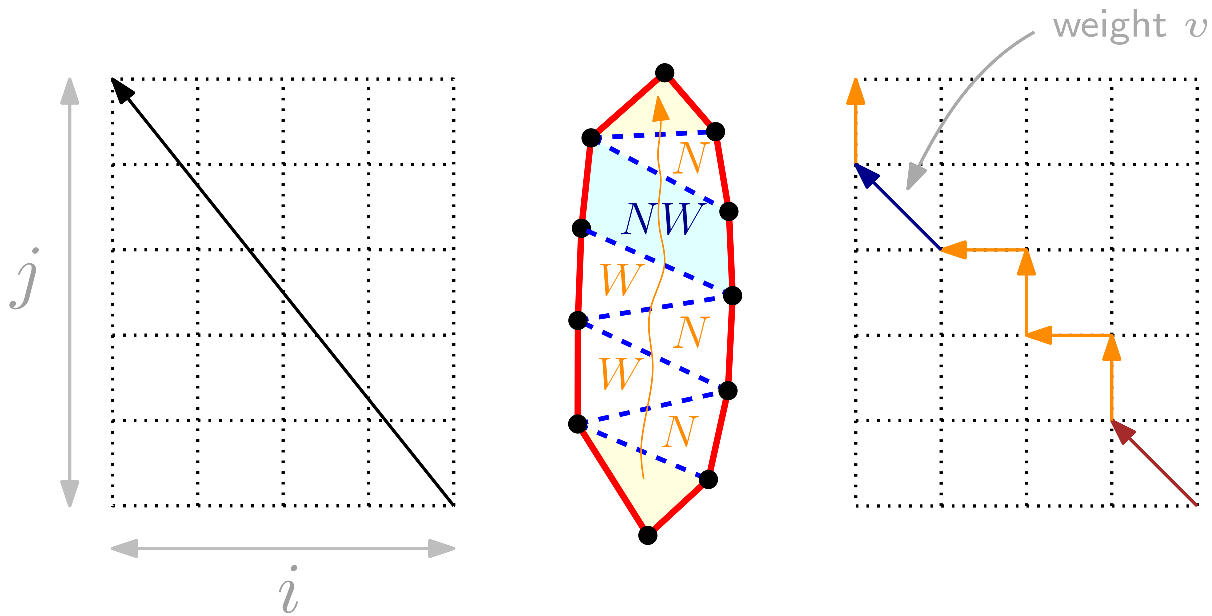
$$\sum_n s_n(v)t^n = t + 2t^2 + 6t^3 + (24+v)t^4 + (116+12v)t^5$$
$$+(642+114v+2v^2)t^6 + (3938+1028v+48v^2)t^7 + \cdots$$

Encoding by tandem walks





small-step portion for a face

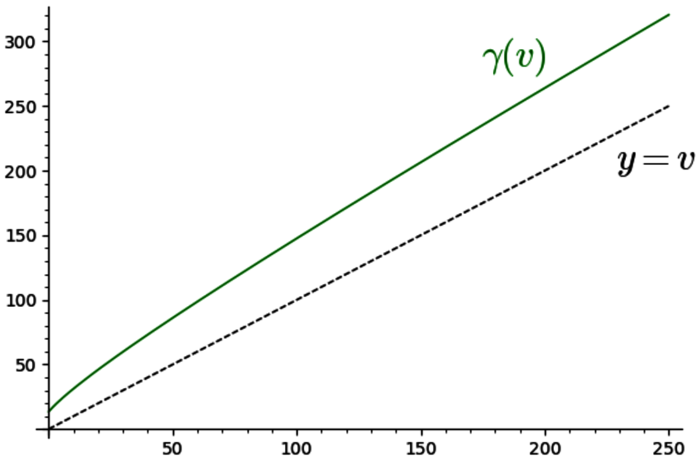


weight on face-step $(-i, j)$: $w(i,j) = \sum_k v^k \binom{i+j-2-k}{k}\binom{i+j-2-2k}{i-1-k}$

Asymptotic behaviour

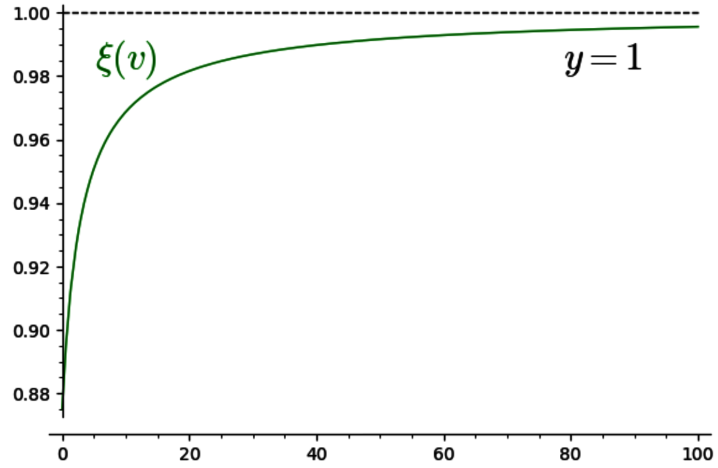
For $v > 0$, $$s_n(v) \sim c(v)\ \gamma(v)^n\ n^{-\alpha(v)}$$

$\alpha(v) = 1 + \frac{\pi}{\theta(v)}$



$$\gamma(v) = \frac{1}{2(2+v)}\left(2v^2+18\,v+27+(9+4v)^{3/2}\right)$$

$\gamma(0) = 27/2$



$$\xi(v) = \frac{1}{4(2+v)^2}\left(4v^2+14v+11+\sqrt{9+4v}\right)$$

$\xi(0) = 7/8$

$\xi(v) = \cos(\theta(v))$

Asymptotic behaviour

[F-Narmanli-Schaeffer'21]

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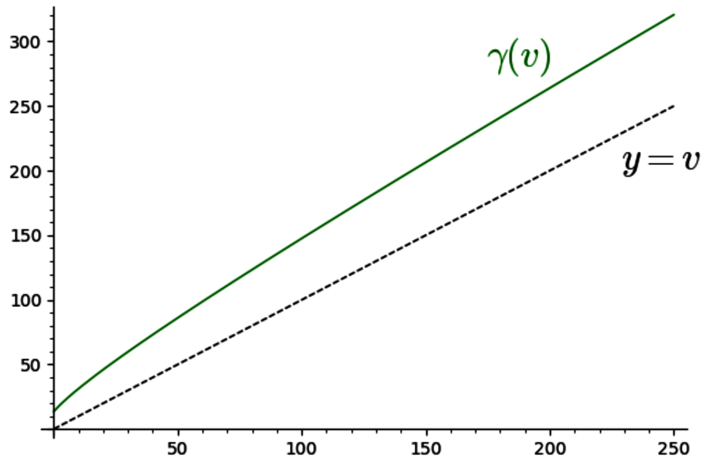

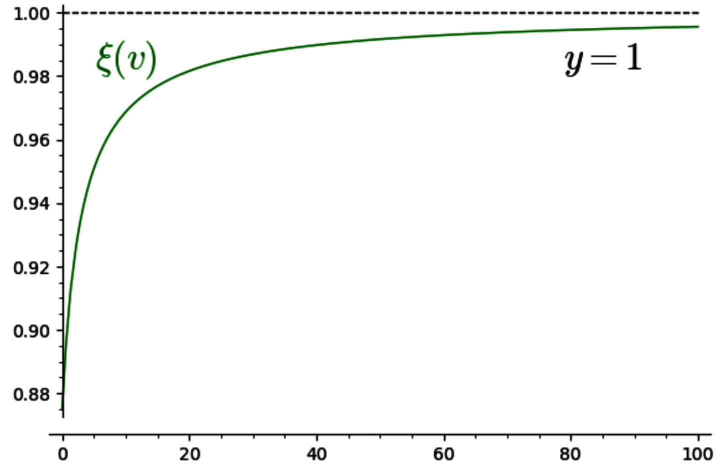

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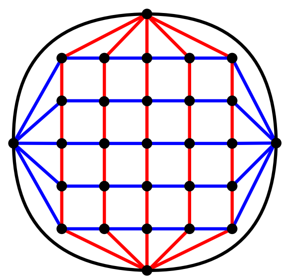

Regime $v \to \infty$ $\qquad \gamma(v) \sim v, \quad \alpha(v) \to \infty \qquad \sim$ regular grid

Regime $v = O(1)$ random lattice (universality class evolves with v)

Asymptotic behaviour

[F-Narmanli-Schaeffer'21]

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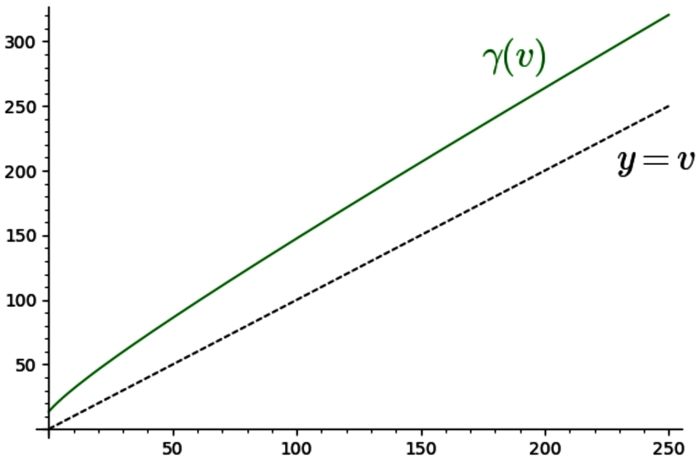

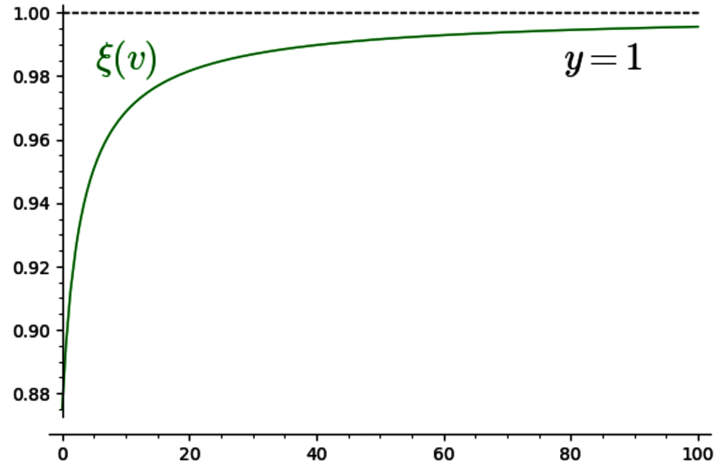

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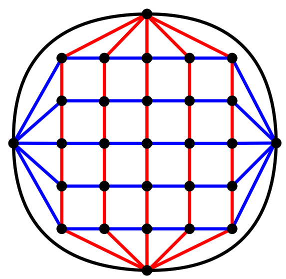

Regime $v = O(1)$ random lattice (universality class evolves with v)

cf [Kazakov, Staudacher, Wynter'96]
(other interpolating model)

Asymptotic behaviour

[F-Narmanli-Schaeffer'21]

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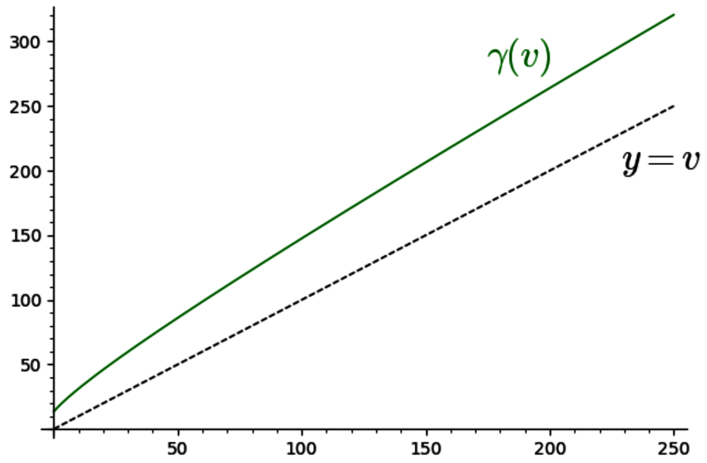

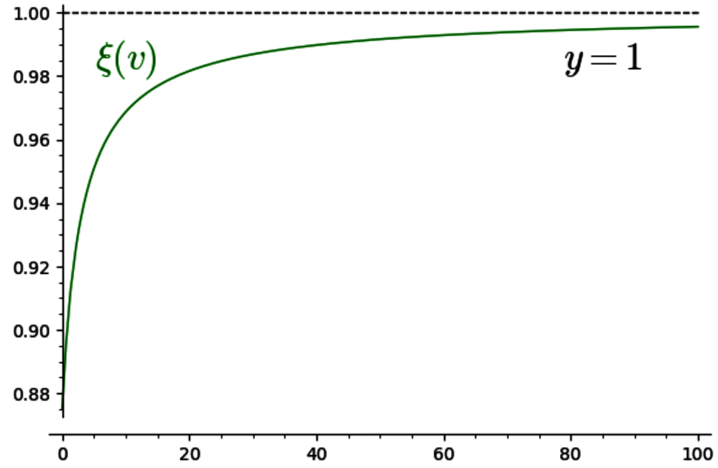

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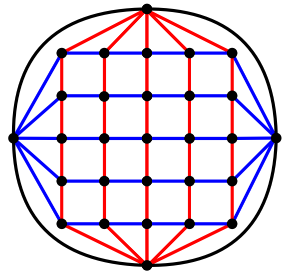
$\xi(v) = \cos(\theta(v))$

$\xi(0) = 7/8$

Rk: $\gamma(v), \xi(v) > 0$ for $v \geq -2$



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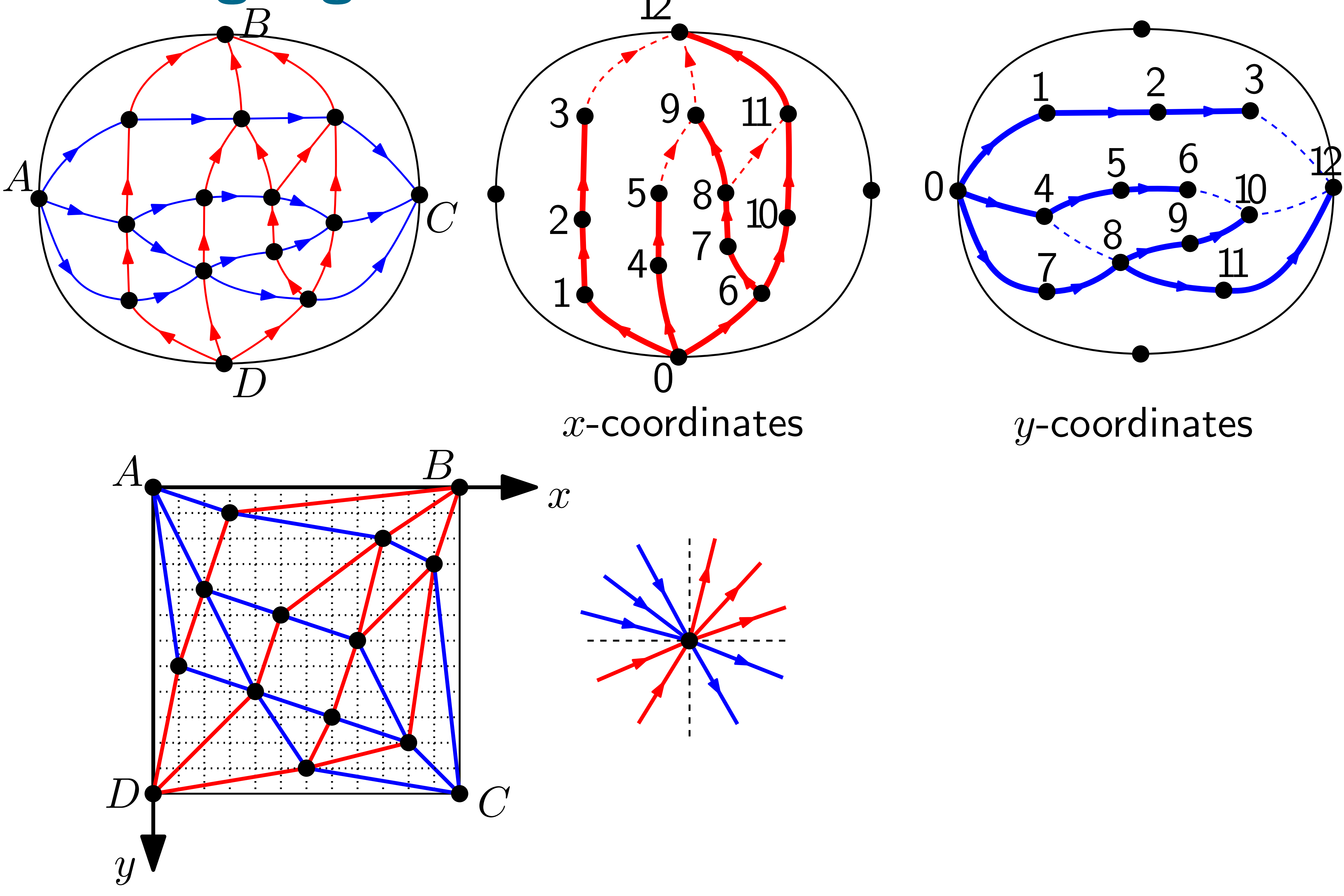
(other interpolating model)

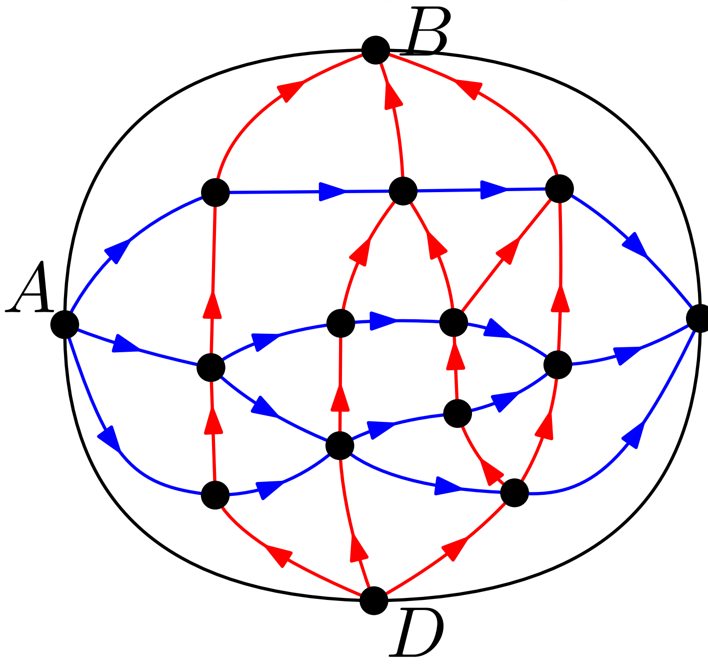
Eulerian quadrangulations

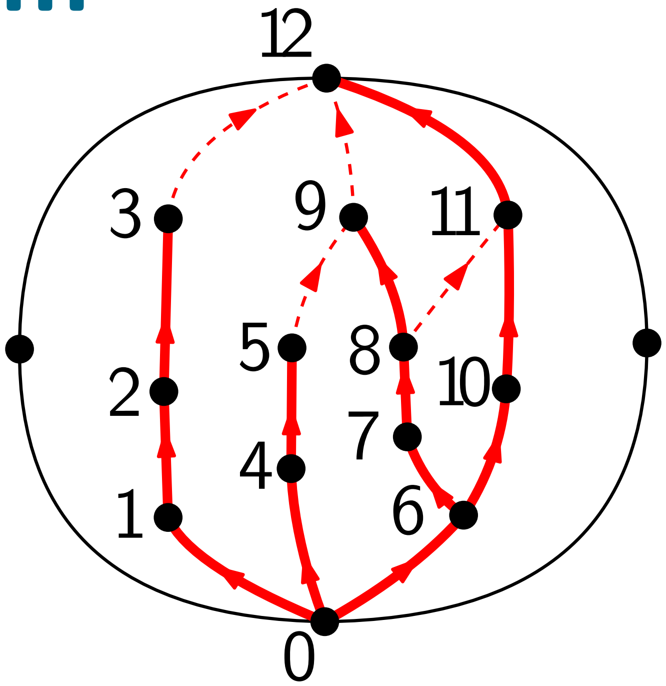
weight v^{-1} per

Drawing algorithm

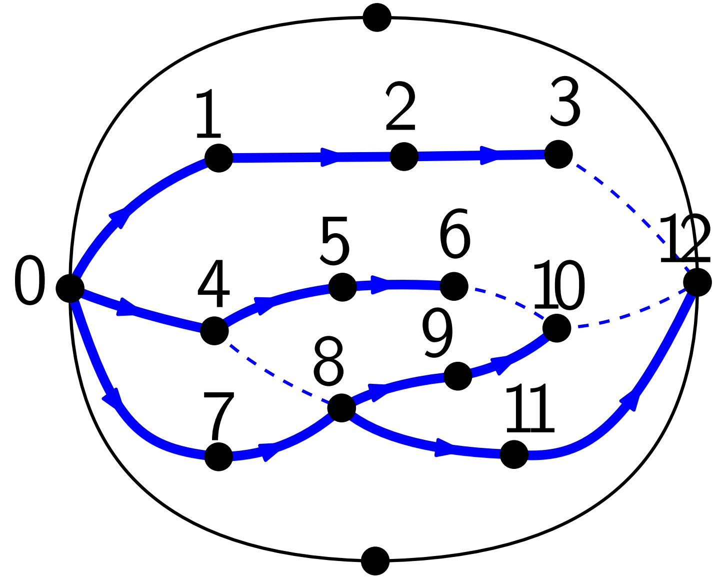


Drawing algorithm

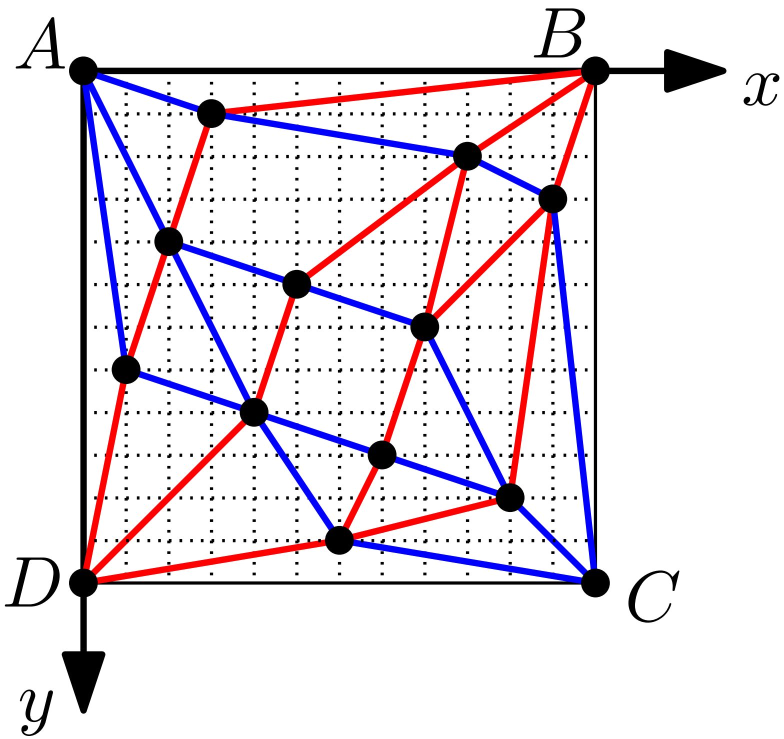


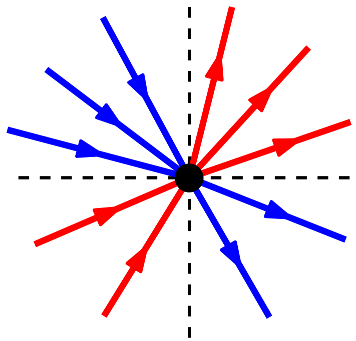


x-coordinates

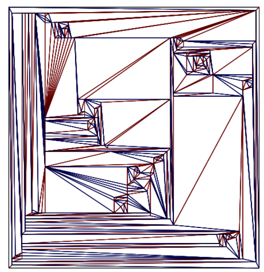


y-coordinates

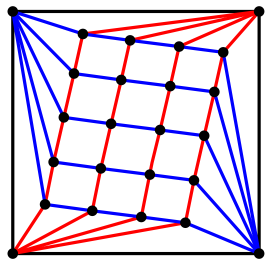




Expected:



for $v = O(1)$



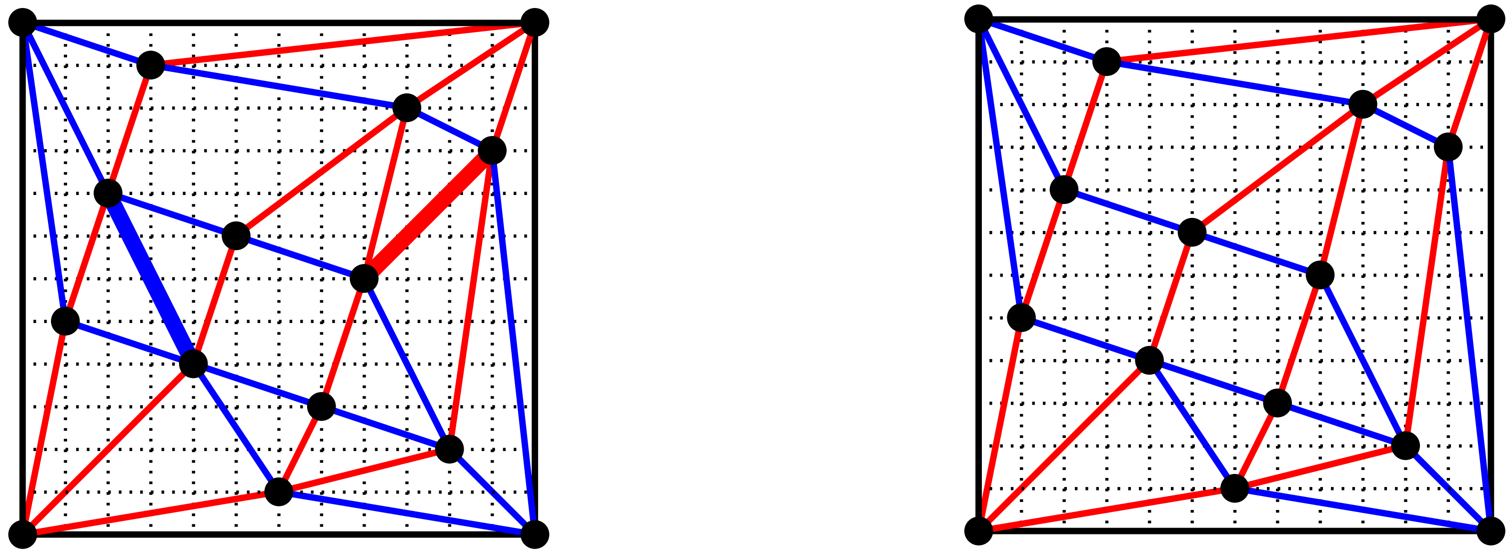
for $v >> 1$

Reduction of a transversal structure

Reduction of a transversal structure (that possibly has quadrangles)

Apply in parallel

Property: A transversal structure gives same drawing as its reduced form



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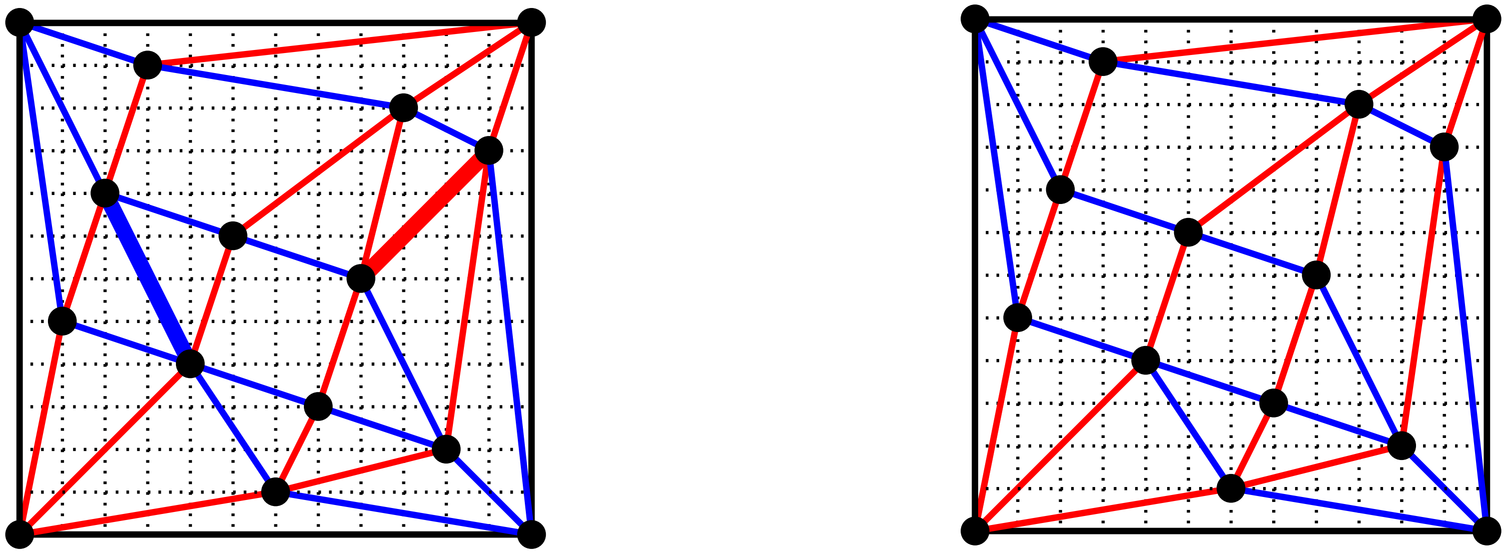
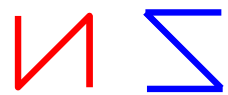
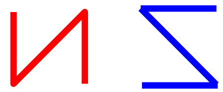


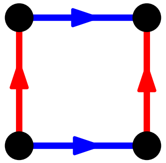
Property: Drawing is bijection from reduced transversal structures

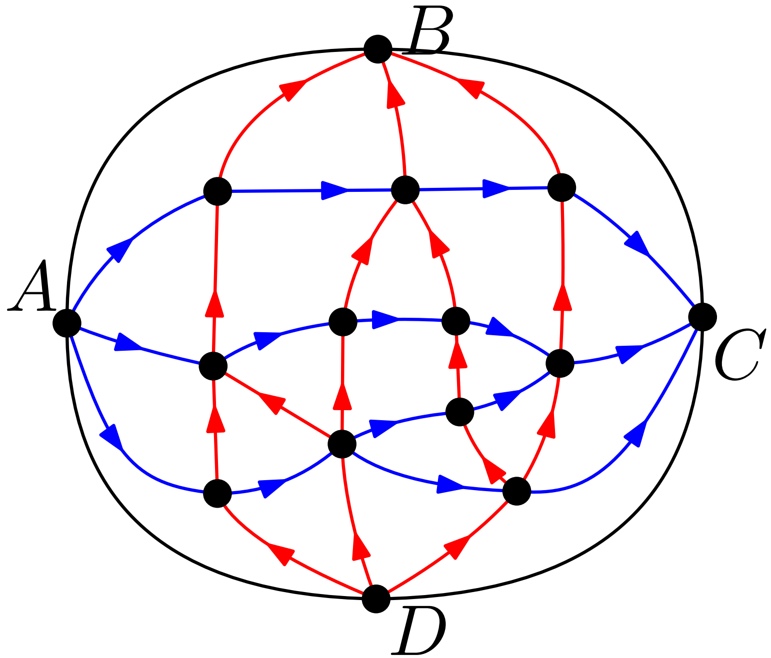
no 

to semi-Baxter permutations (no $3\underline{1\,4}2$)

v-weighted modele on reduced transversal structures

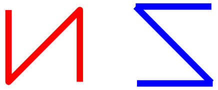
reduced $=$ no 

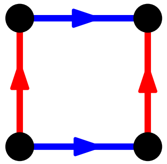
weight v per 

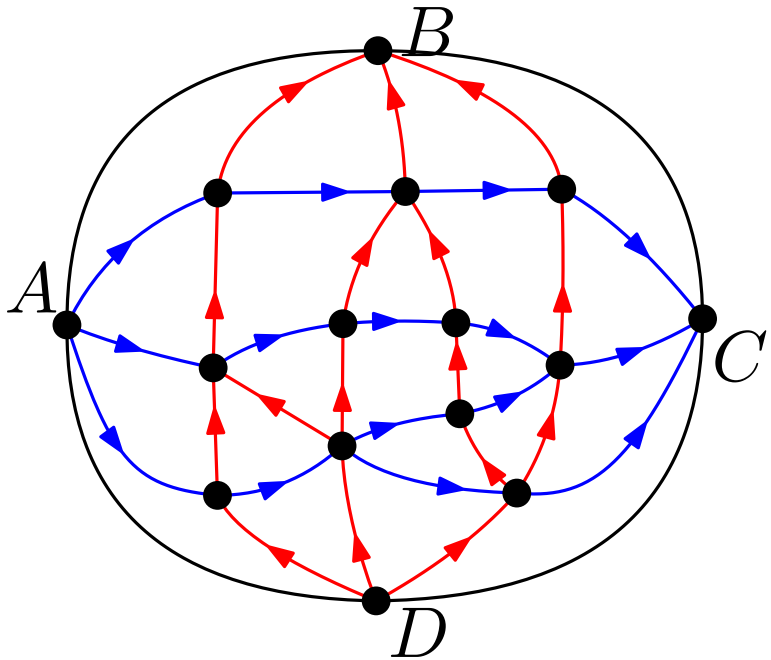


weight v^3

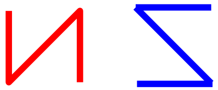
v-weighted modele on reduced transversal structures

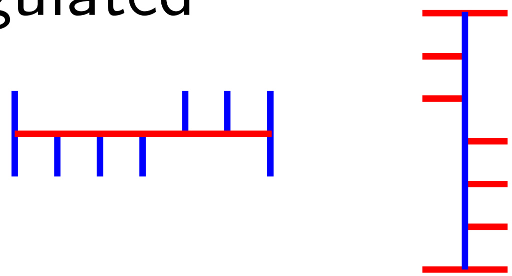
reduced $=$ no 

weight v per 



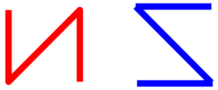
weight v^3

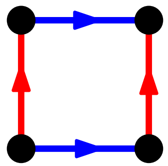
Rk: $v = 0$ $\Leftrightarrow$ no  and triangulated

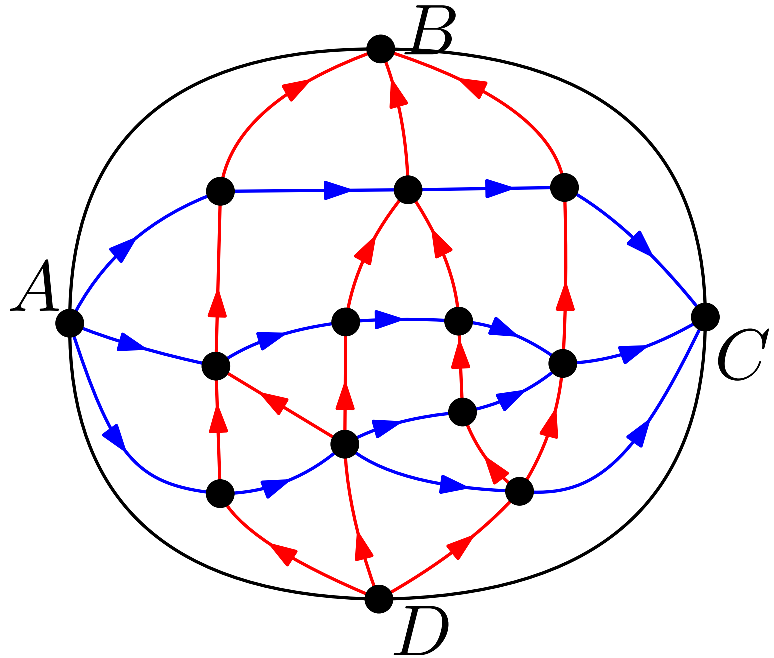
$\Leftrightarrow$ weak rectangulations 

(Baxter numbers)

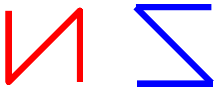
v-weighted modele on reduced transversal structures

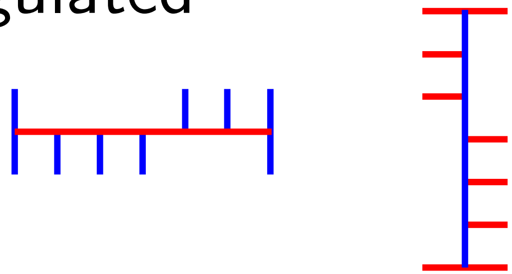
reduced $=$ no 

weight v per 



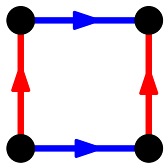
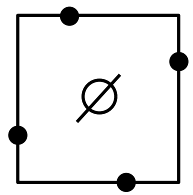
weight v^3

Rk: $v = 0 \Leftrightarrow$ no  and triangulated

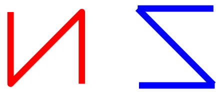
$\Leftrightarrow$ weak rectangulations 

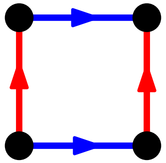
(Baxter numbers)

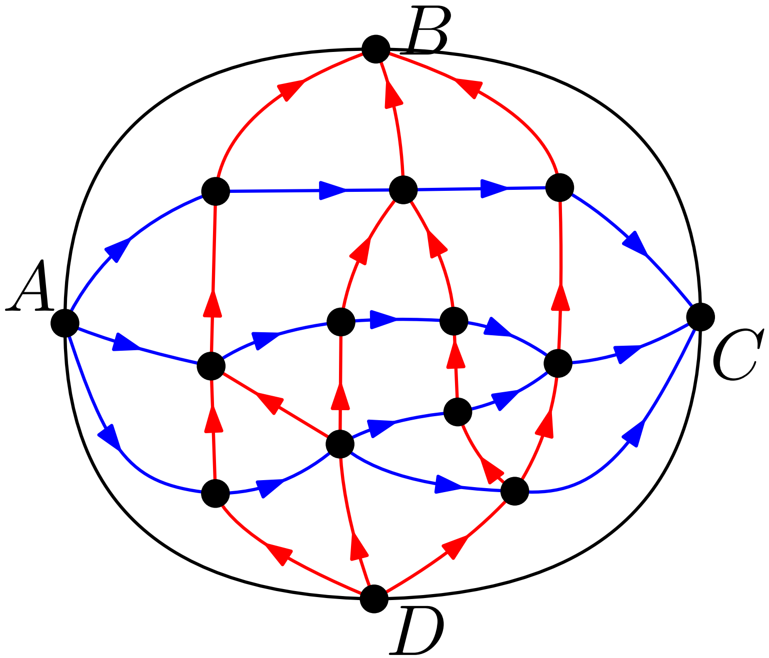
reduced transversal structure $\longleftrightarrow$ semi-Baxter permutation

 $\longleftrightarrow$ pattern 

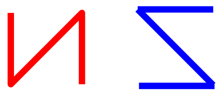
v-weighted modele on reduced transversal structures

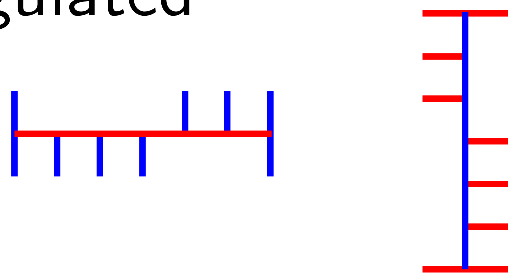
reduced $=$ no 

weight v per 

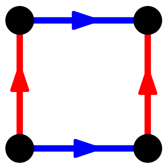
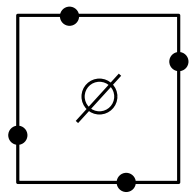


weight v^3

Rk: $v = 0 \;\Leftrightarrow$ no  and triangulated

$\Leftrightarrow$ weak rectangulations (Baxter numbers) 

reduced transversal structure $\longleftrightarrow$ semi-Baxter permutation

 $\longleftrightarrow$ pattern 

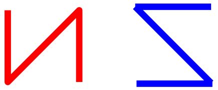
$v = 0 \;\Leftrightarrow$ Baxter permutations (no $3\underline{1\,4}2$ $\;2\underline{4\,1}3$)

Link between the two v-weighted modeles

Reduction of a transversal structure (that possibly has quadrangles)

Apply in parallel

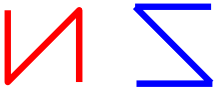
reduced = no



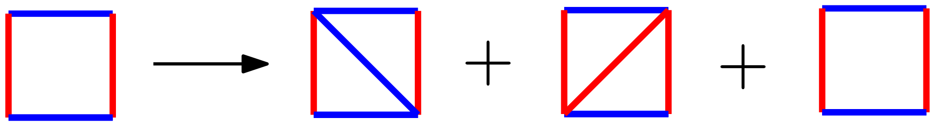
Link between the two v-weighted modeles

Reduction of a transversal structure (that possibly has quadrangles)

Apply in parallel

reduced $=$ no 

Property: $\tilde{s}_n(v) :=$ analogue of $s_n(v)$ on reduced transversal structures

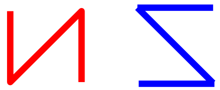
$$\boxed{s_n(v) = \tilde{s}_n(2+v)}$$ cf 

So $s_n(v)$ has combinatorial meaning for $v \geq -2$

Link between the two v-weighted modeles

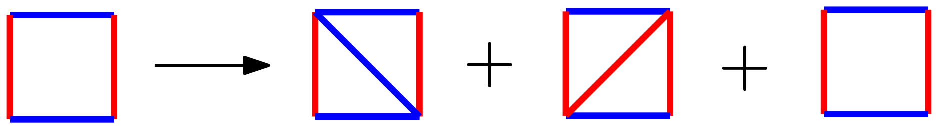
Reduction of a transversal structure (that possibly has quadrangles)

Apply in parallel

reduced = no 

Property: $\tilde{s}_n(v) :=$ analogue of $s_n(v)$ on reduced transversal structures

$$s_n(v) = \tilde{s}_n(2+v)$$

cf 

So $s_n(v)$ has combinatorial meaning for $v \geq -2$

$v = -2$ Baxter structures, asymptotics in n^{-4}

$v = -1$ semi-Baxter structures, asymptotics in n^{-6}