On the number of intervals in the Tamari lattices

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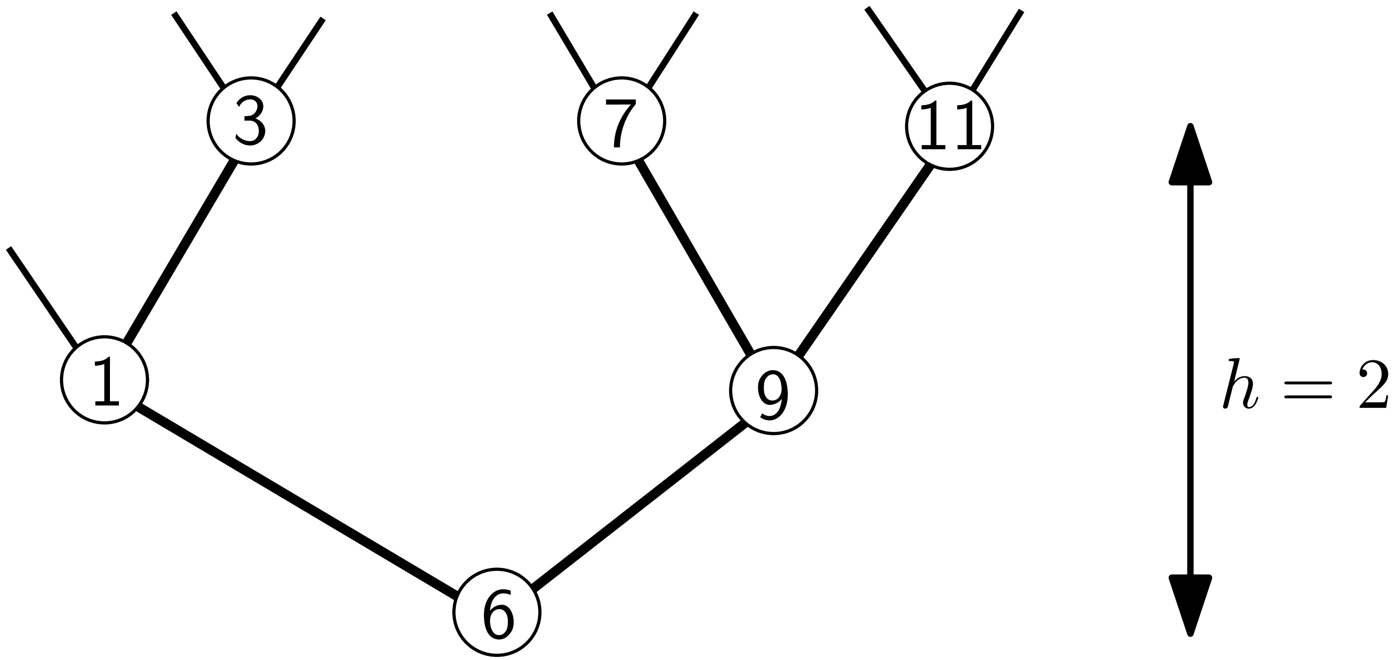
Joint work with M. Bousquet-Mélou (LaBRI) and L.F. Préville-Ratelle (UQAM)

Rocquencourt, Oct 2011

Binary search trees

A binary search tree (BST) is a data structure to store (comparable) items

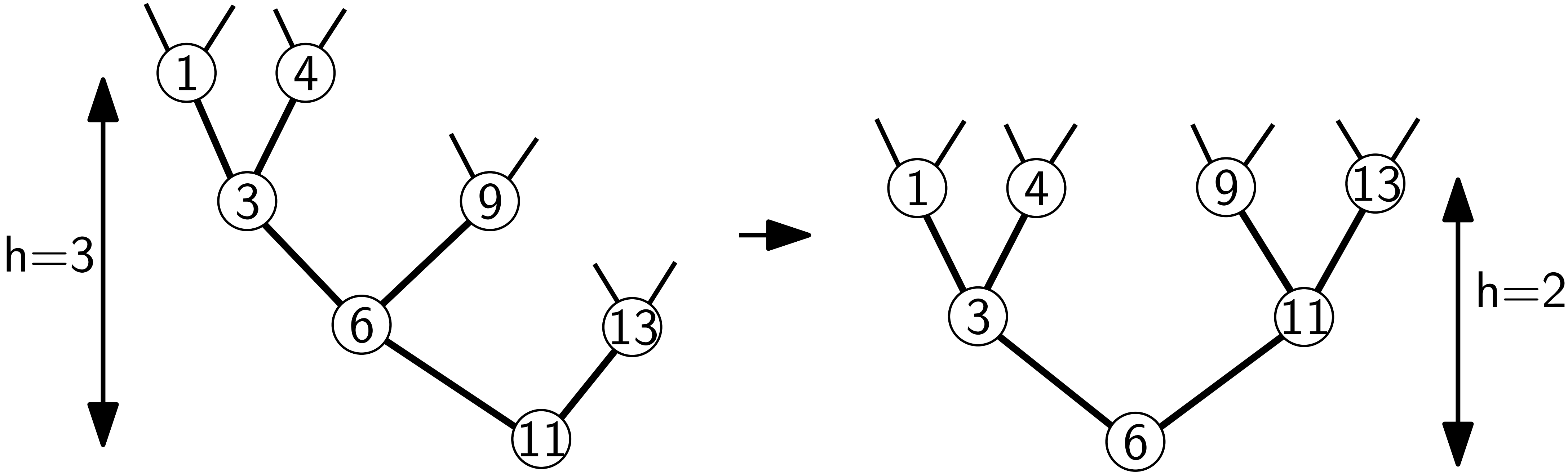


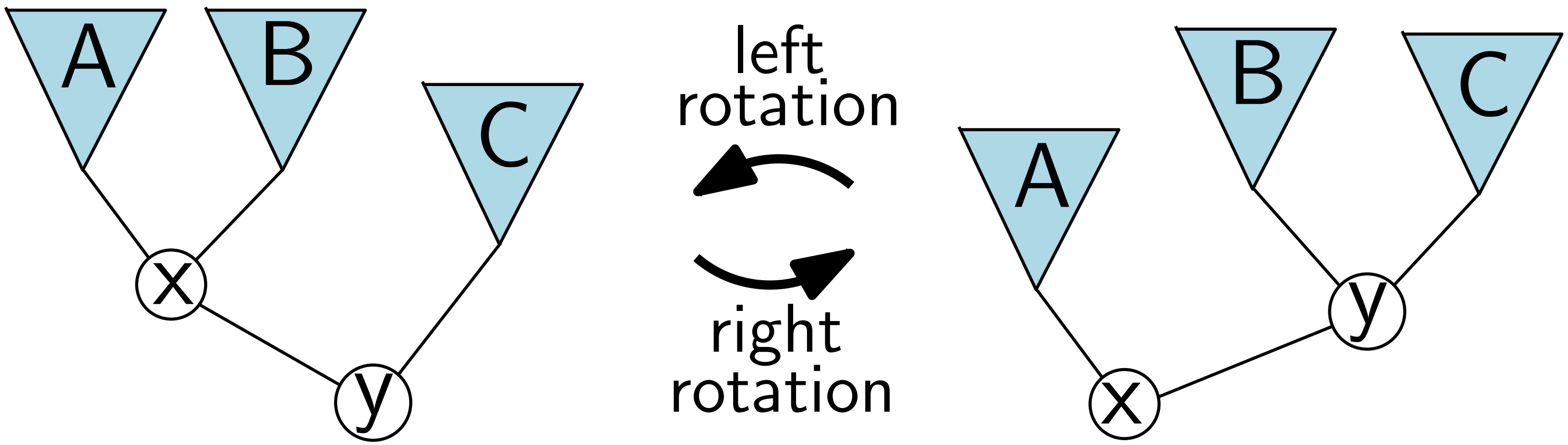
Items are in increasing order "from left to right" in the BST

Insertion, deletion, search of an item are done in time $O(h)$

Balancing a BST

Rotation operations can be used to decrease the height





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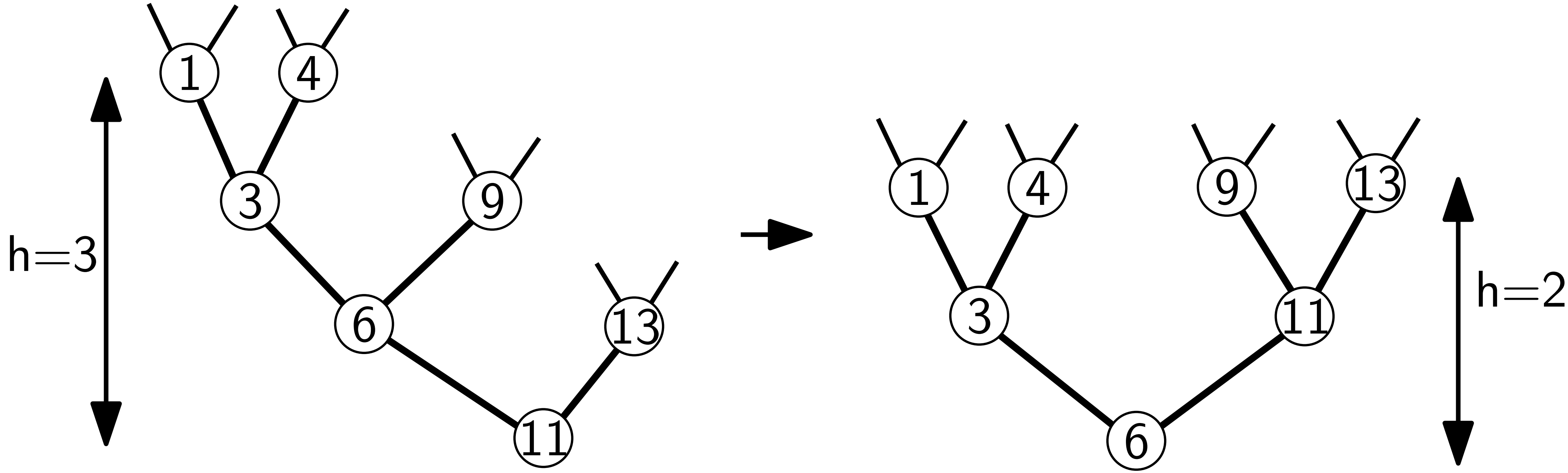
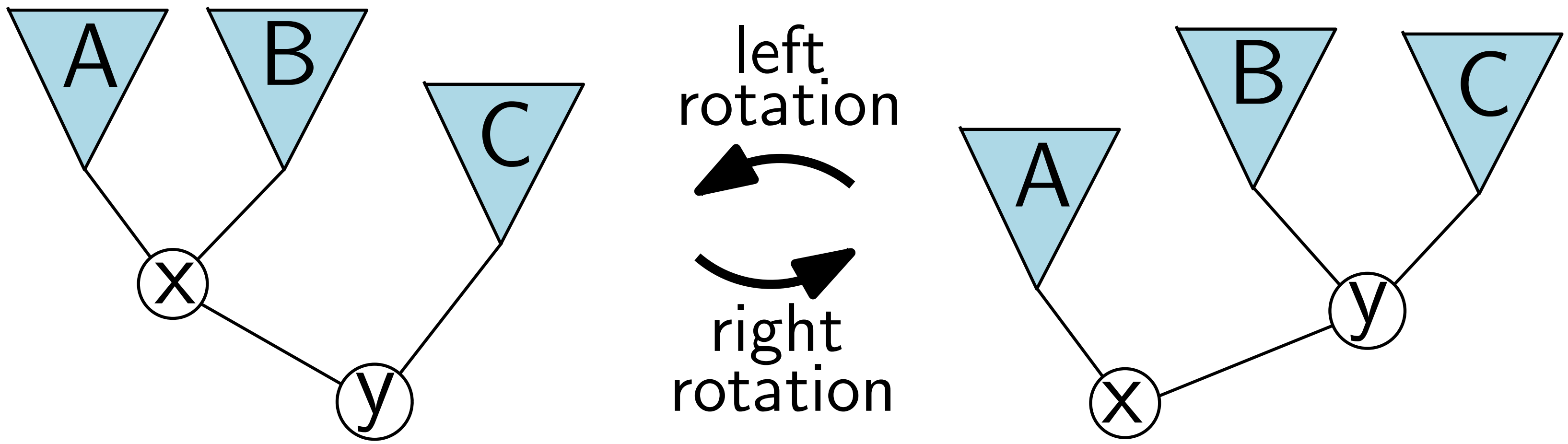
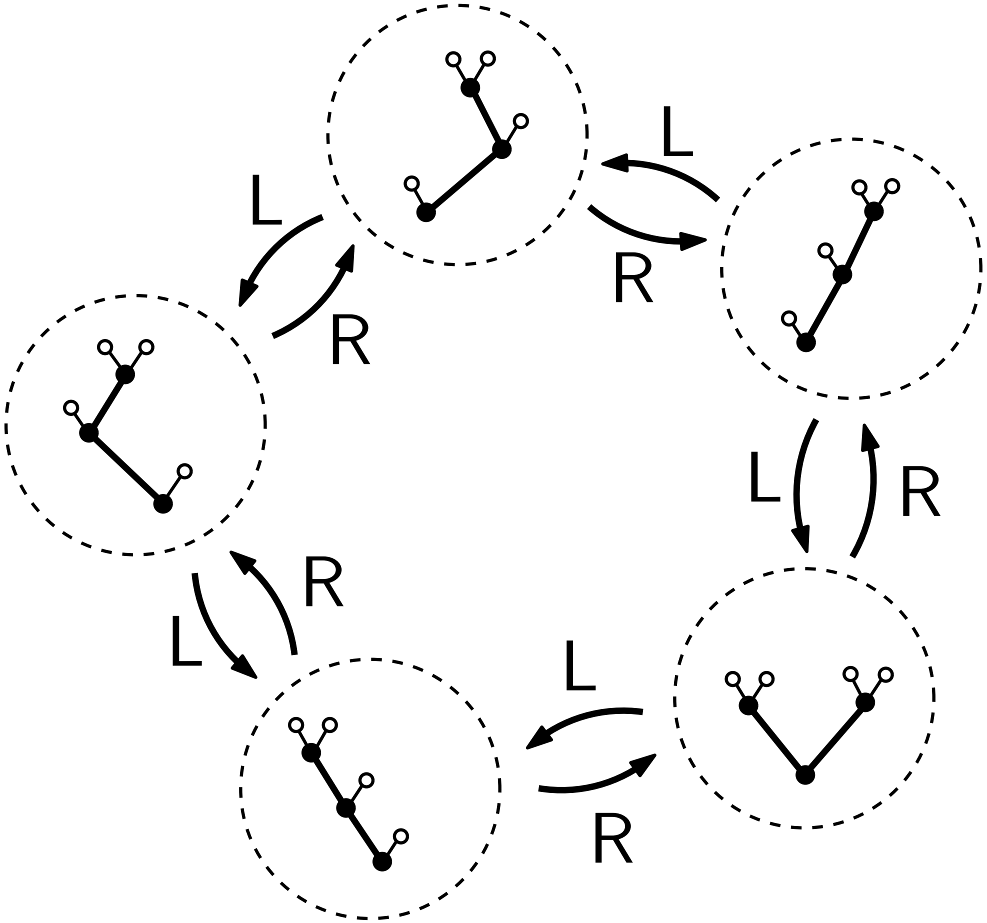


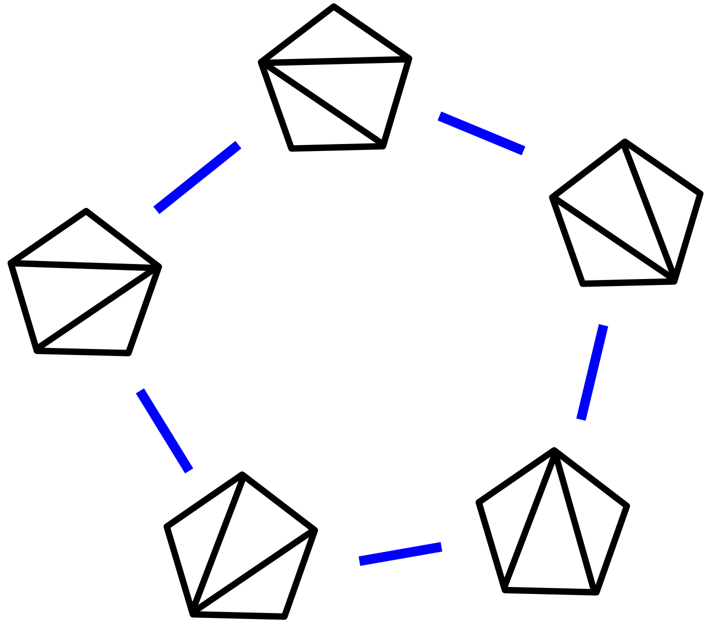


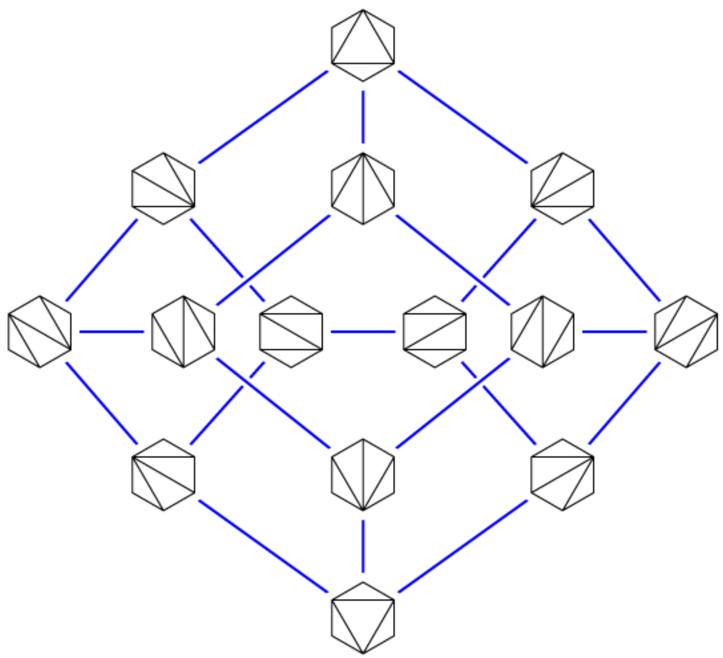
$\Rightarrow$ efficient data structures (AVL): maintain height in $O(\log(n))$

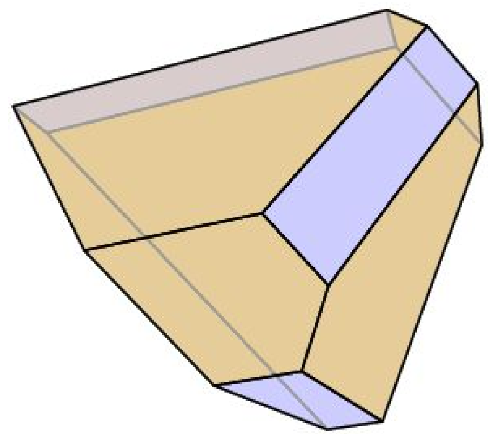
The adjacency graph for rotation-relations



cf the associahedron

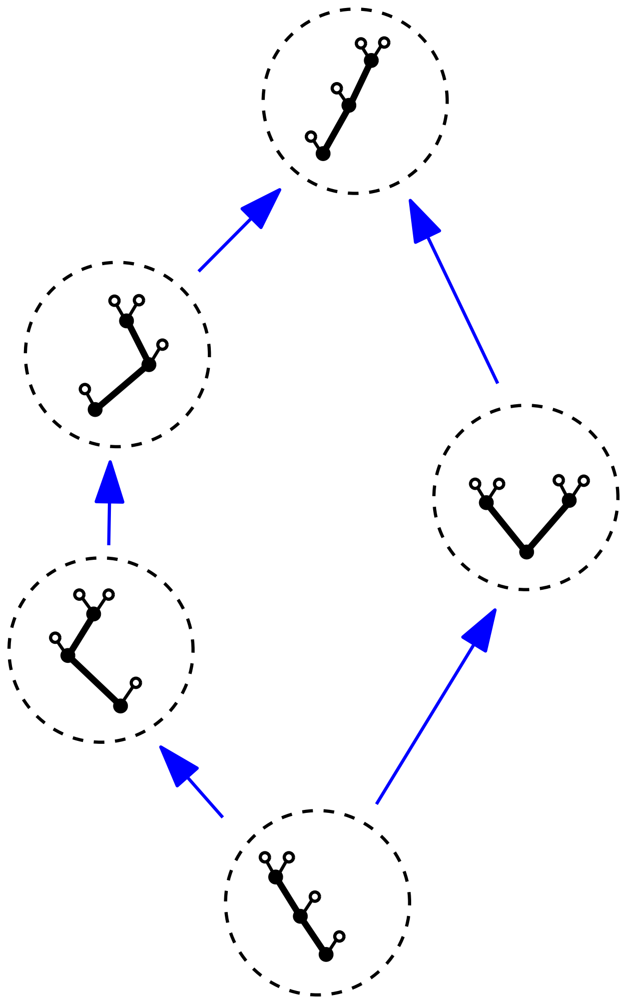




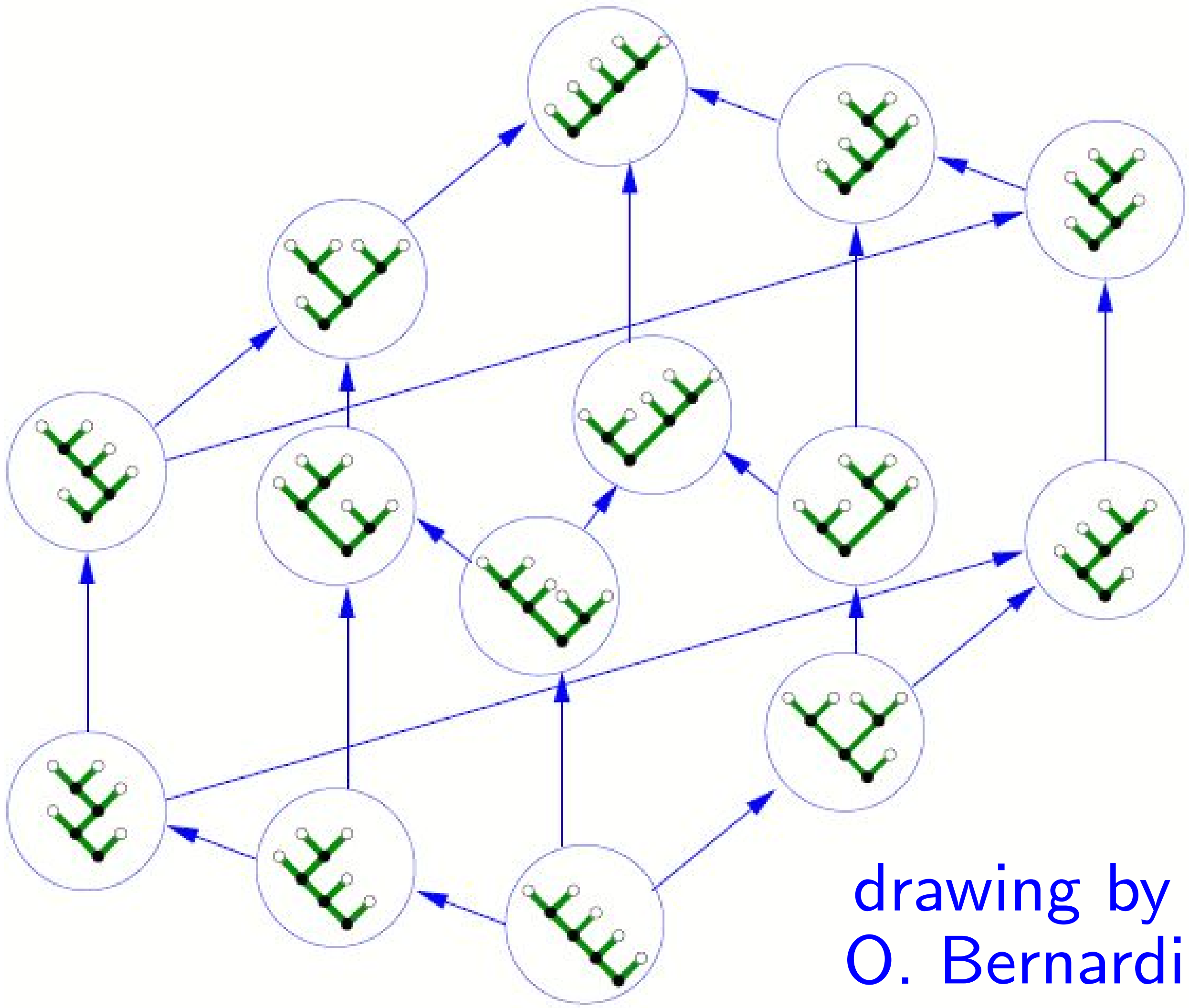


The Tamari lattice

The Tamari lattice $\mathcal{T}_n$ is the partial order on binary trees with n nodes where the covering reation corresponds to right rotation



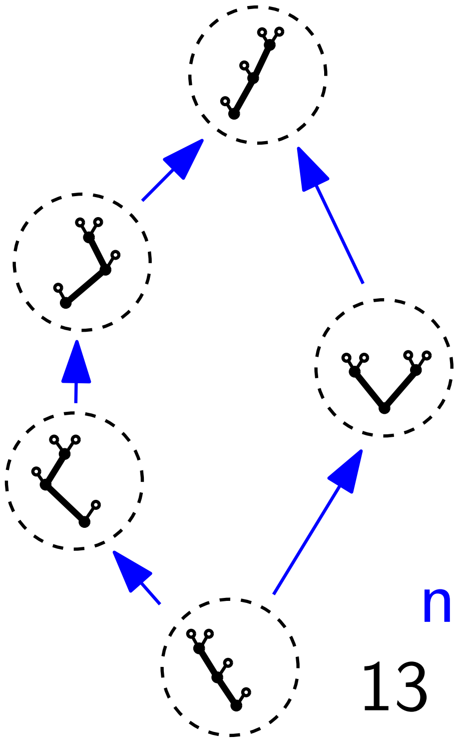
n=3



n=4

Enumeration of intervals in the Tamari lattice

An interval in $\mathcal{T}_n$ is a pair (t, t') such that $t \leq t'$



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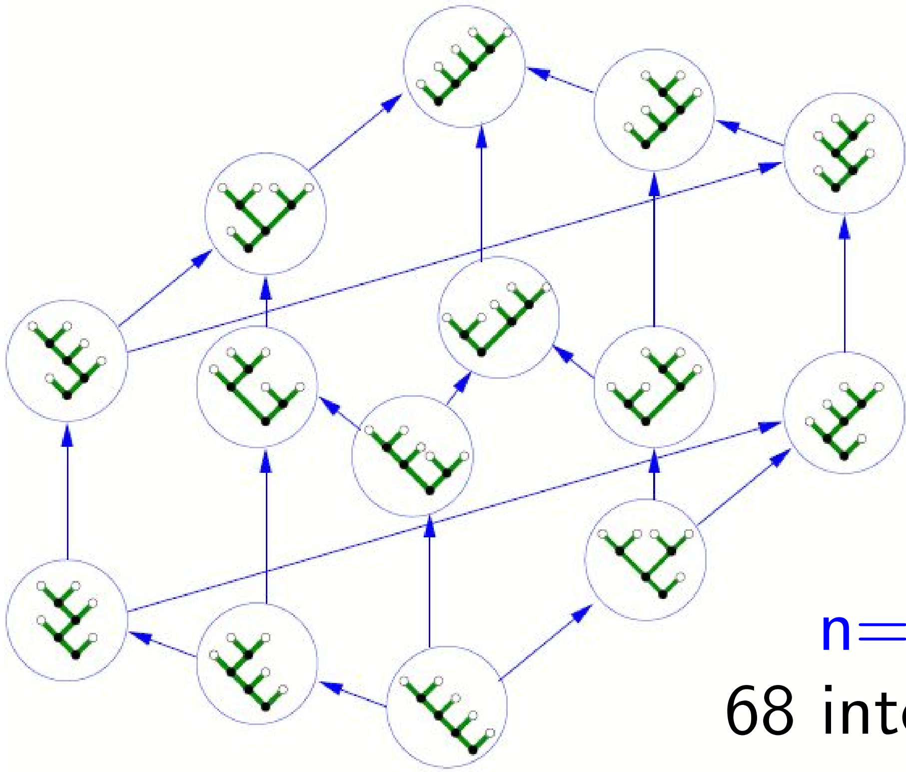
13 intervals



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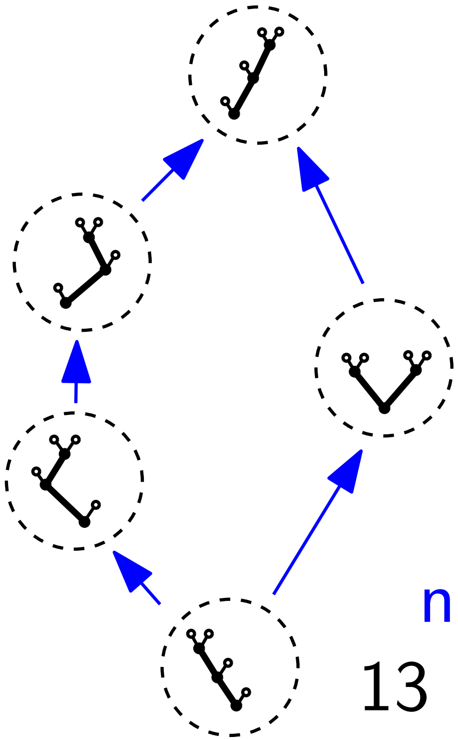

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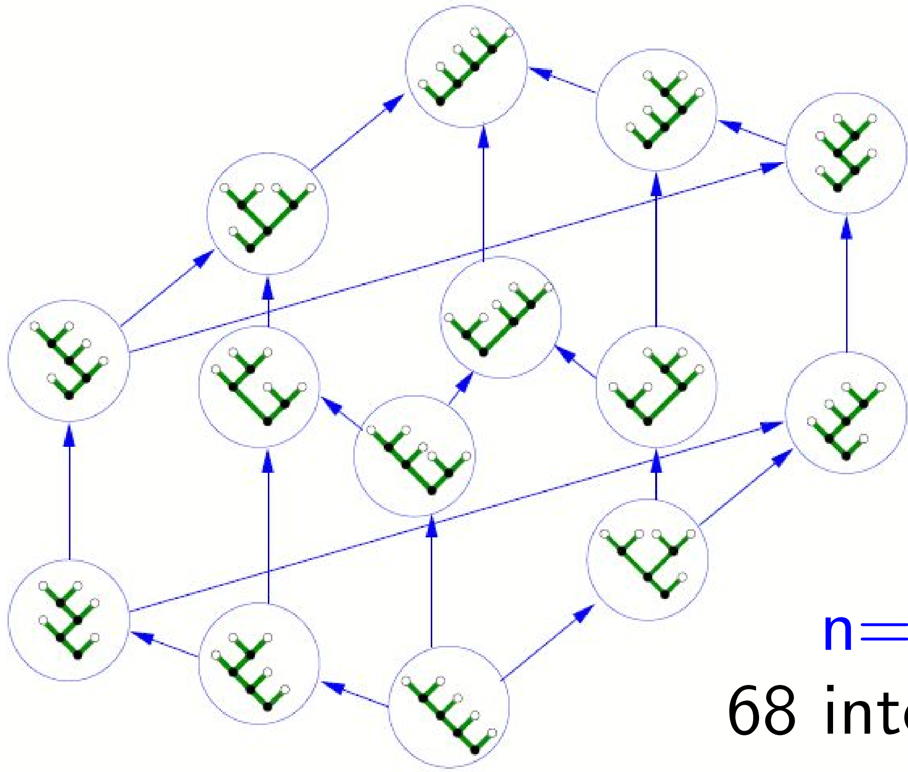

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Theorem [Chapoton'06]: there are $\frac{2}{n(n+1)}\binom{4n+1}{n-1}$ intervals in $\mathcal{T}_n$

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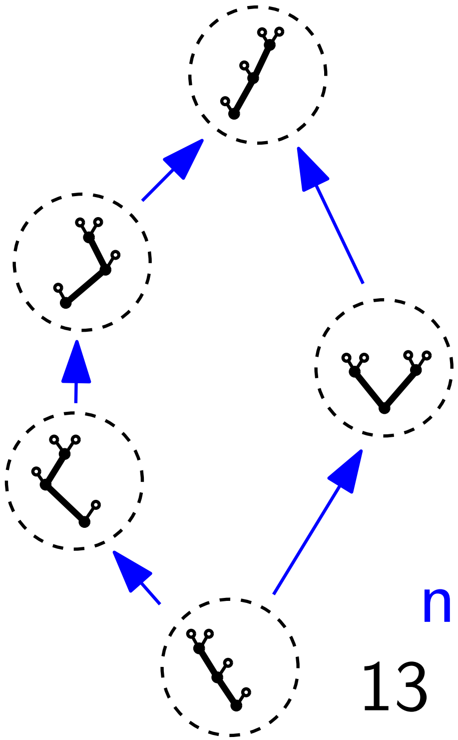

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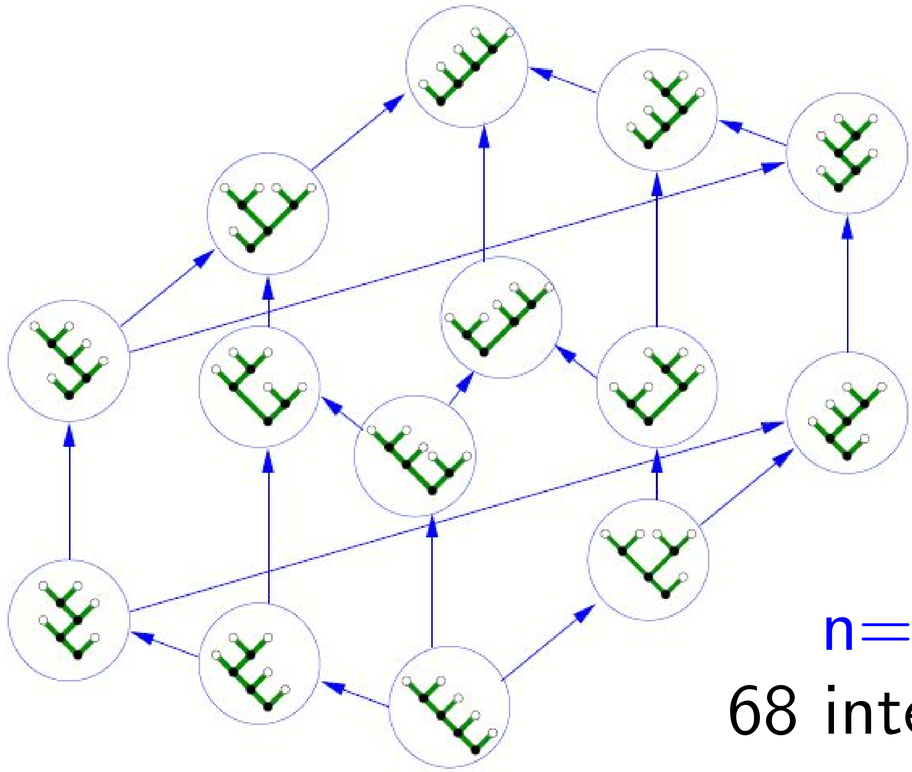

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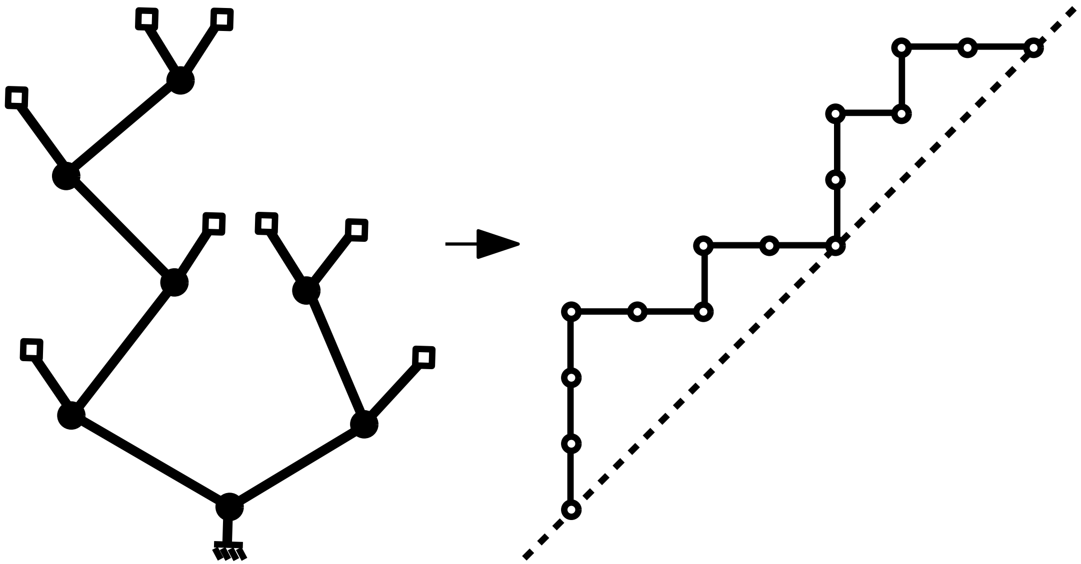
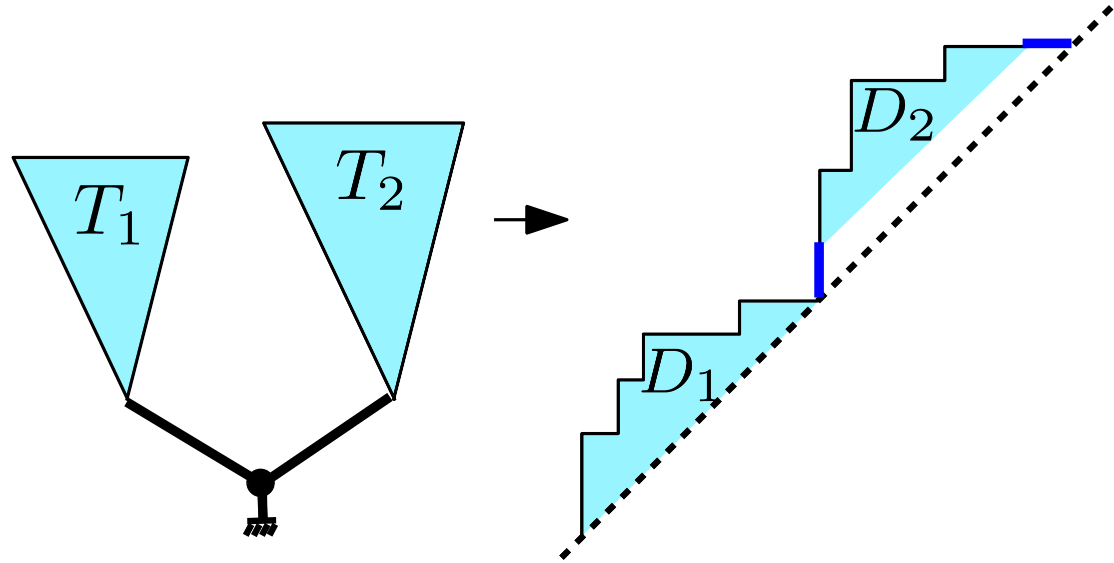
Talk overview:

- 3 proofs of Chapoton's result
- generalization to so-called "m-Tamari" lattices

Reformulation of the Tamari lattice for Dyck paths





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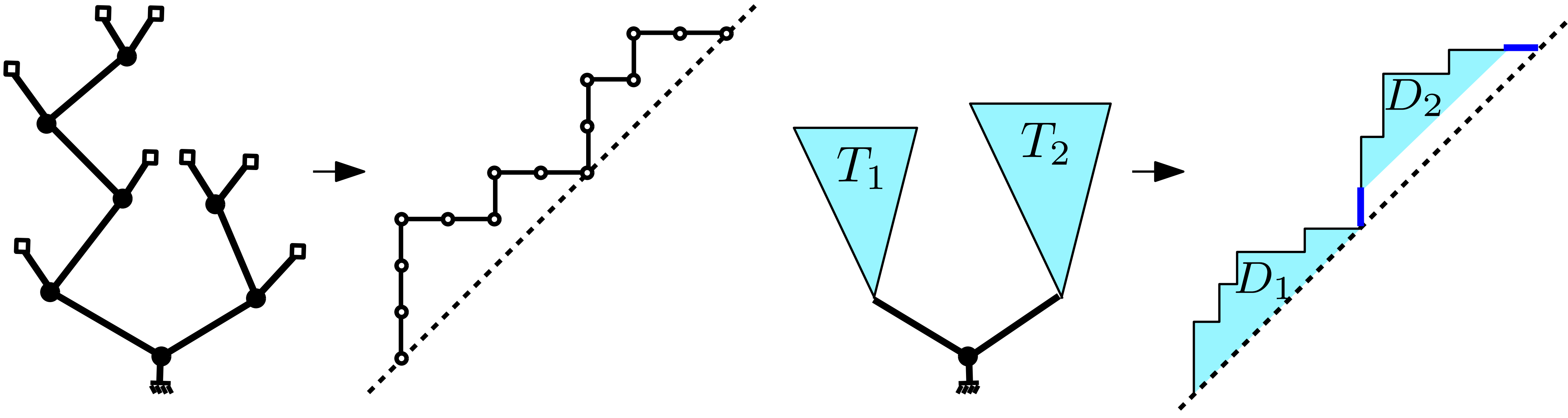

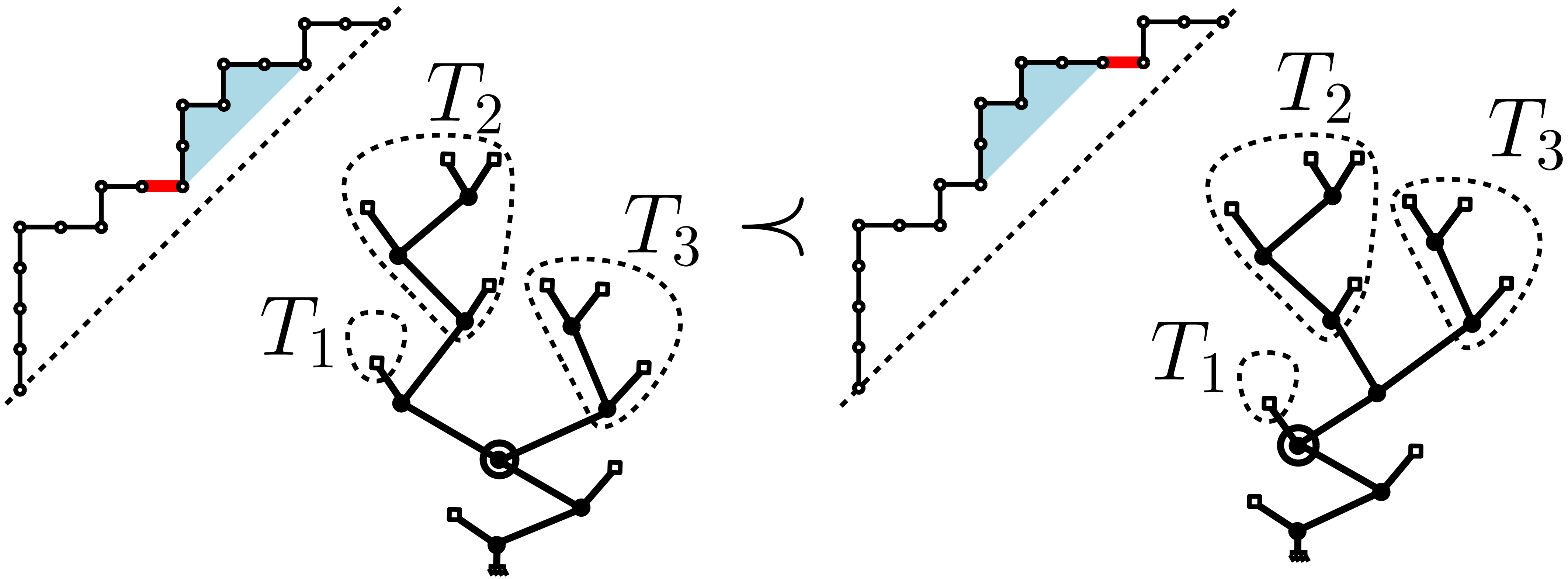

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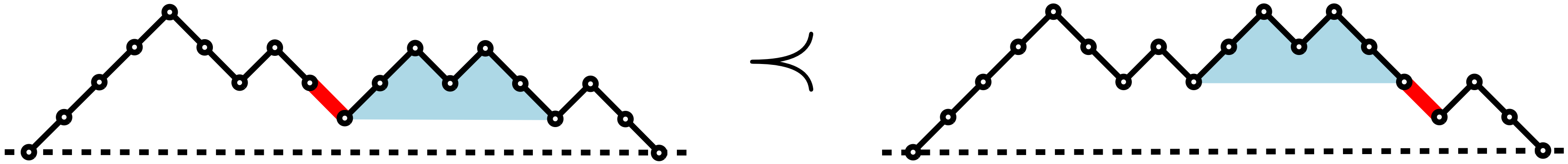

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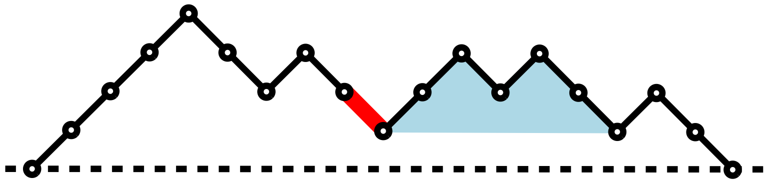

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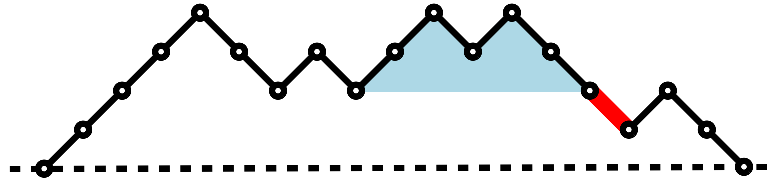
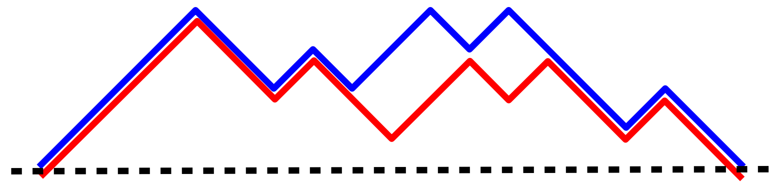


Rk: if $t \leq t'$ in $\mathcal{T}_n$, then t is below t'



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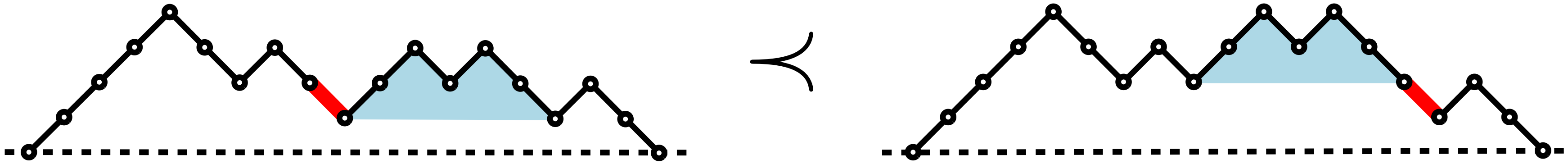

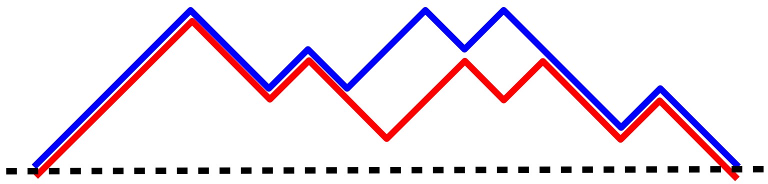

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the converse is not true !

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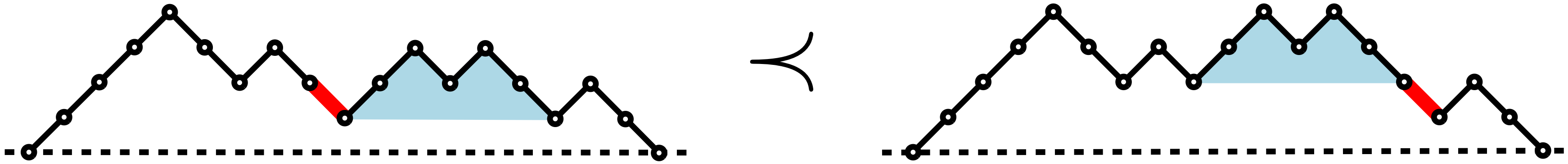

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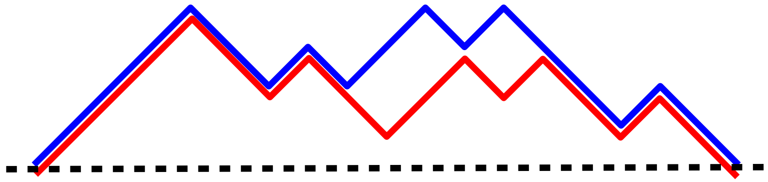
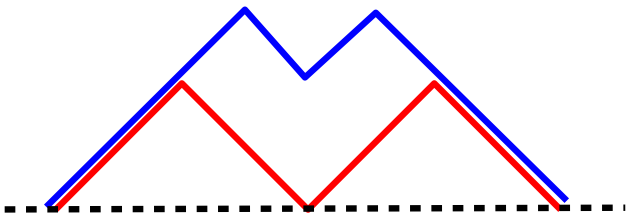


Q: How to test if a pair  is an interval in $\mathcal{T}_n$?

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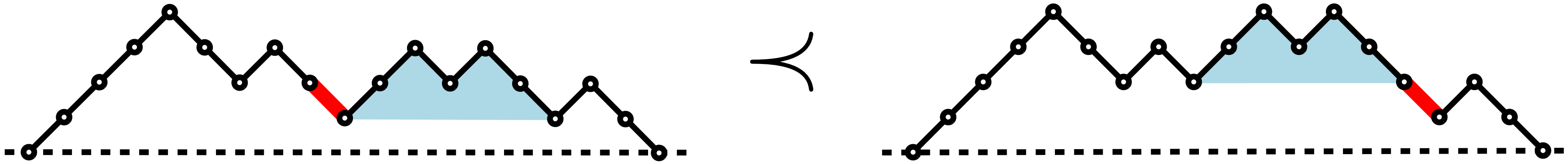

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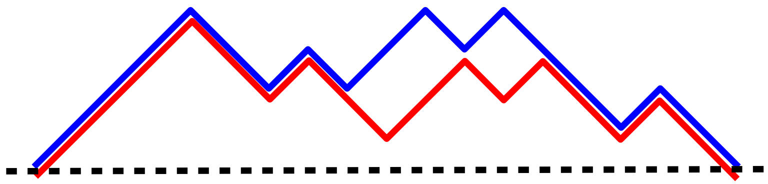

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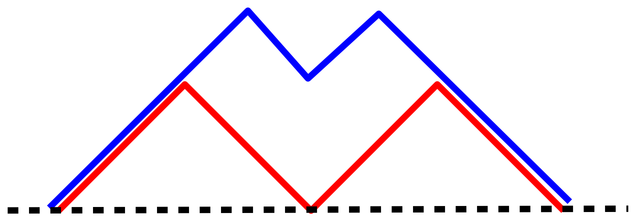

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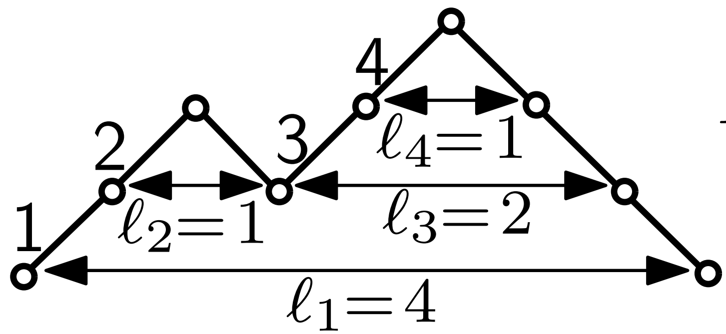
Length-vector L_D of D:



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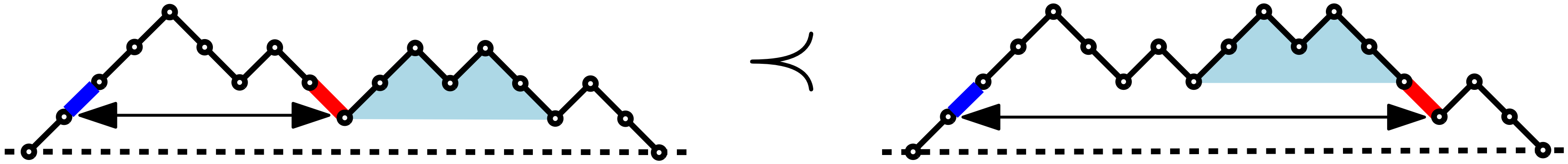

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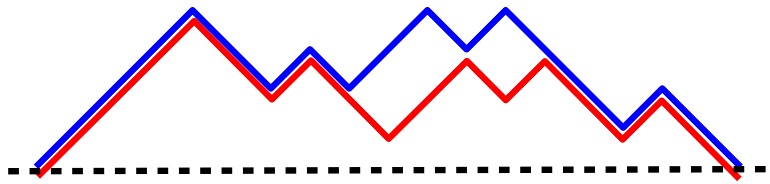

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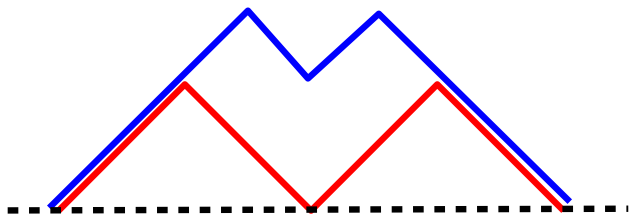

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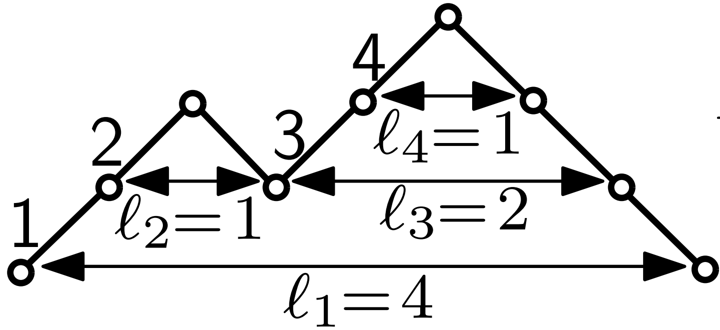

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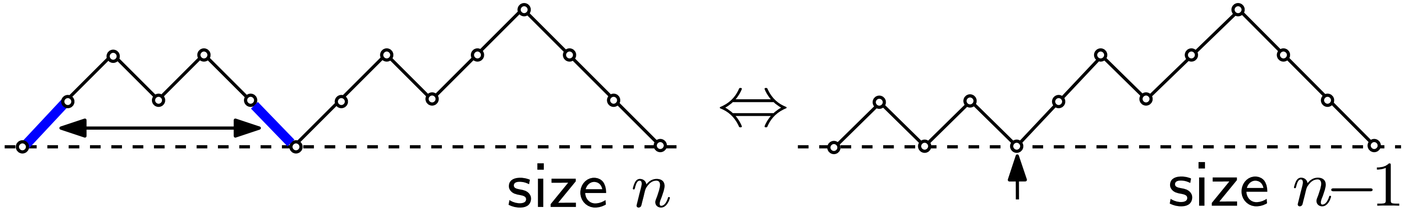
Lem: $D \leq D'$ in $\mathcal{T}_n$ iff $L_D \leq L_{D'}$

Recursive method for intervals in the Tamari lattice

- Reduction of a Dyck path:



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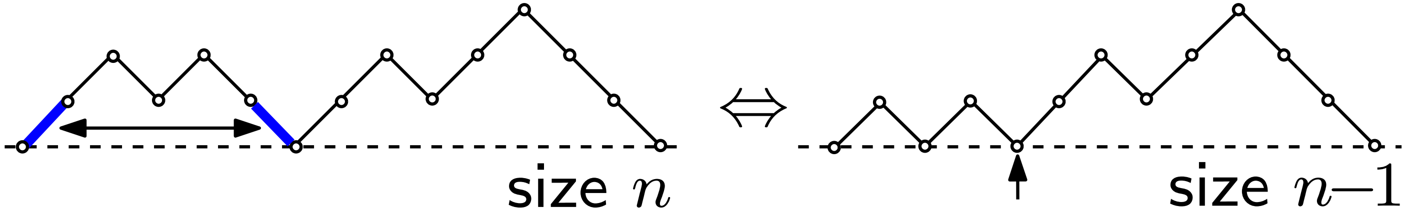

(removes 1st component in length-vector)

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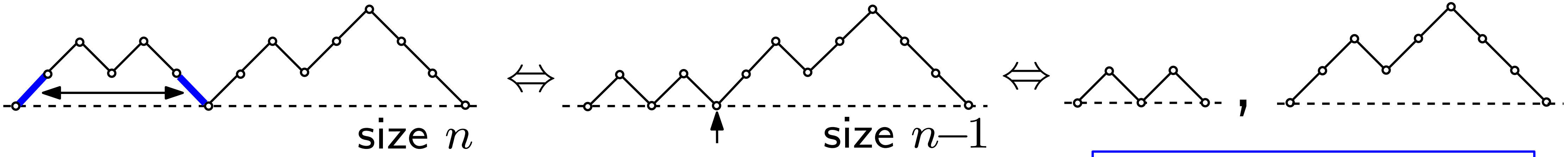

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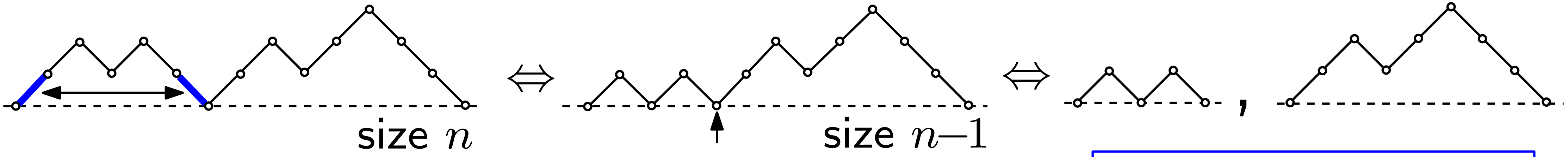

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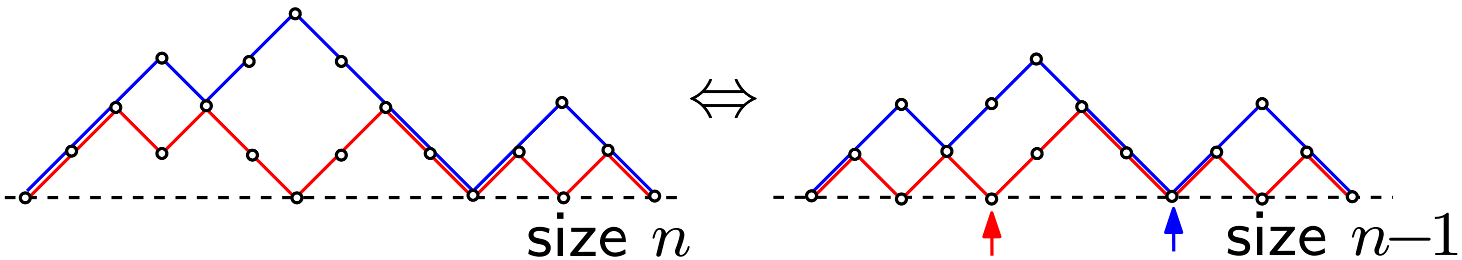

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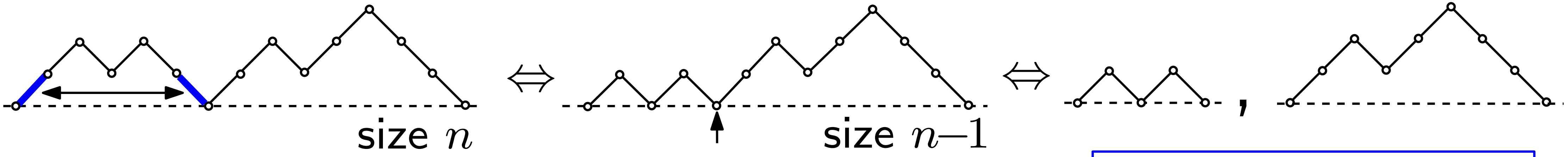

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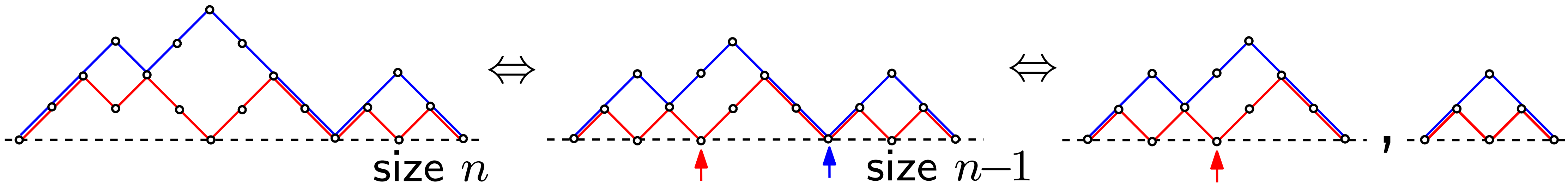

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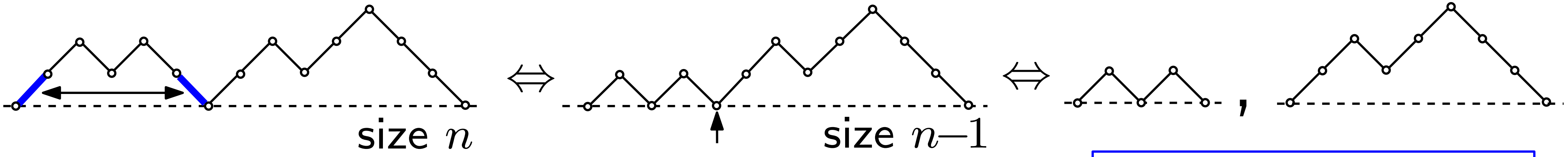

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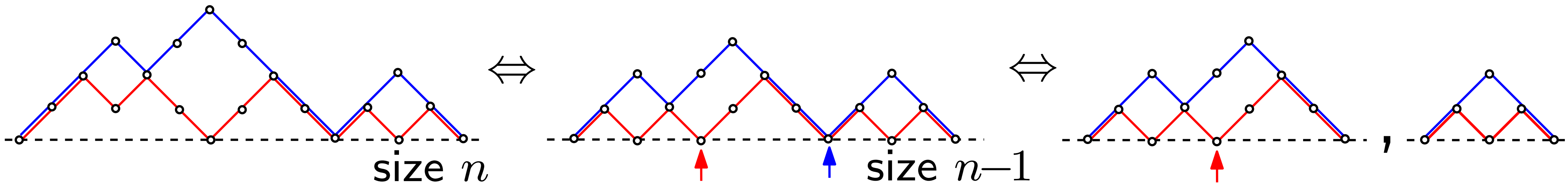

Let $a_{n,i} = \#($ intervals in $\mathcal{T}_n$ with i bottom contacts$)$

Let $F(t,x) := \sum_{n,i} a_{n,i} t^n x^i$. Then $F(t,1) = 1 + t \cdot F_x(t,1)F(t,1)$

Recursive method for intervals in the Tamari lattice

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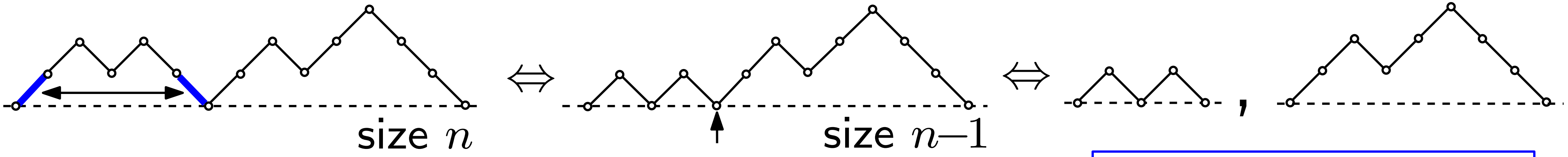

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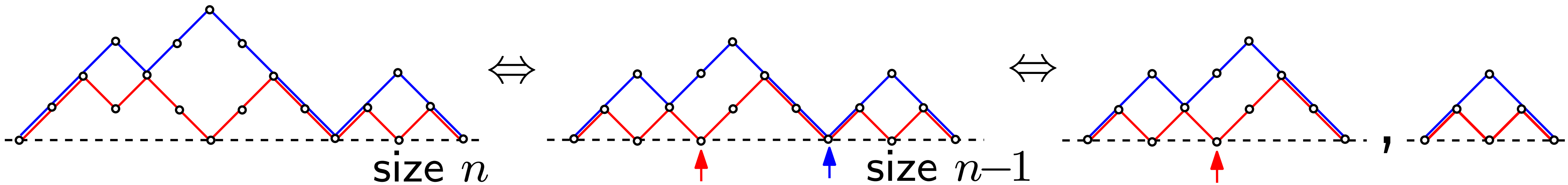

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More generally:

$$F(t,x) = x + t \cdot \mathrm{subs}(x^i = x + \cdots + x^i, F(t,x)) \cdot F(t,x)$$

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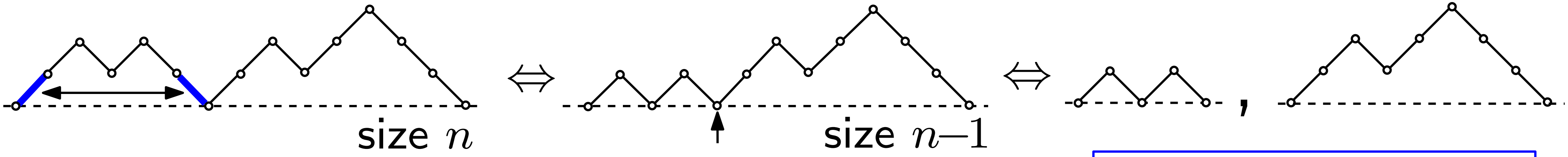

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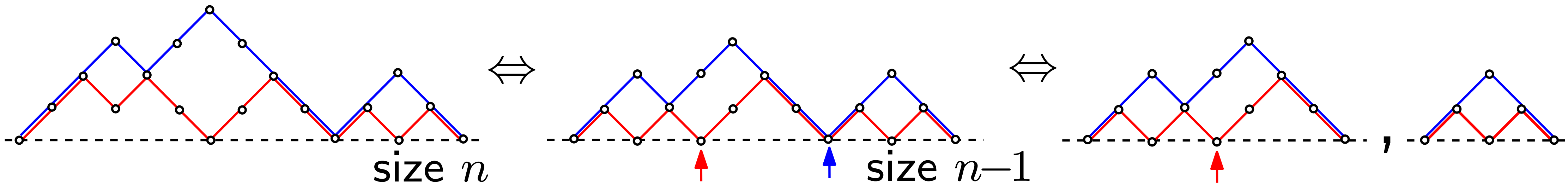

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$$\boxed{F(t,x) = x + txF(t,x)\frac{F(t,x) - F(t,1)}{x-1}}$$

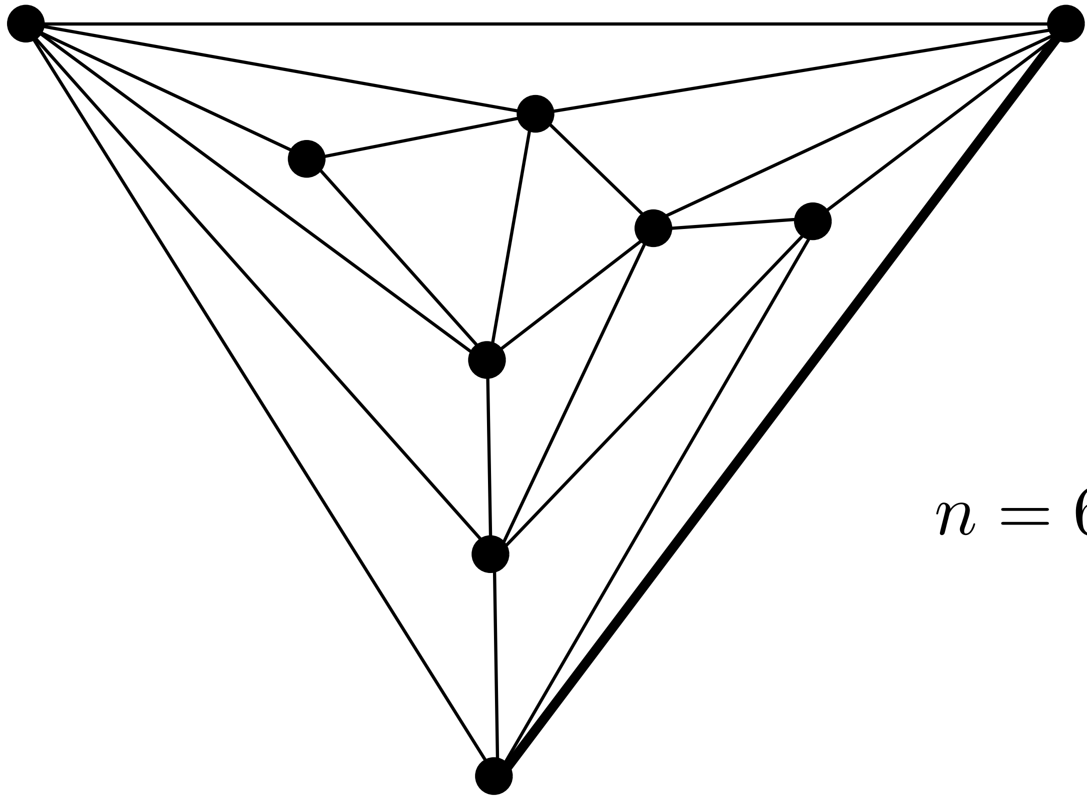
Hence the number of intervals in $\mathcal{T}_n$ is $[t^n]F(t,1)$, with

$$\boxed{F(t,x) = x + txF(t,x)\frac{F(t,x) - F(t,1)}{x-1}}$$

We present 3 methods for finding $[t^n]F(t,1)$ from the equation above:

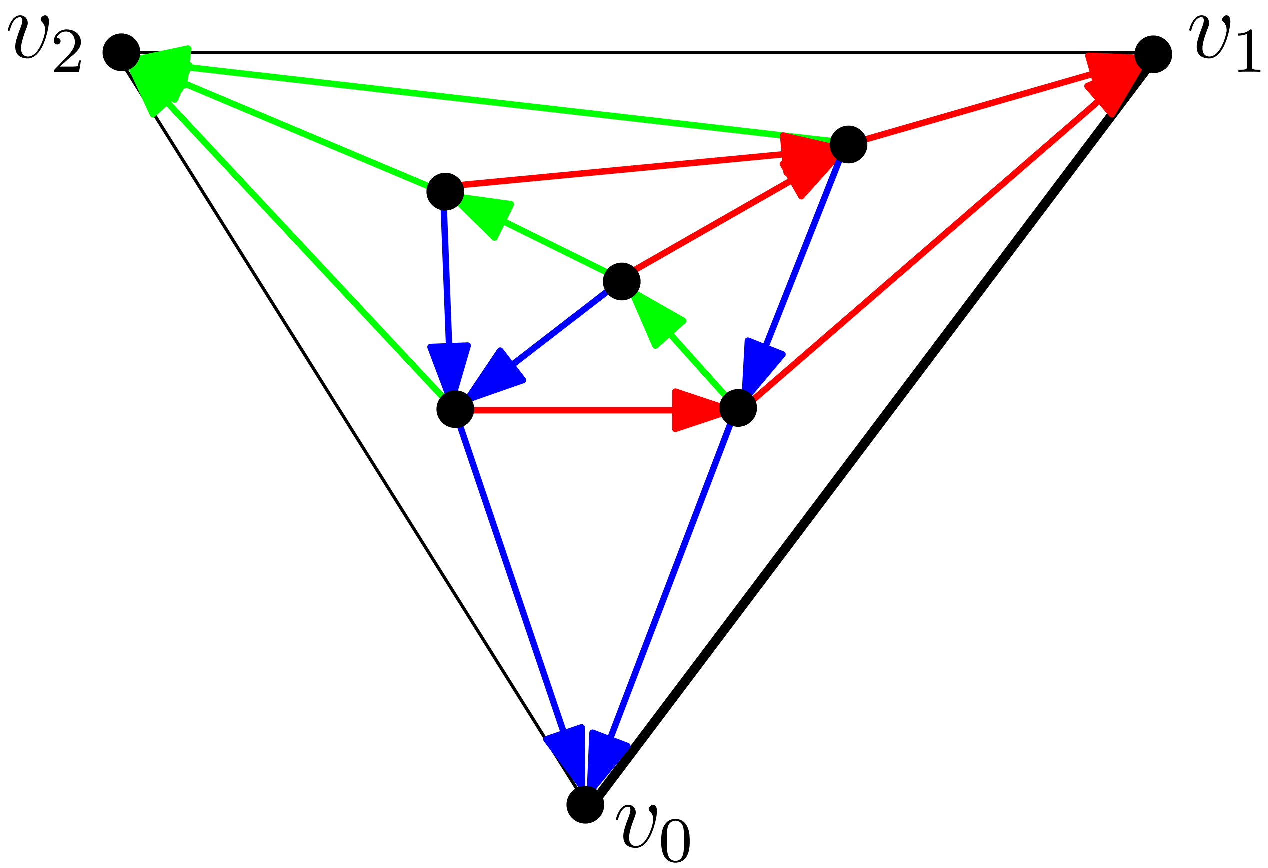
- bijective
- Solve the equation using the quadratic method
- Solve the equation by guessing/checking

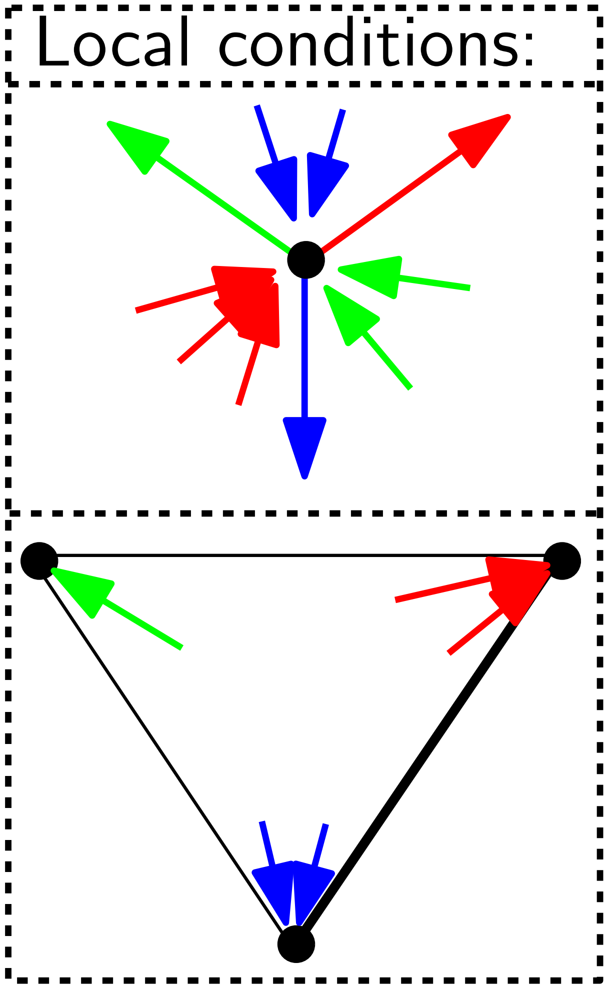
Triangulations



$n = 6$ internal vertices

Schnyder woods





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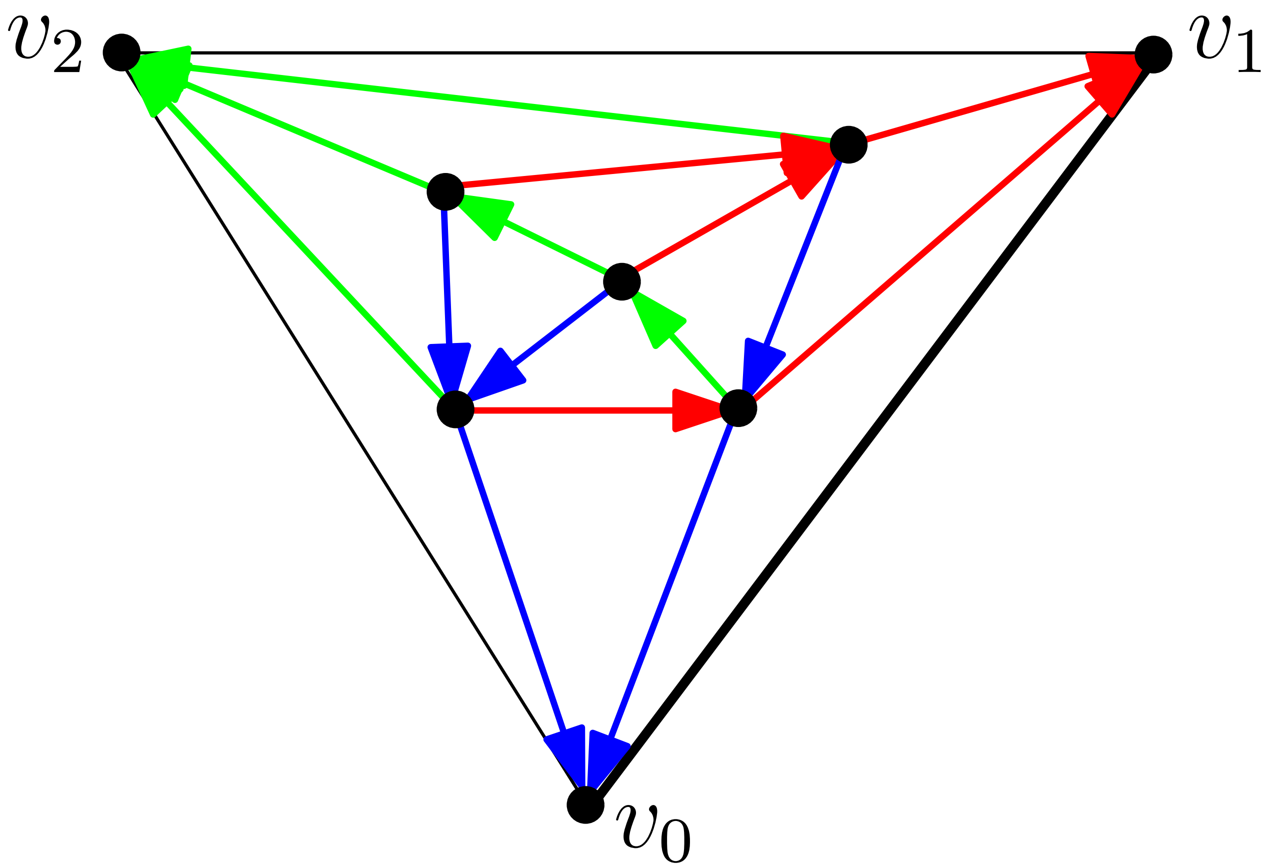

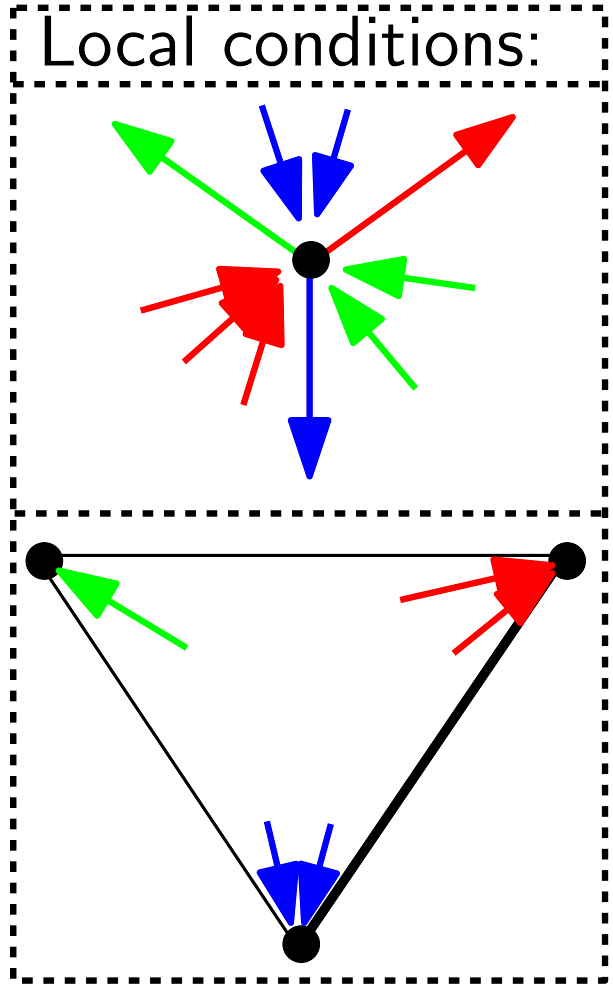

The Schnyder wood is called minimal if it has no clockwise circuit

Theo [Schnyder'90, Brehm'03]:

Any triangulation has a unique minimal Schnyder wood

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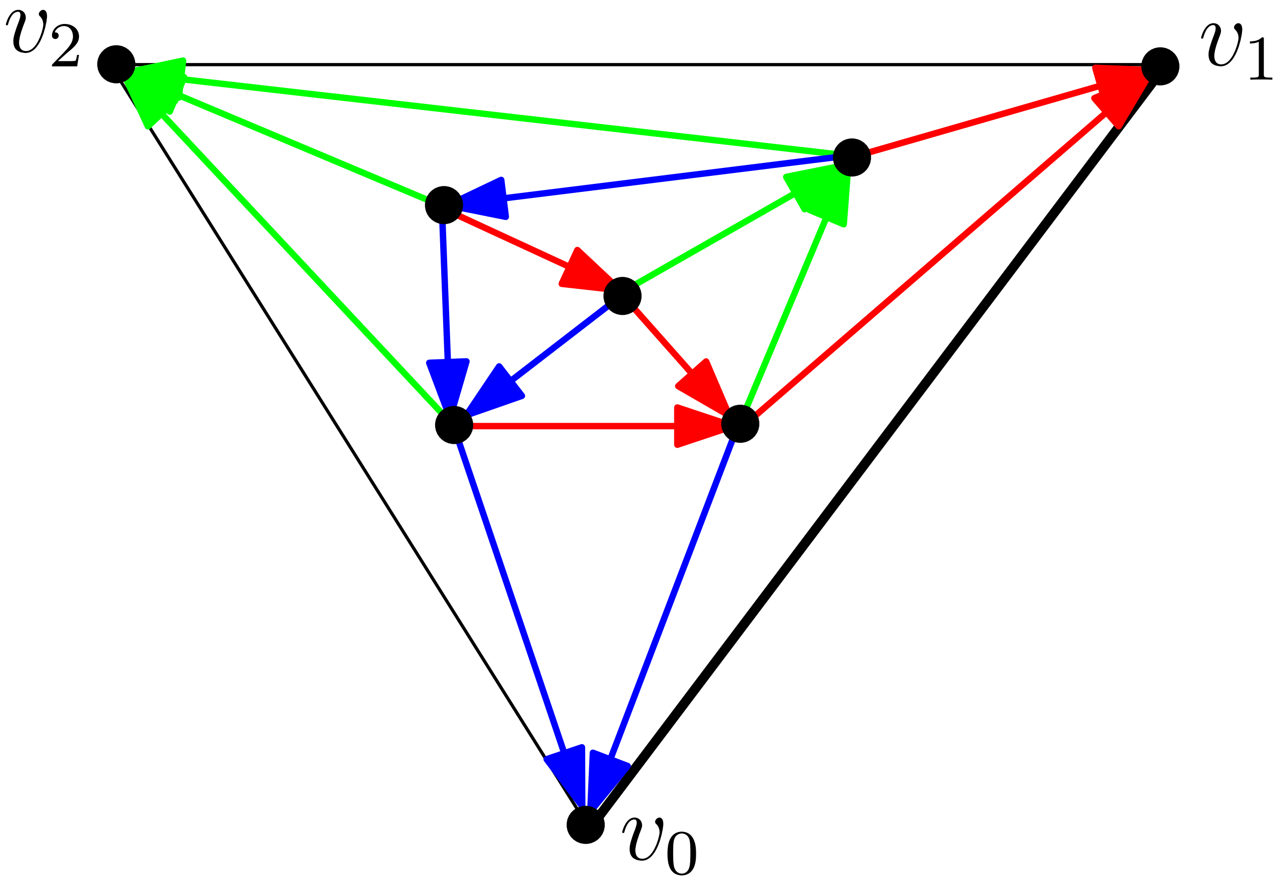

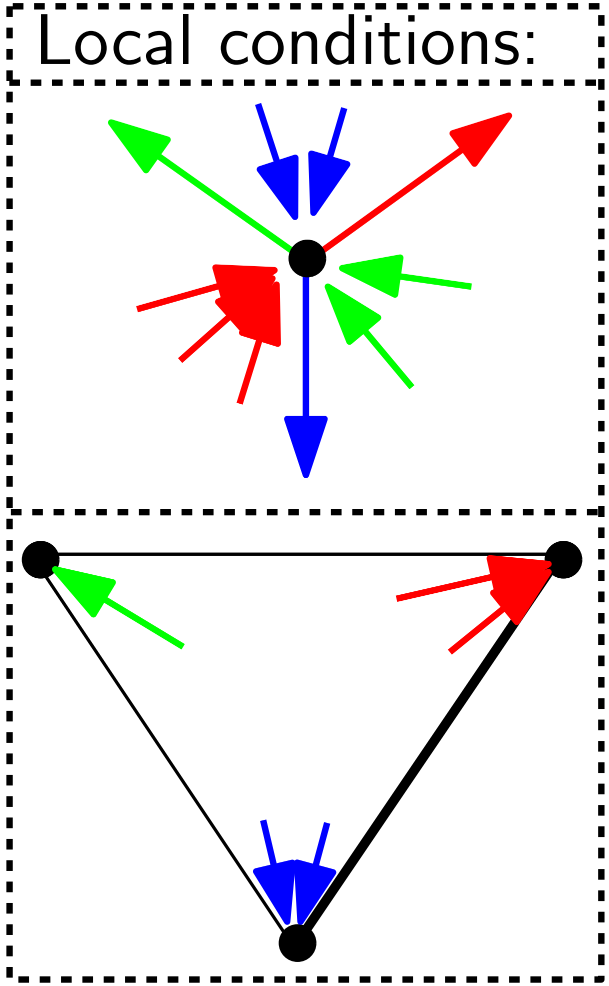

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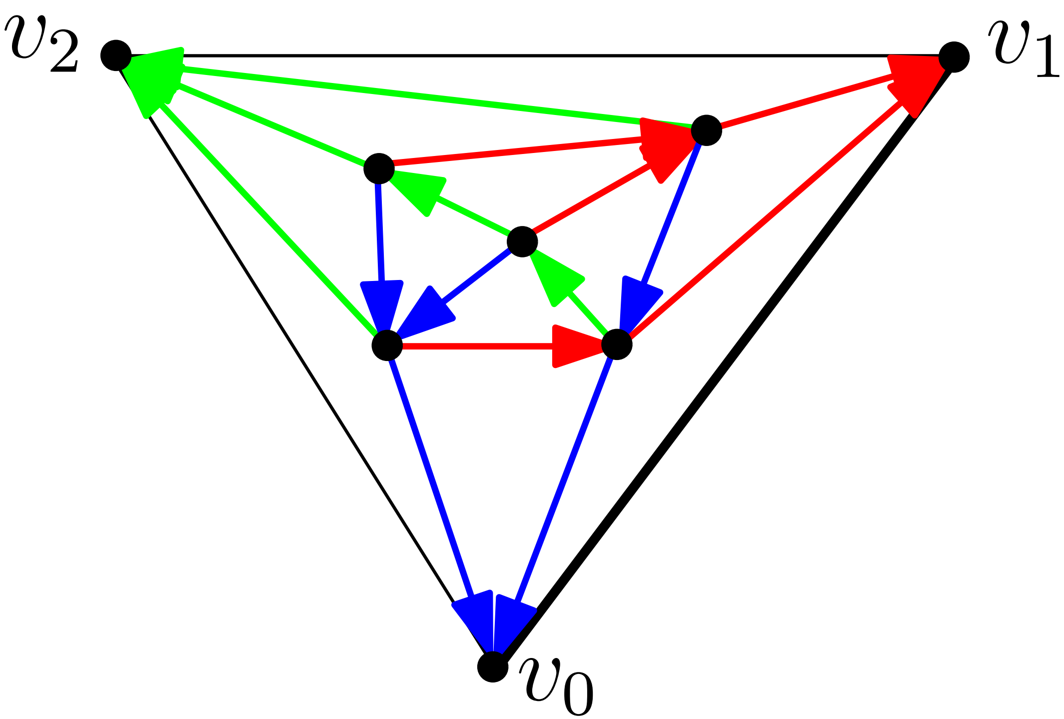
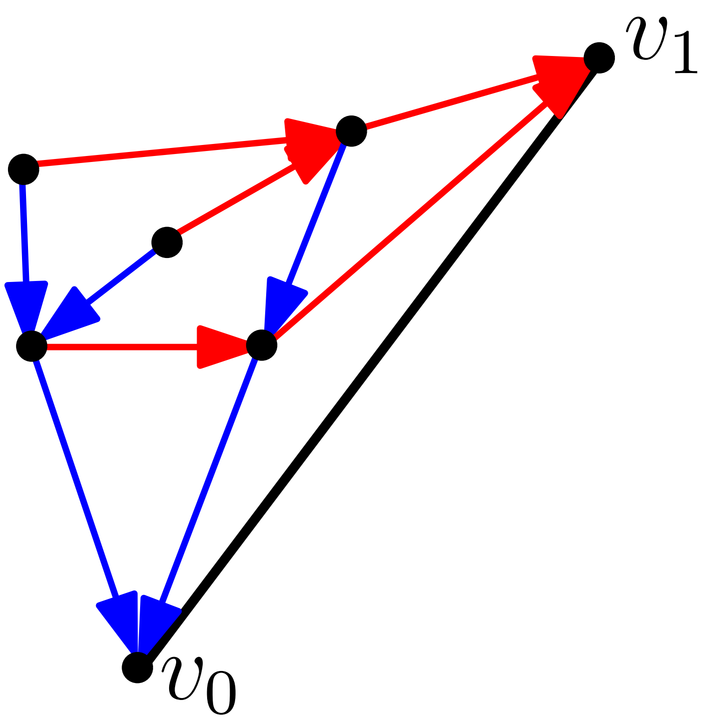
Any triangulation has a unique minimal Schnyder wood

The red-blue induced structure





There is no loss of information in deleting the green edges

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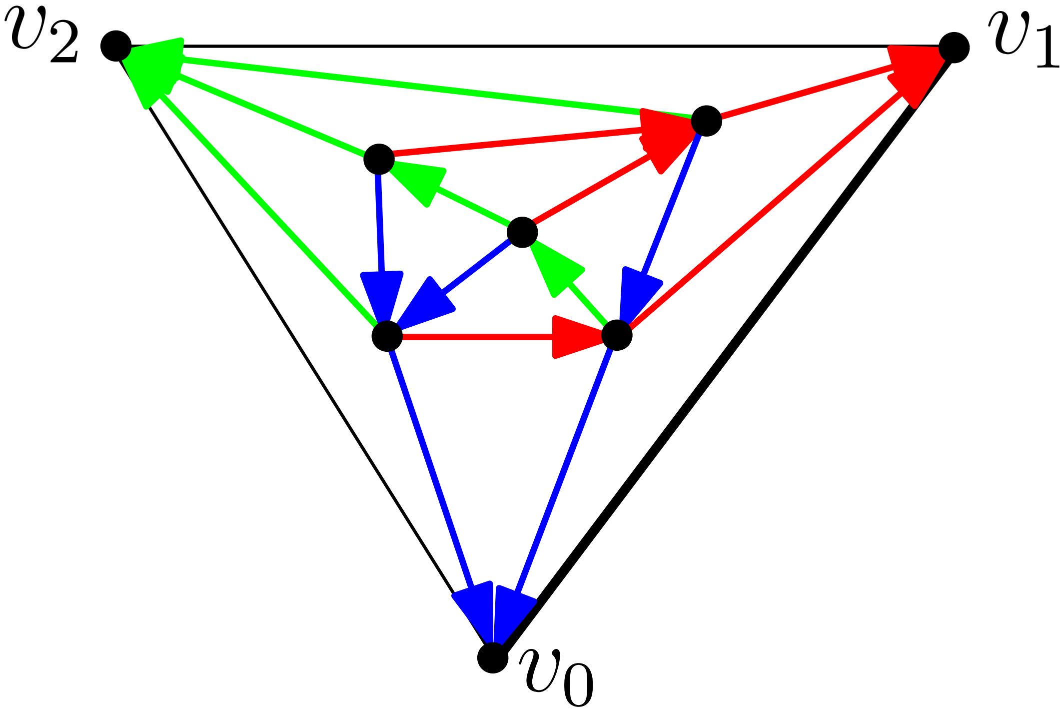

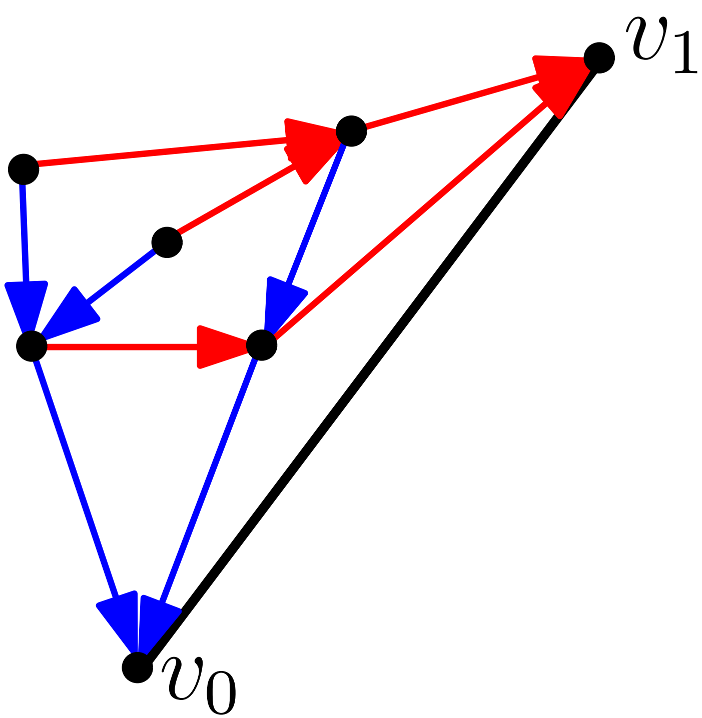

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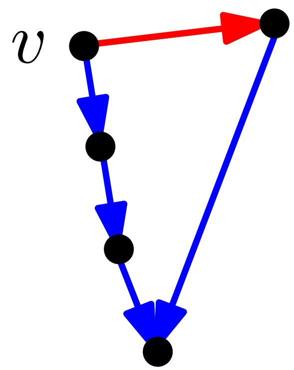
no clockwise circuit
(i.e., minimal)

$\Leftrightarrow$



$\forall v$ interval vertex
the blue parent of
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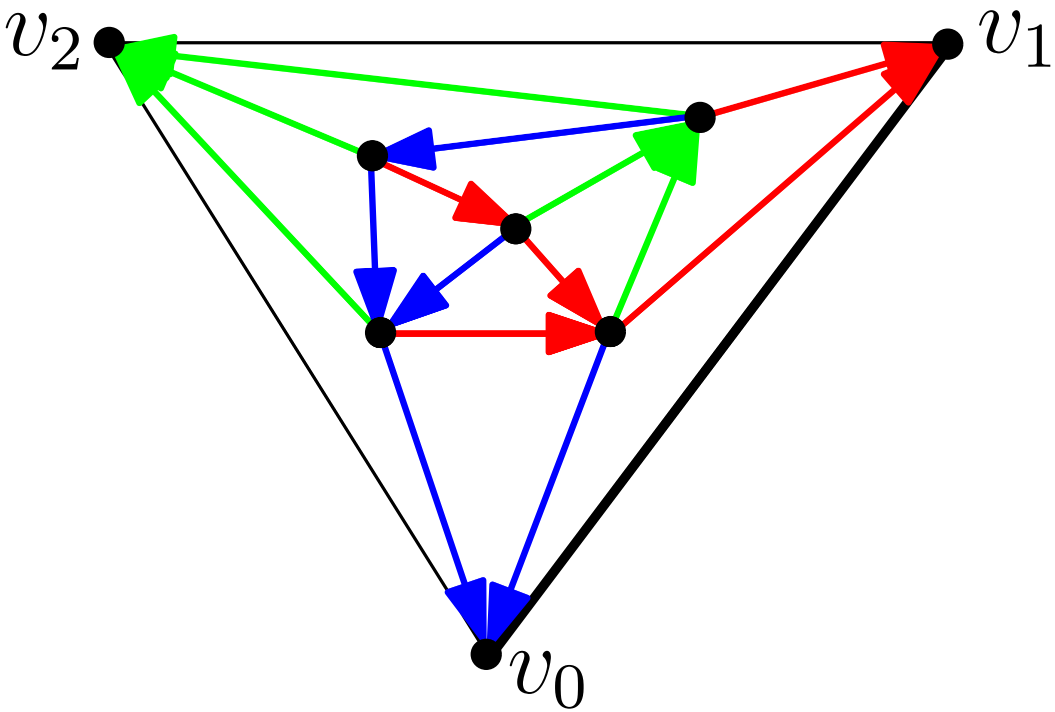

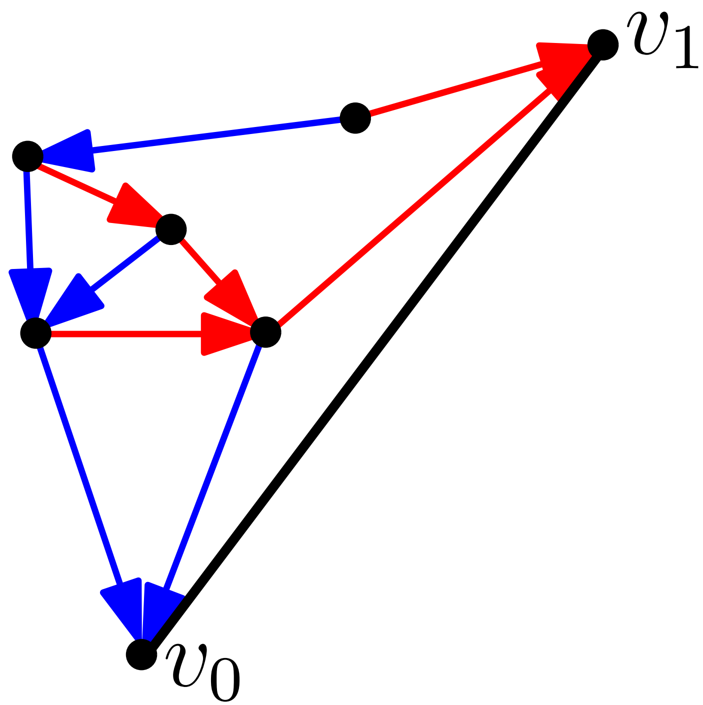

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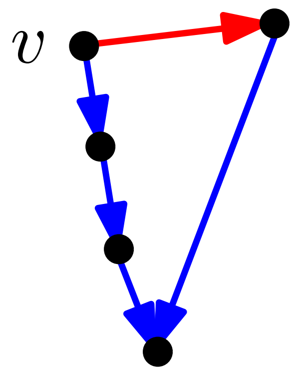
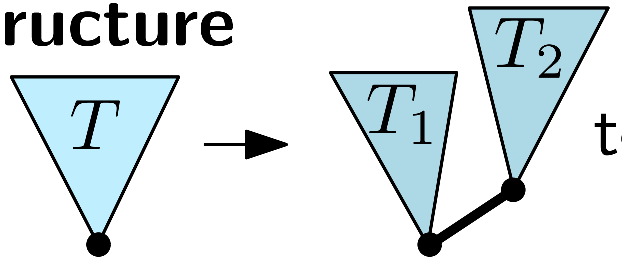


$\forall v$ interval vertex the blue parent of the red parent is a blue ancester

Decomposing a minimal red-blue structure

cf L.-F. Préville-Ratelle. Idea: apply  to the blue tree

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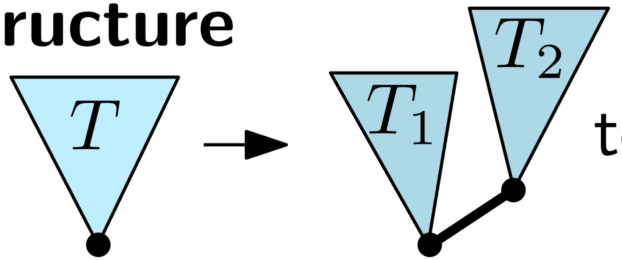
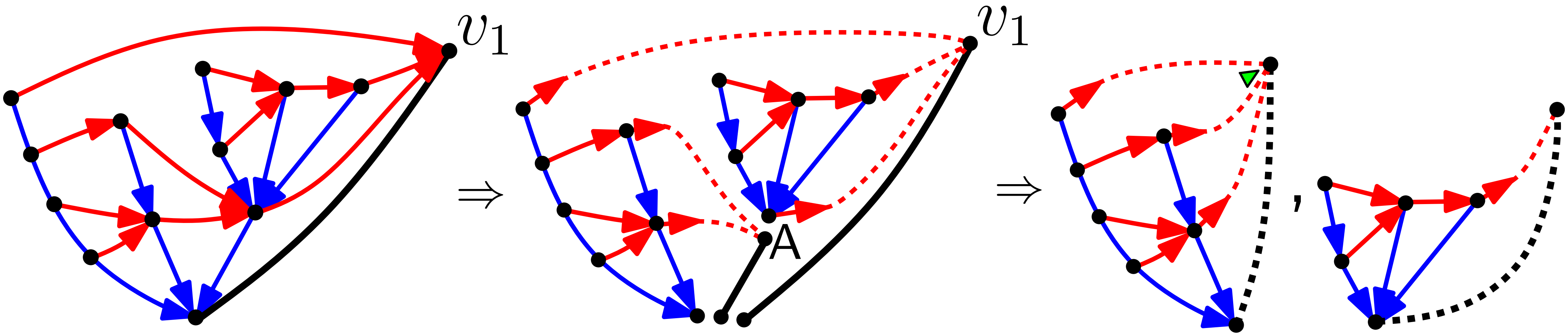
cf L.-F. Préville-Ratelle. Idea: apply  to the blue tree



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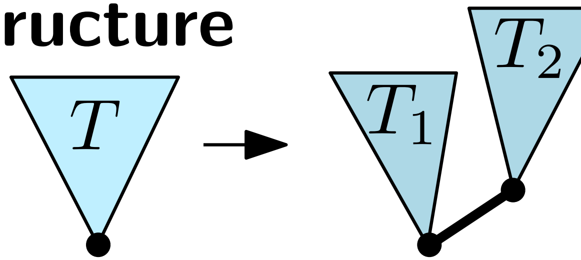

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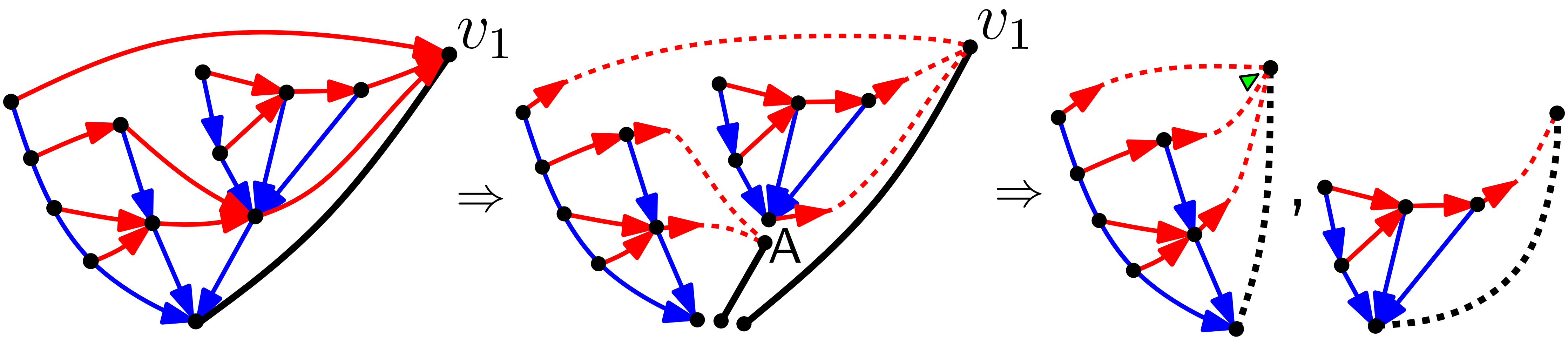

Let $a_{n,i} = \#($ triangulations with $n+3$ vertices, $\deg(v_1) = i+1$ $)$

Let $F(t,x) := \sum_{n,i} a_{n,i} t^n x^i$.

Then
$$F(t,x) = x + txF(t,x)\frac{F(t,x) - F(t,1)}{x-1}$$

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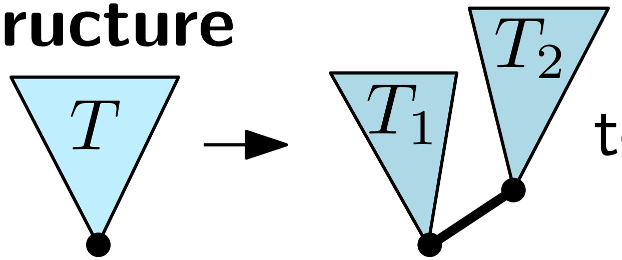
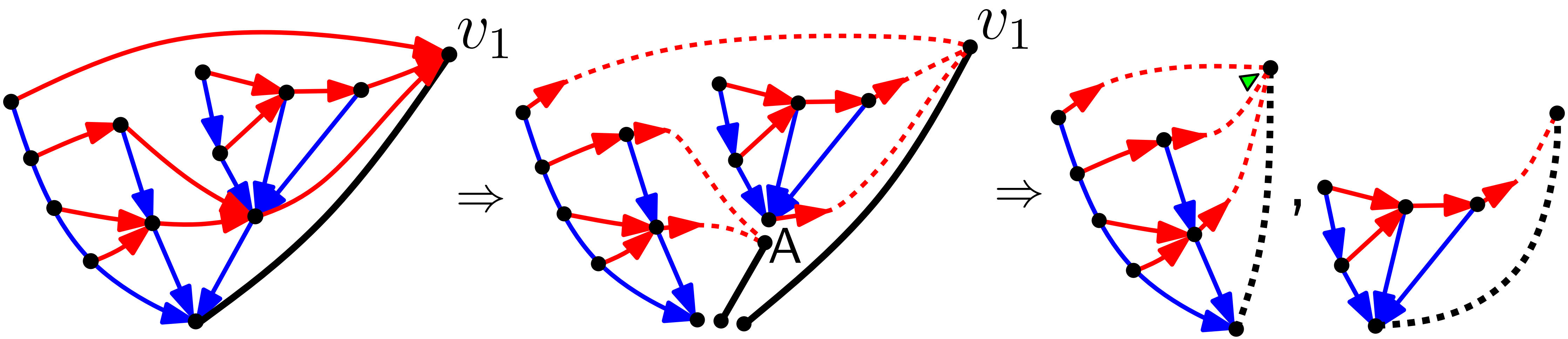
cf L.-F. Préville-Ratelle. Idea: apply  to the blue tree



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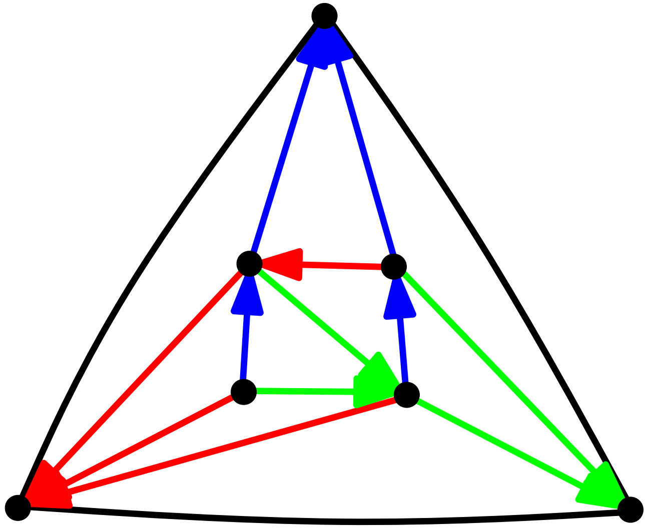
$\Rightarrow$ #(intervals in $\mathcal{T}_n$) = #(triangulations n internal vertices)

(also direct bijection in [Bernardi,Bonichon'09])

Bijective counting of triangulations

Moreover, minimal Schnyder woods give a bijection to count triangulations



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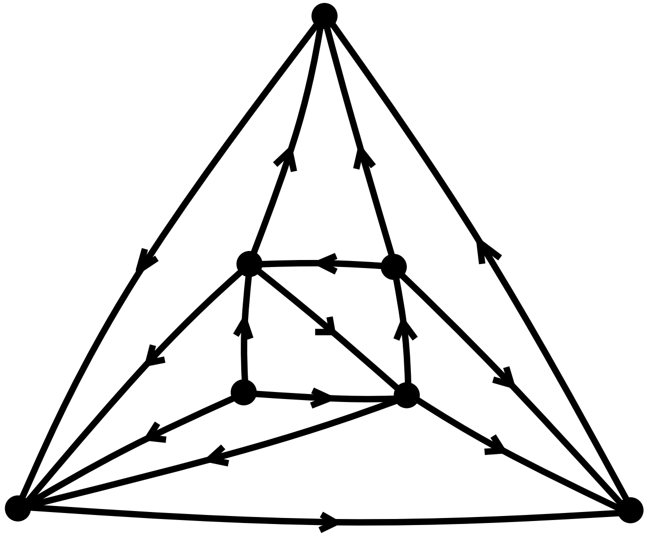

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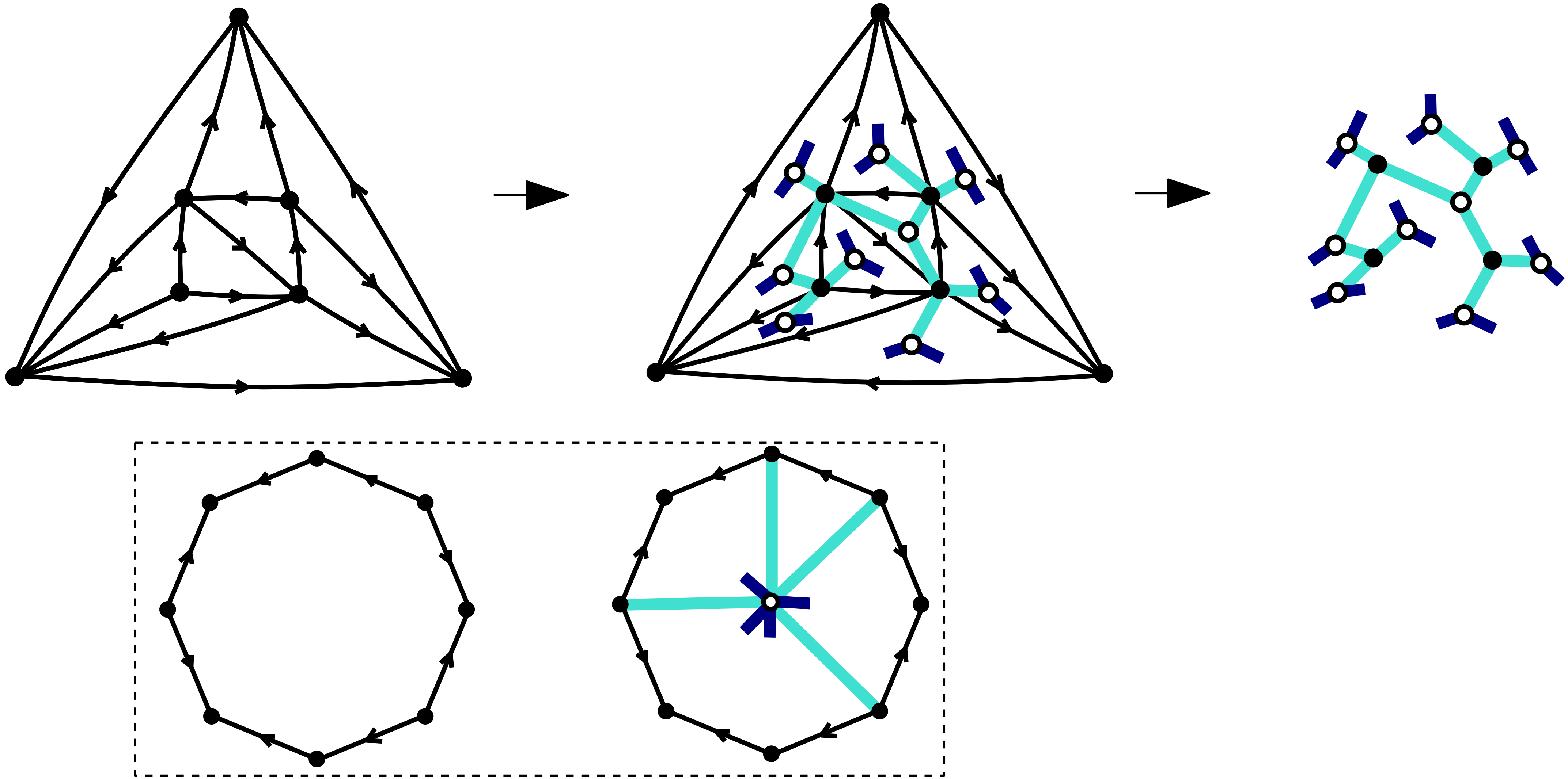

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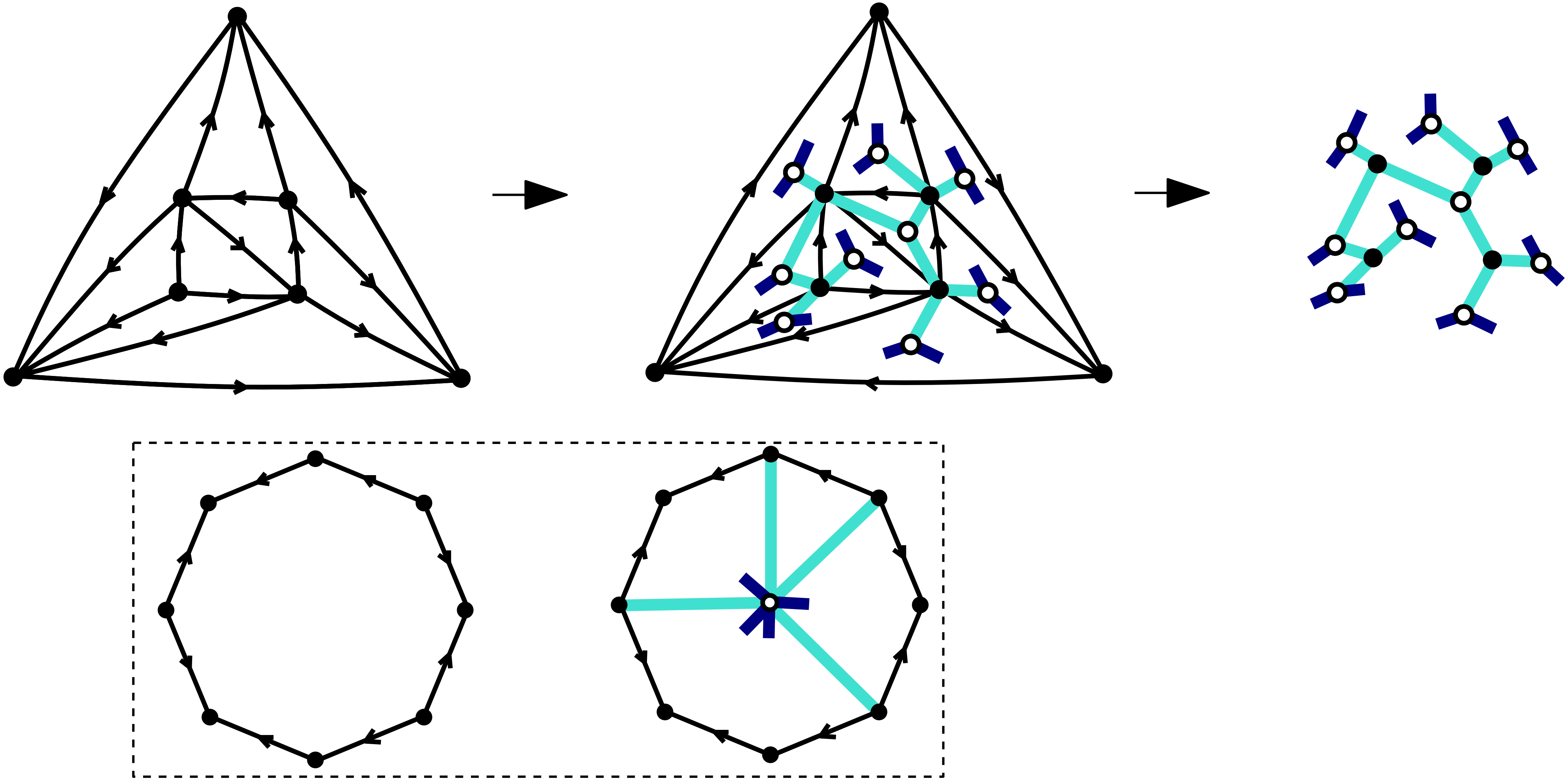

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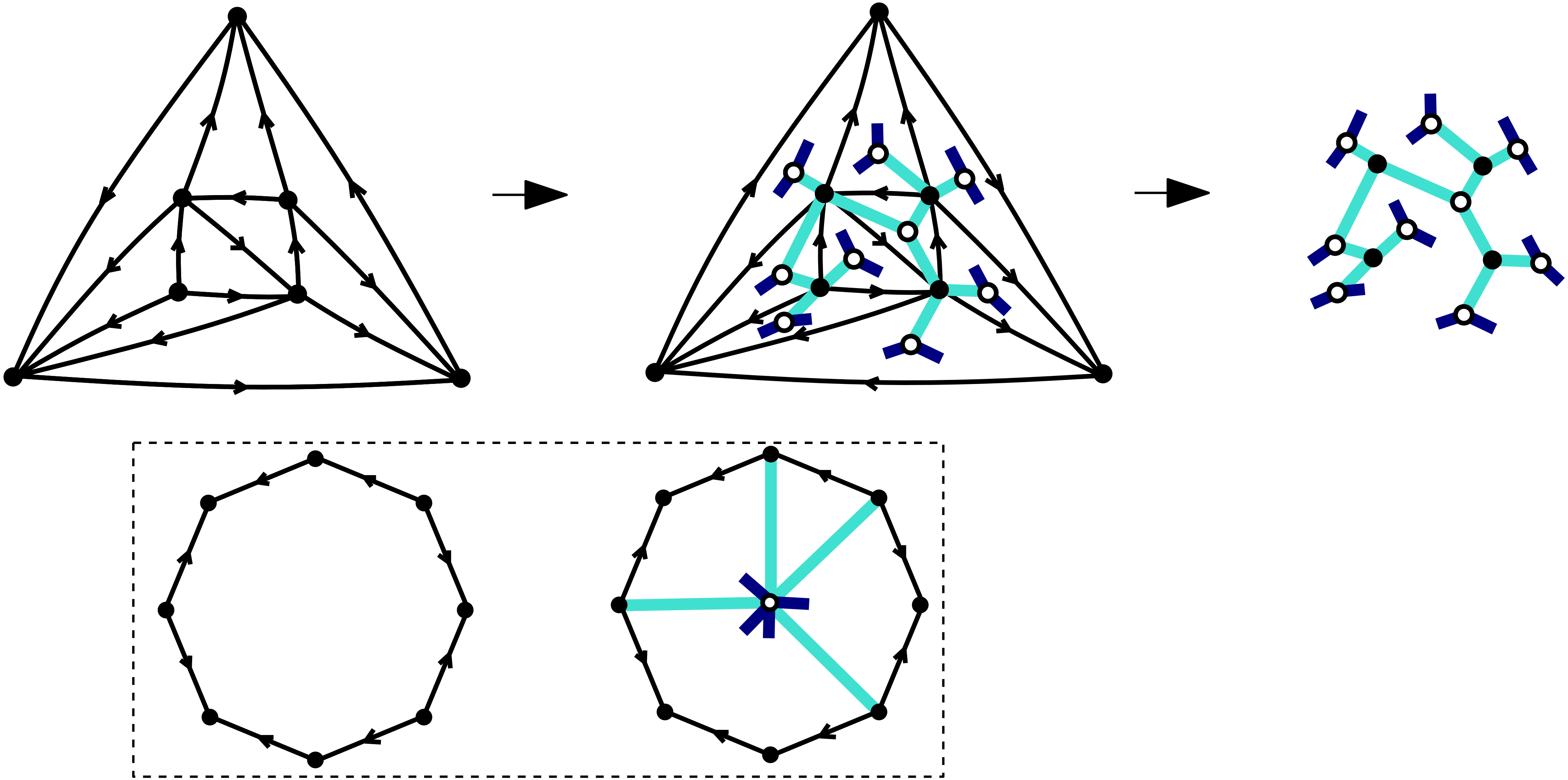

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& intervals in $\mathcal{T}_n$

The quadratic method [Tutte, Brown'60s, Bousquet-Mélou,Jehanne'06]

Method to solve directly

$$\boxed{F(t,x) = x + txF(t,x)\frac{F(t,x) - F(t,1)}{x-1}} \quad \text{(E)}$$

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$$\Rightarrow \boxed{[t^n]f = \tfrac{2}{n(n+1)}\tbinom{4n+1}{n-1}}$$ (with gfun)

Guessing/checking

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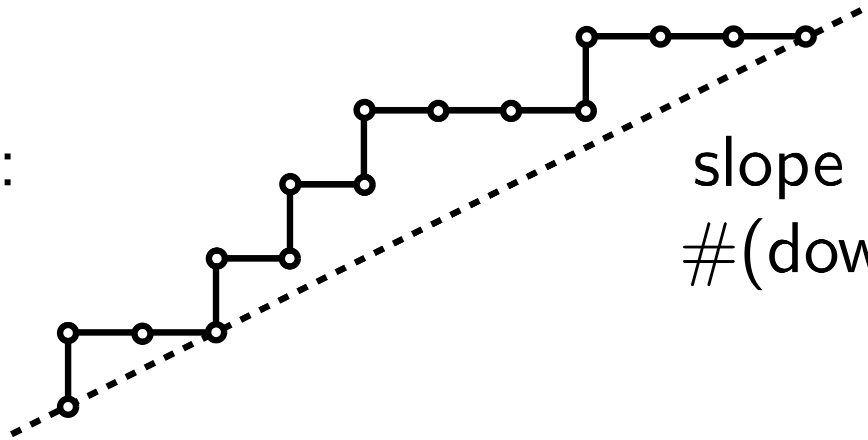
(with Lagrange inversion formula)

Generalization to "m-Tamari lattices", for any $m \geq 1$

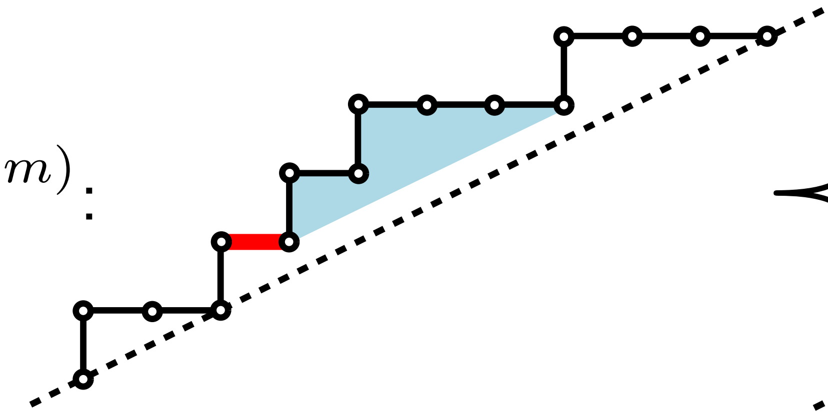
m-Dyck path:



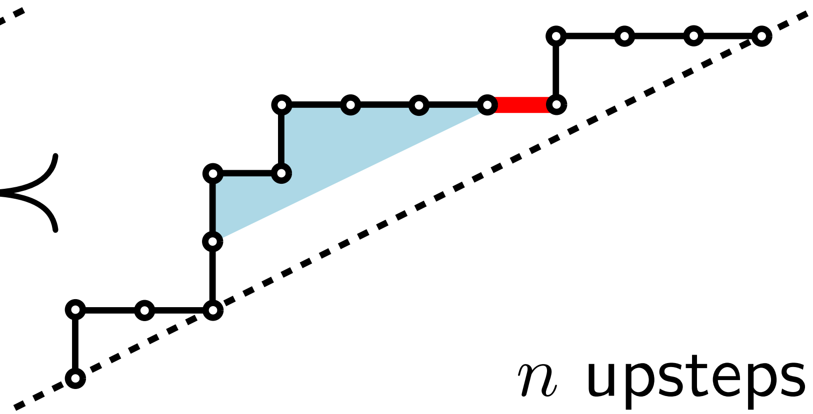
slope $= 1/m$

#(downsteps)$= m \cdot$ #(upsteps)

m-Tamari lattice $\mathcal{T}_n^{(m)}$:



$\prec$



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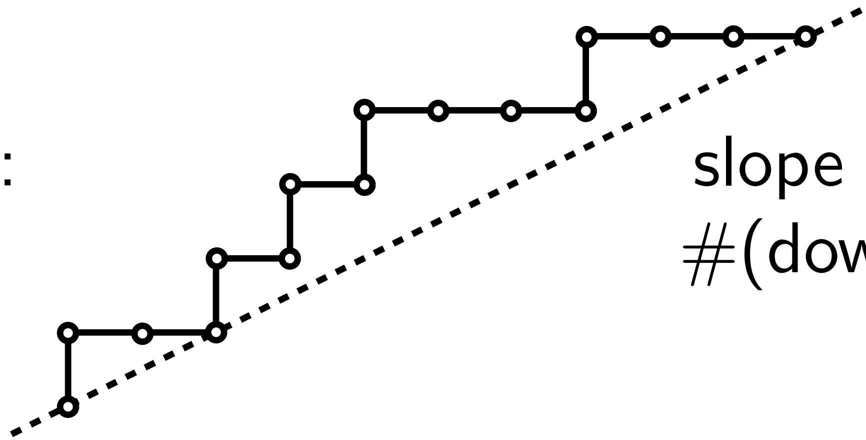

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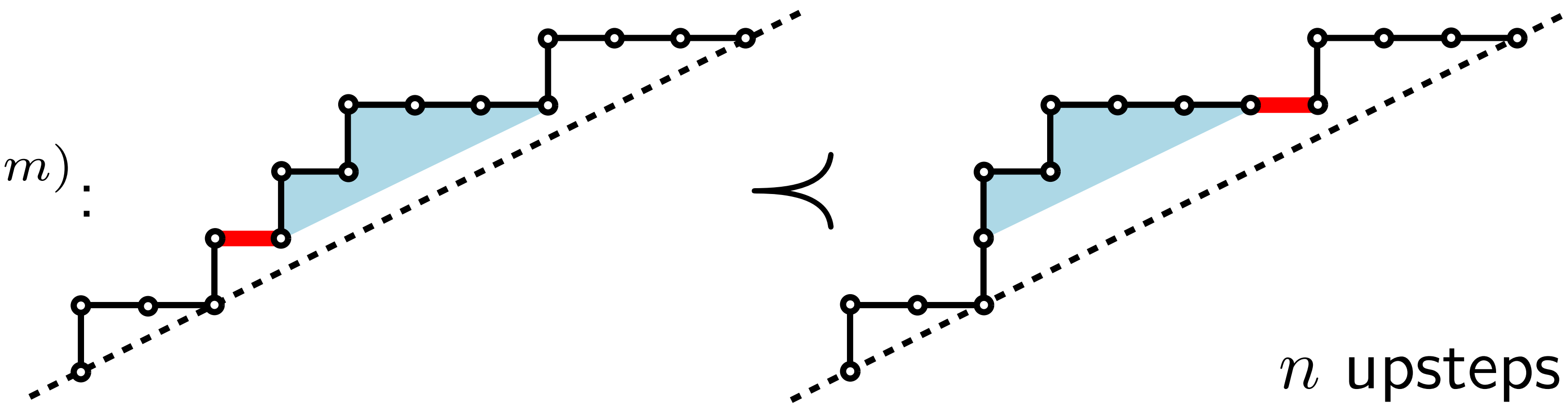

Conjecture [Bergeron]: $\mathcal{T}_n^{(m)}$ has $\dfrac{m+1}{n(mn+1)}\dbinom{(m+1)^2 n + m}{n-1}$ intervals

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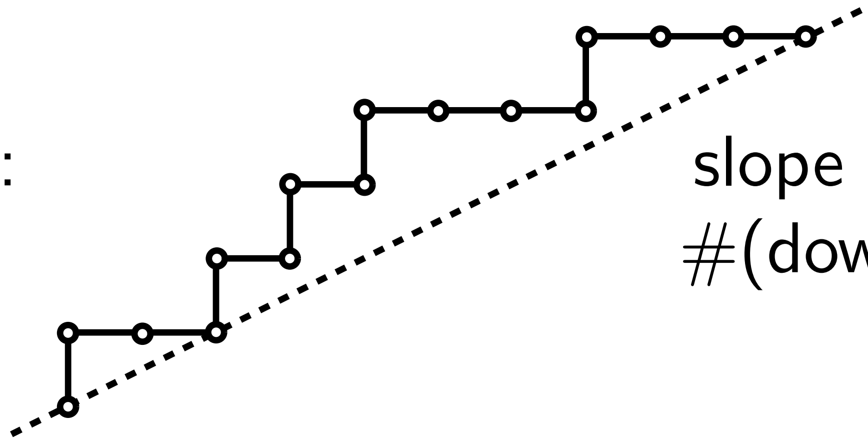

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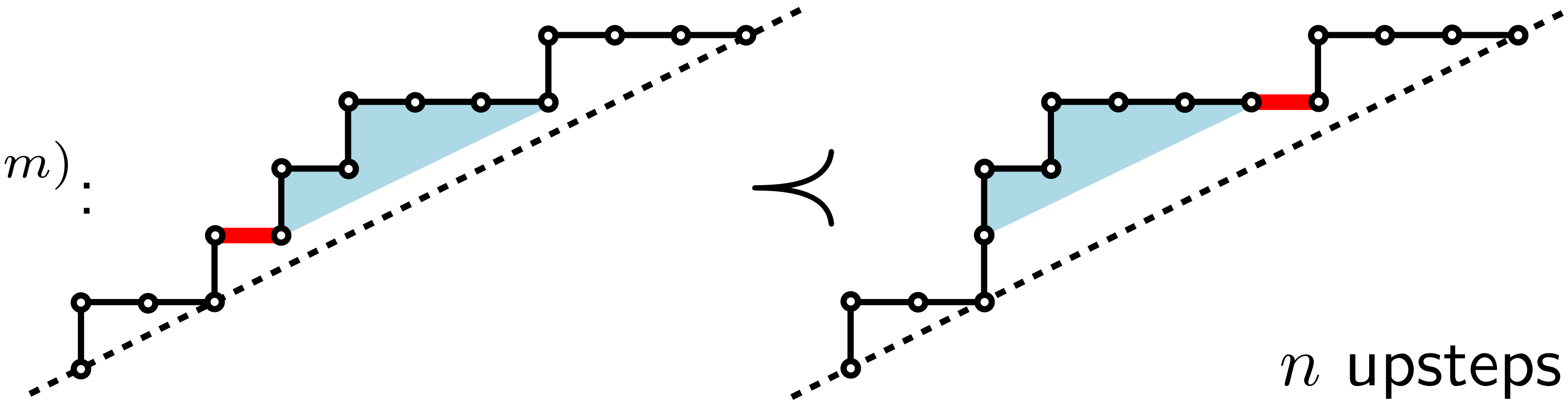

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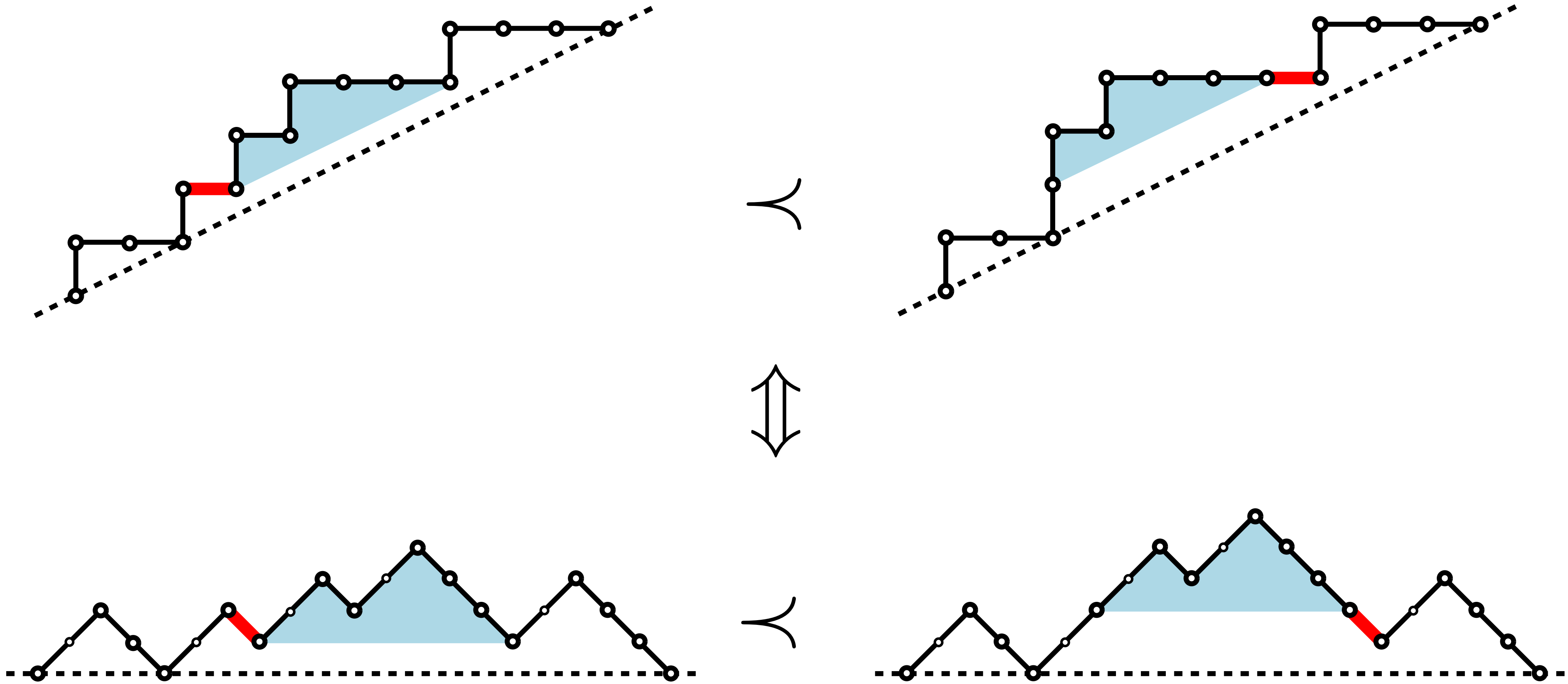
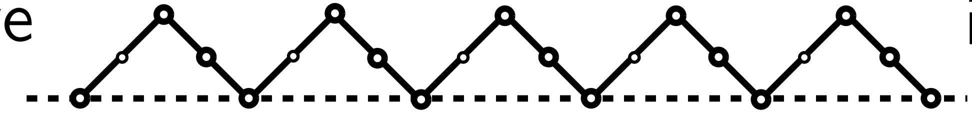
Now proved in [Bousquet-Mélou, F, Préville-Ratelle'11]

A slight reformulation

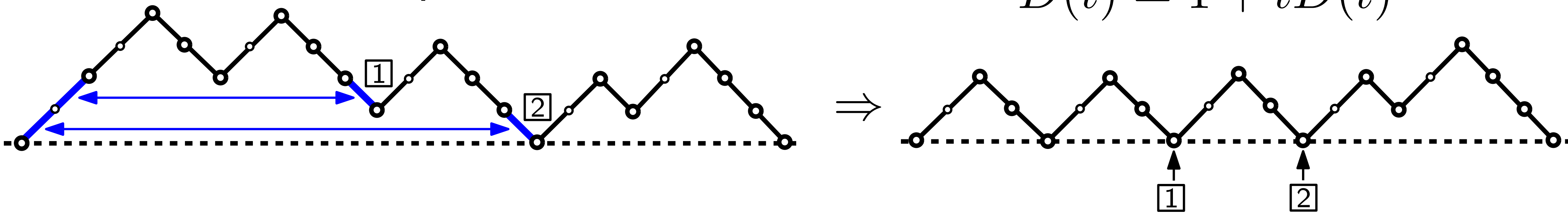


$\mathcal{T}_n^{(m)} \simeq$ sublattice of paths above  in $\mathcal{T}_{nm}$

The functional equation for $m \geq 1$

- Reduction of one path:

$$D(t) = 1 + tD(t)^3$$



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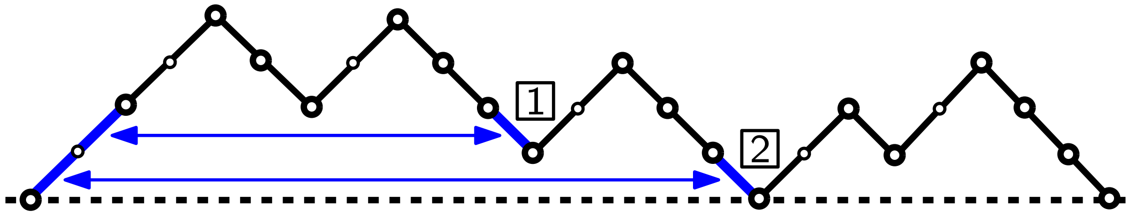
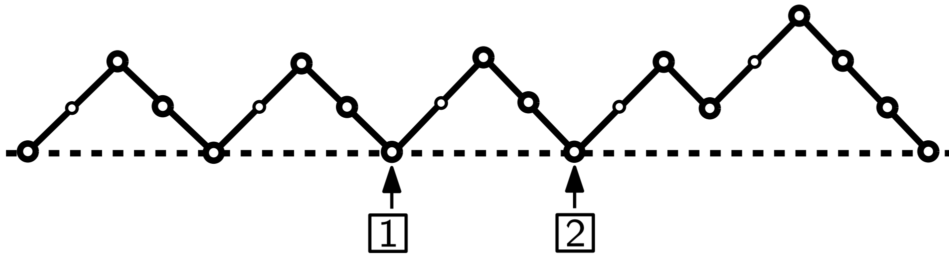
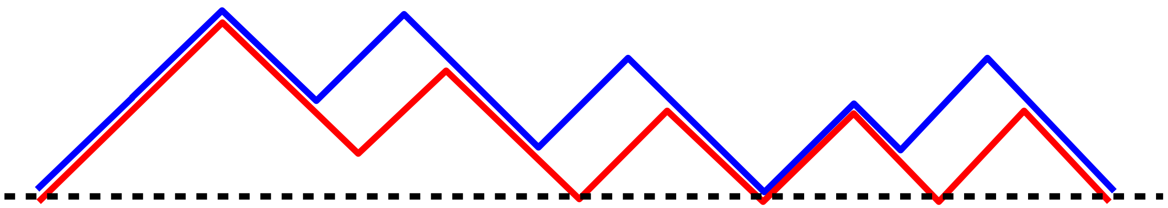


$$D(t) = 1 + tD(t)^3$$

$\Rightarrow$



- Reduction of an interval:



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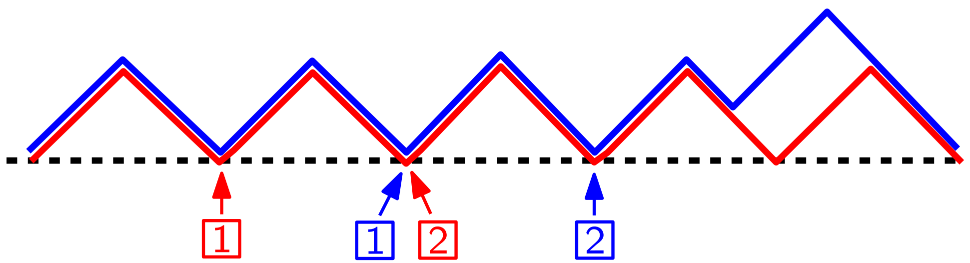

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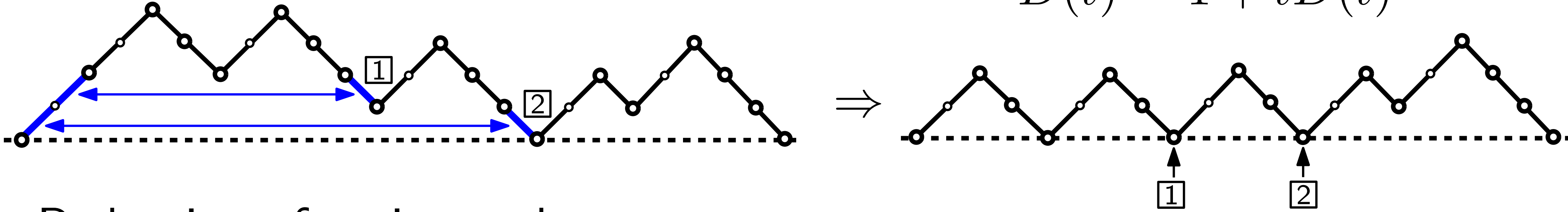

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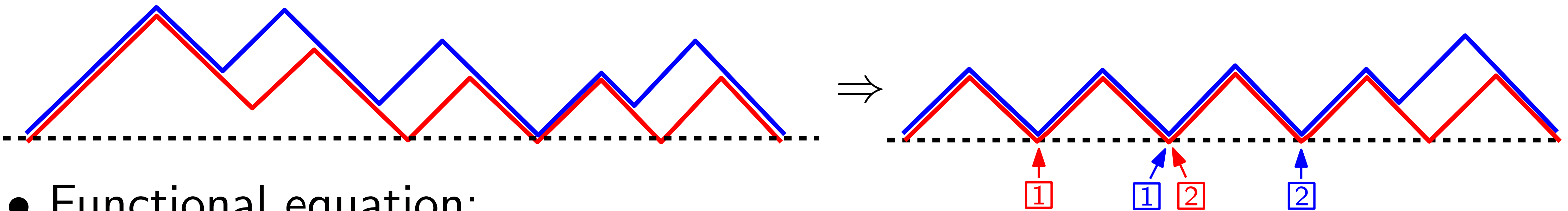

- Functional equation:

Let $a_{n,i} = \#($ intervals in $\mathcal{T}_n^{(m)}$ with i bottom contacts$)$

Let $F(t,x) := \sum_{n,i} a_{n,i} t^n x^i$

$$\boxed{F(t,x) = x + tx\Delta^m(F(t,x))}$$

where $\Delta : G(t,x) \to H(t,x) := F(t,x)\frac{G(t,x)-G(t,1)}{x-1}$

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We guess (using gfun:seriestoalgeq,algcurves:parametrization):

change of variable: $t = z(1-z)^{m^2+2m}, \quad x = \dfrac{1+u}{(1+zu)^{m+1}},$

F is rational in (z,u): $\boxed{F = \dfrac{(1+u)(1+zu)}{u(1-z)^{m+2}} \cdot \left(\dfrac{(1+u)}{(1+zu)^{m+1}} - 1\right)}.$

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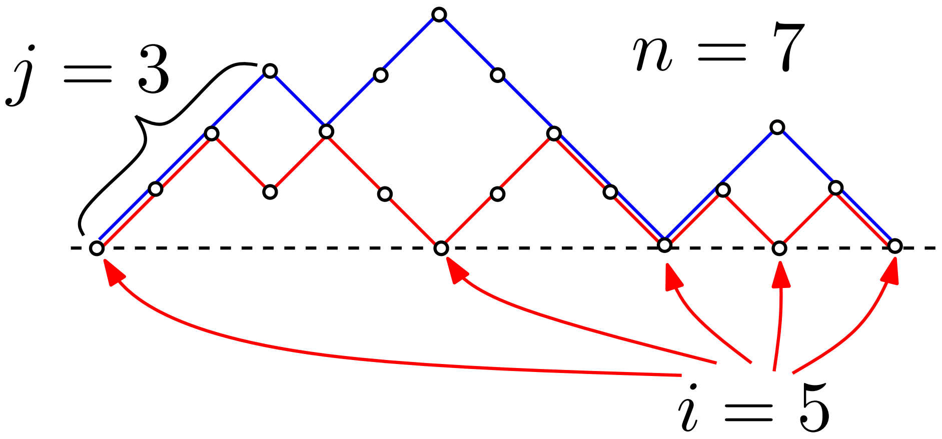
Lagrange inv. formula $\Rightarrow$ $$\boxed{[t^n]F(t,1) = \frac{m+1}{n(mn+1)}\binom{(m+1)^2 n + m}{n-1}}$$

Other results



Can express trivariate generating function:

$F(t, x, y) = \sum_{n,i,j} a_{n,i,j} t^n x^i y^j$

Surprising symmetry in x and y

change var.: $t = z(1-z)^{m^2+2m}, \quad x = \dfrac{1+u}{(1+zu)^{m+1}}, \quad y = \dfrac{1+v}{(1+zv)^{m+1}}.$

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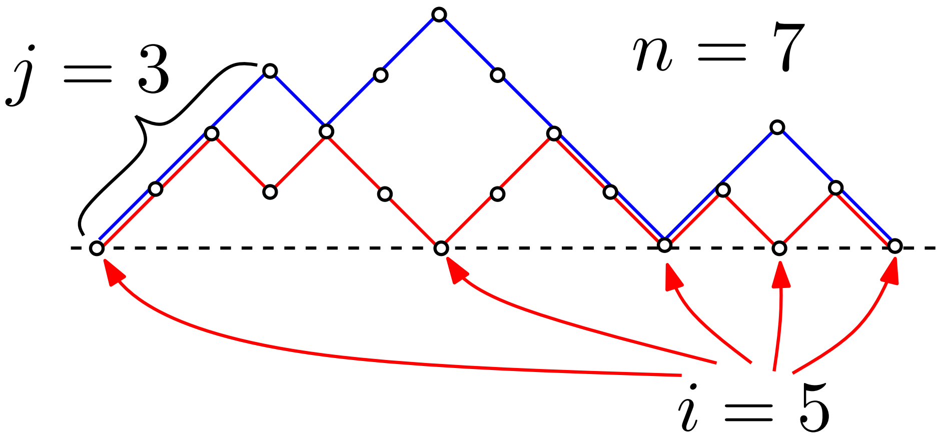

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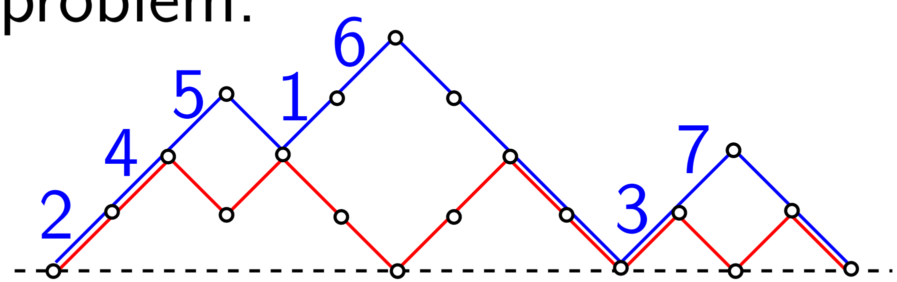
Labelled problem:



Conjecture [Bergeron'10]:

$$a_n^{(m)} = (m+1)^n (mn+1)^{n-2}$$

now proved in [Bousquet-Mélou, Chapuy, Préville-Ratelle'11]
(also guessing/checking, but on non-algebraic expressions)