Enumeration of corner polyhedra and 3-connected Schnyder labelings

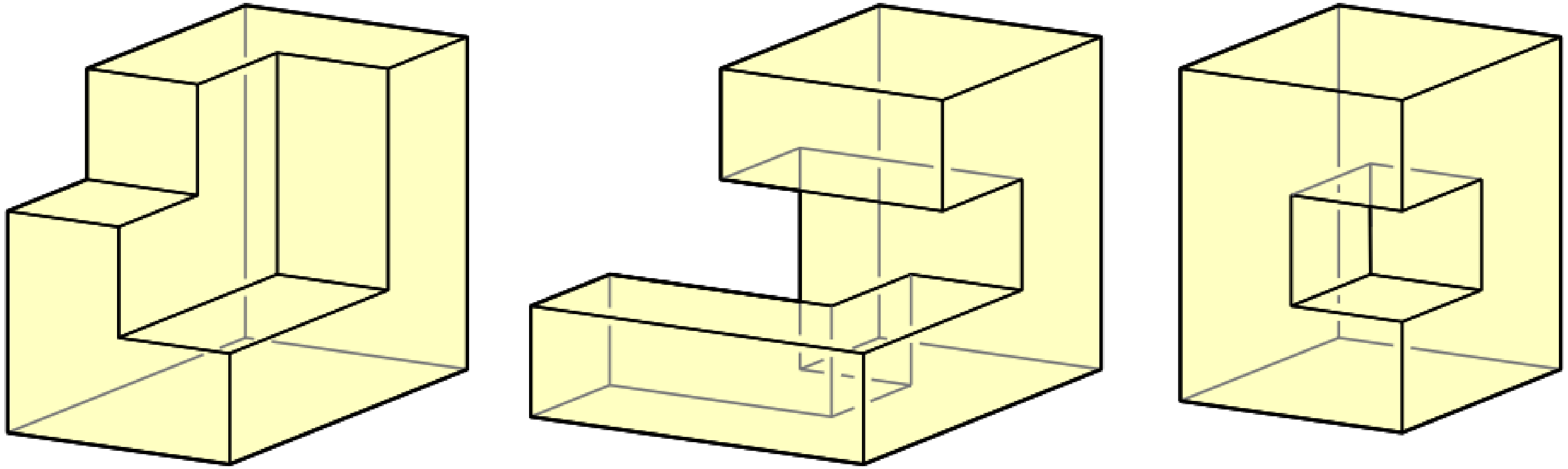
Éric Fusy (LIGM, Univ. Gustave Eiffel)

Joint work with Erkan Narmanli and Gilles Schaeffer

rencontres ANR 3dMaps, LIPN, 2024

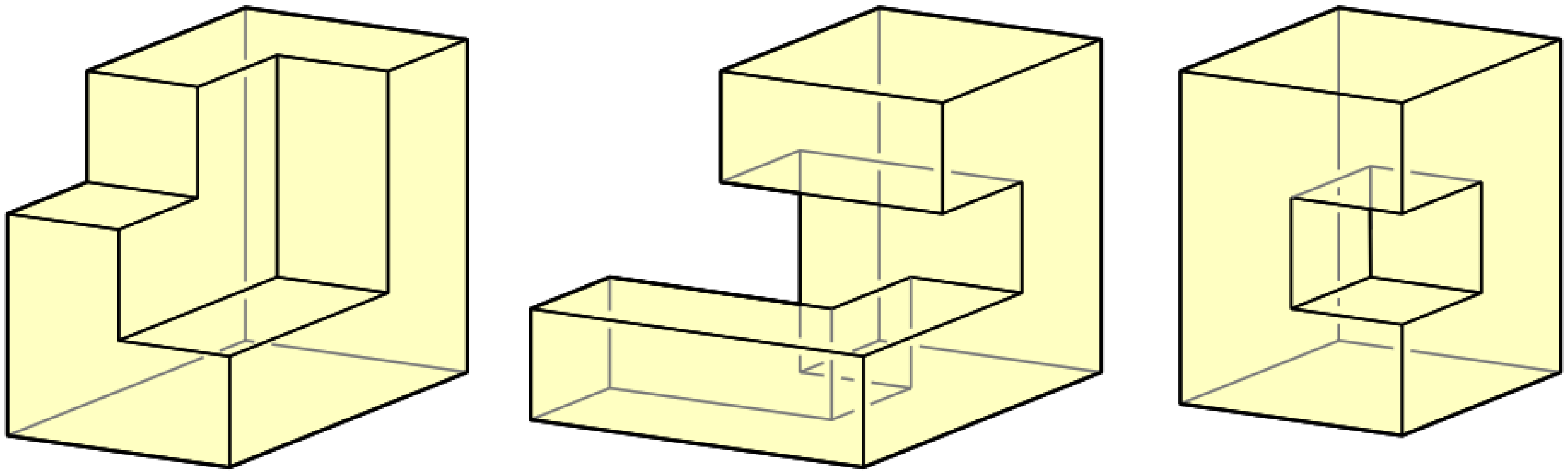
Simple orthogonal polyhedra [Eppstein-Mumford'09]

simple orthogonal polyhedron = 3d polyhedron such that, at each vertex three axis-aligned segments meet



Simple orthogonal polyhedra [Eppstein-Mumford'09]

simple orthogonal polyhedron = 3d polyhedron such that, at each vertex three axis-aligned segments meet

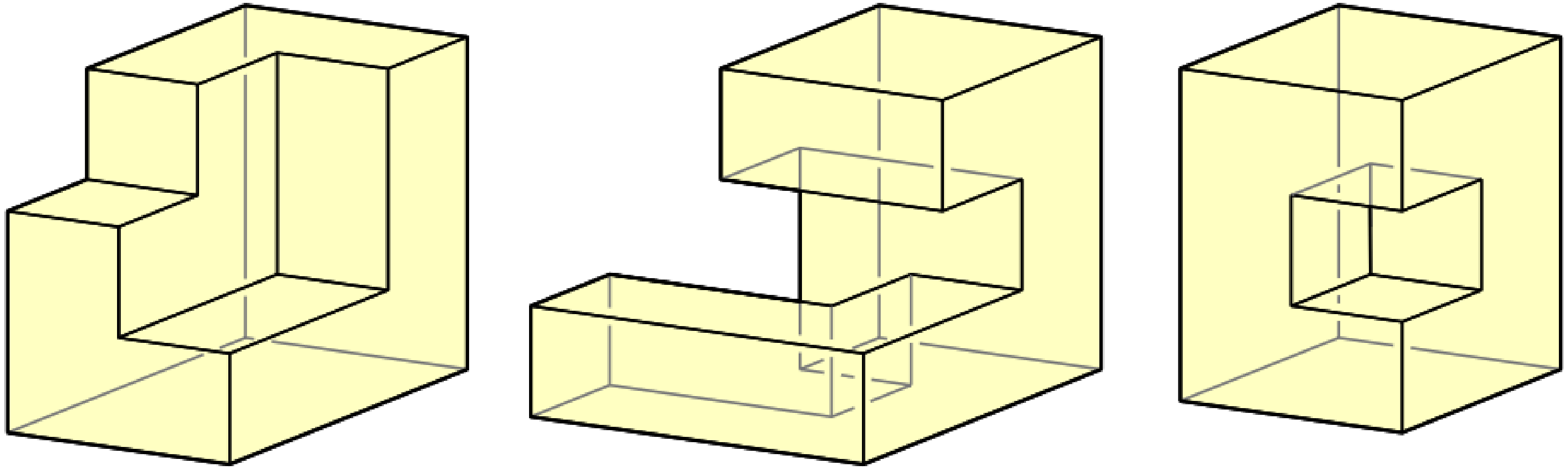


Rk: boundary forms a cubic (and bipartite) map on the sphere

Simple orthogonal polyhedra

[Eppstein-Mumford'09]

simple orthogonal polyhedron = 3d polyhedron such that, at each vertex three axis-aligned segments meet

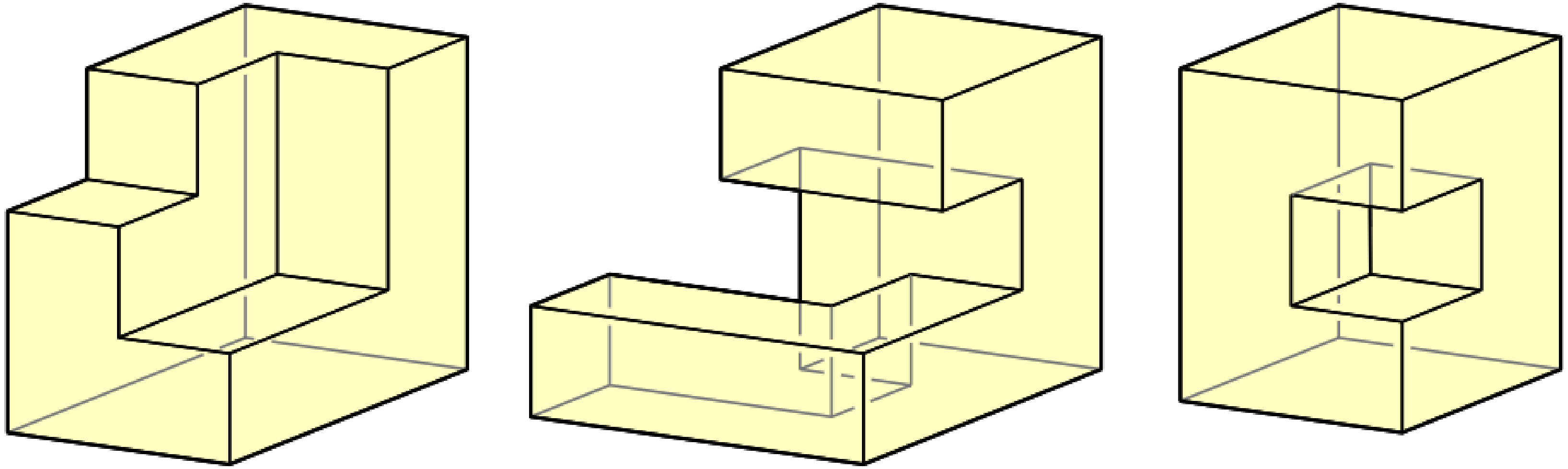


Rk: boundary forms a cubic (and bipartite) map on the sphere

Q: Which cubic bipartite planar maps admit a realization as a simple orthogonal polyhedron?

Simple orthogonal polyhedra [Eppstein-Mumford'09]

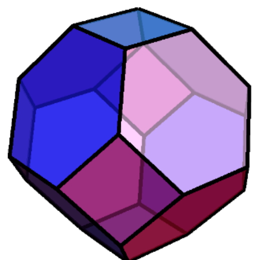
simple orthogonal polyhedron = 3d polyhedron such that, at each vertex three axis-aligned segments meet



Rk: boundary forms a cubic (and bipartite) map on the sphere

Q: Which cubic bipartite planar maps admit a realization as a simple orthogonal polyhedron?

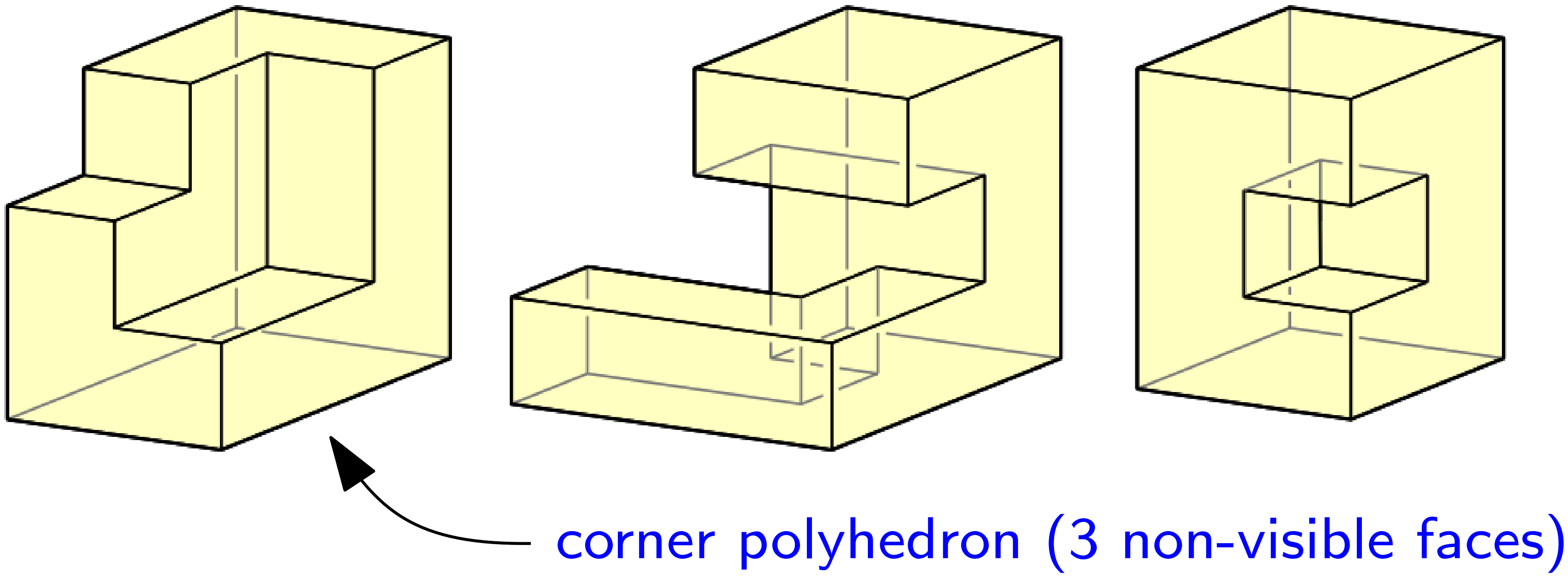
(cf Steinitz theorem for convex polyhedra)



Simple orthogonal polyhedra

[Eppstein-Mumford'09]

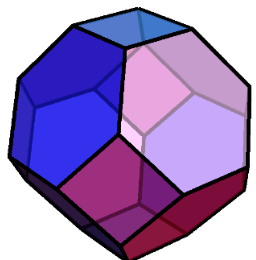
simple orthogonal polyhedron = 3d polyhedron such that, at each vertex three axis-aligned segments meet



Rk: boundary forms a cubic (and bipartite) map on the sphere

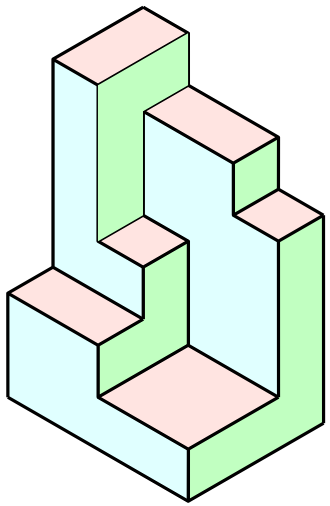
Q: Which cubic bipartite planar maps admit a realization as a simple orthogonal polyhedron?

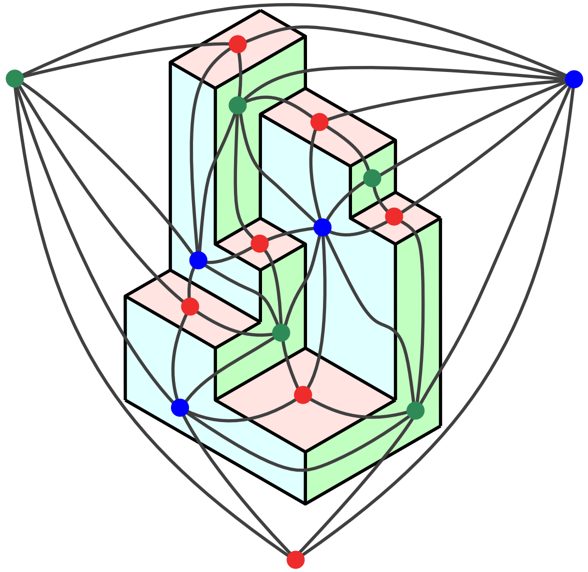
(cf Steinitz theorem for convex polyhedra)

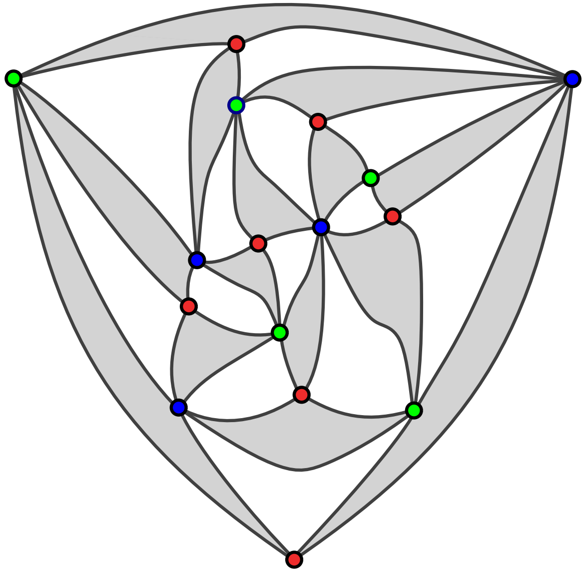


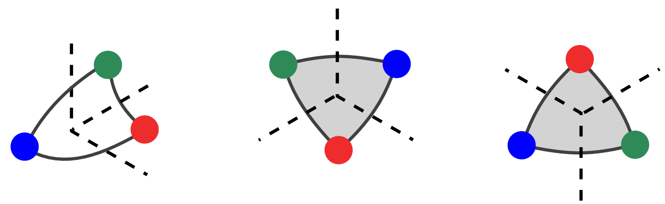
Characterization of corner polyhedra maps

[Eppstein-Mumford'09]



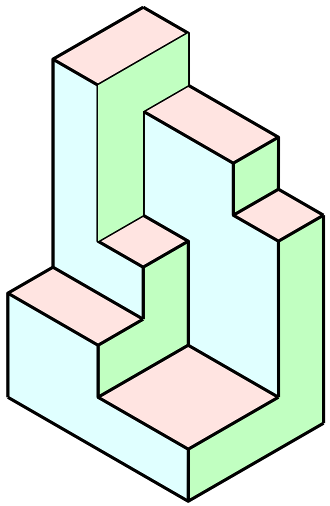


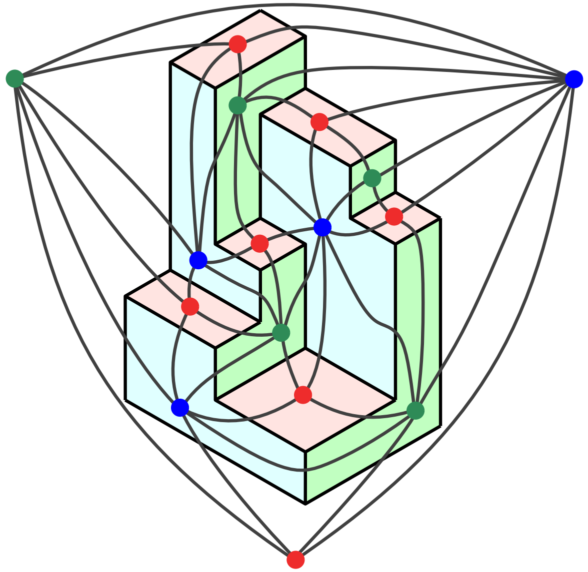


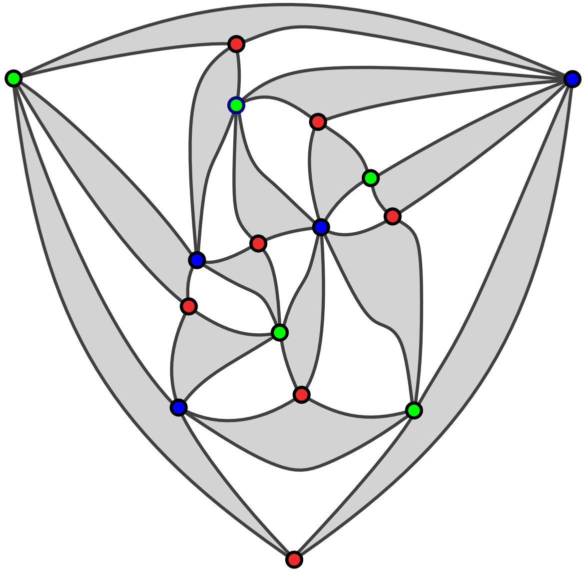


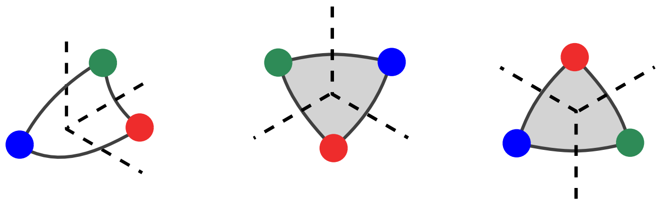
Characterization of corner polyhedra maps

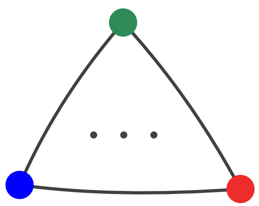
[Eppstein-Mumford'09]





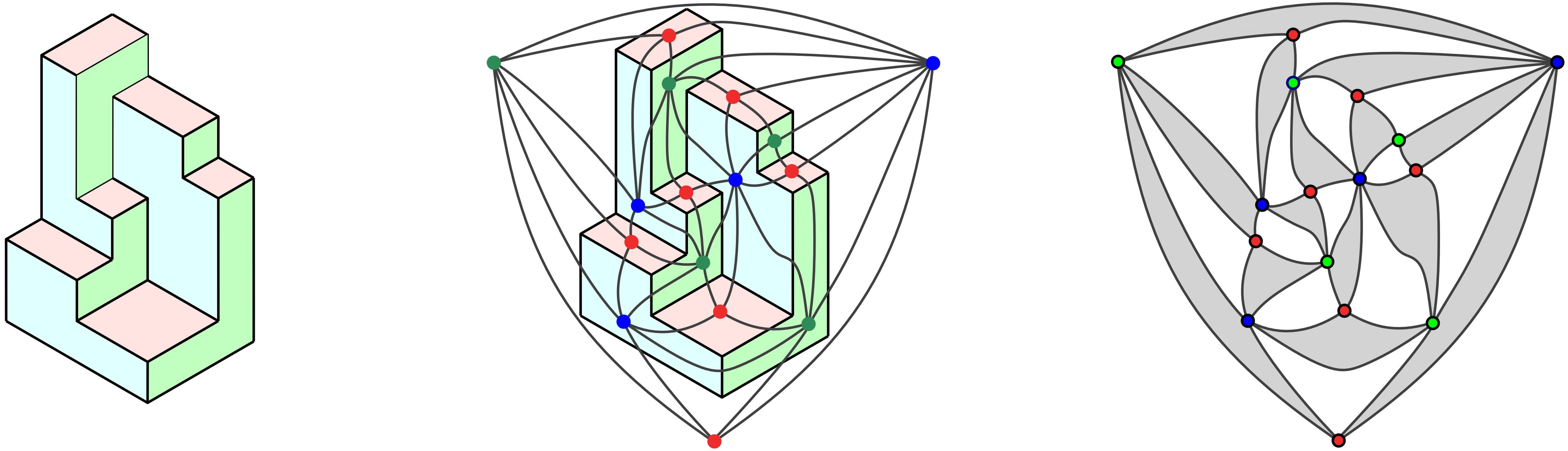


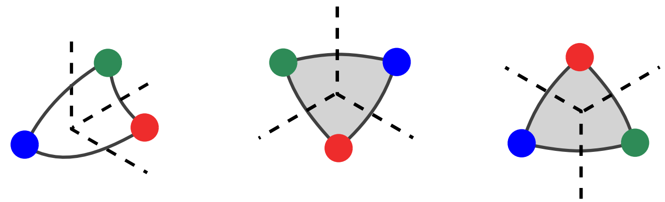


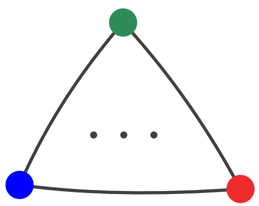
Every  bounds a face

Characterization of corner polyhedra maps

[Eppstein-Mumford'09]





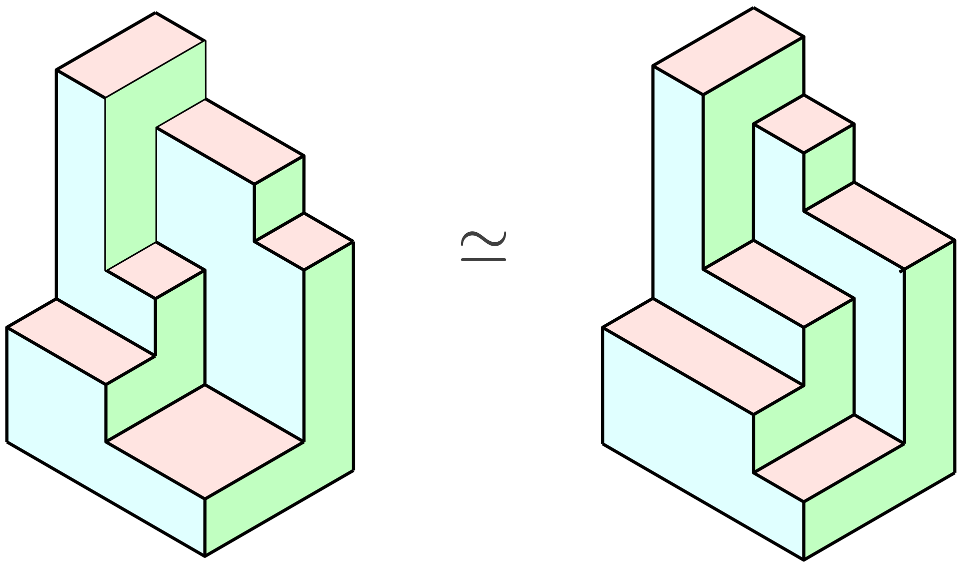
Every  bounds a face

Enumeration of these "corner triangulations": [Dervieux, Poulalhon, Schaeffer'16]

$$C(t) = \sum_n c_n t^n = t^3 + 3t^5 + 4t^6 + 15t^7 + 39t^8 + 120t^9 + \cdots$$

has rational expression in terms of Catalan generating function

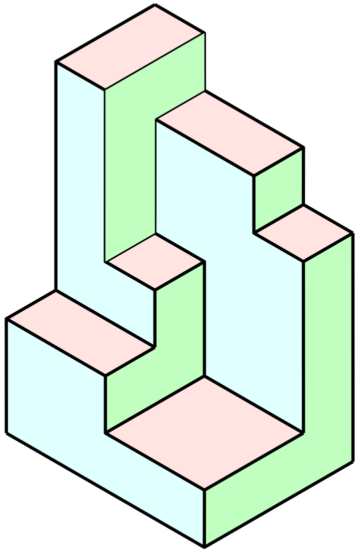
Enumeration of corner polyhedra



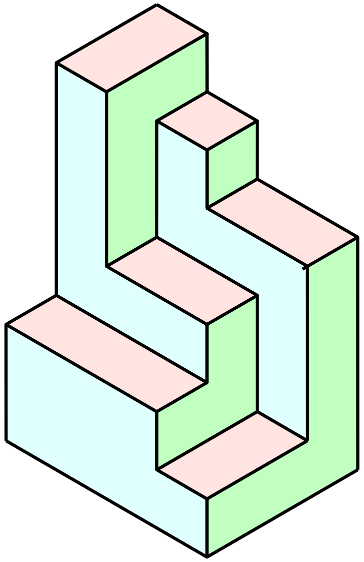
$p_n = \#$ combinatorial types of corner polyhedra of size n

where size $= \#$ flats -3

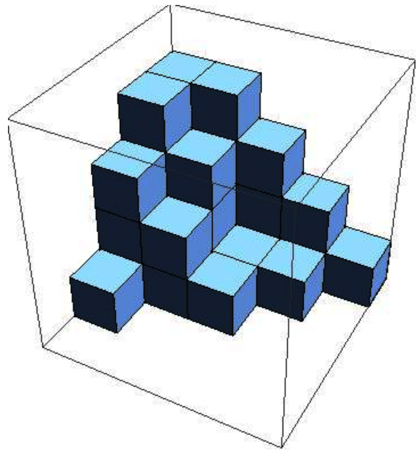
Enumeration of corner polyhedra



$\simeq$



$\neq$ counting plane partitions
by volume



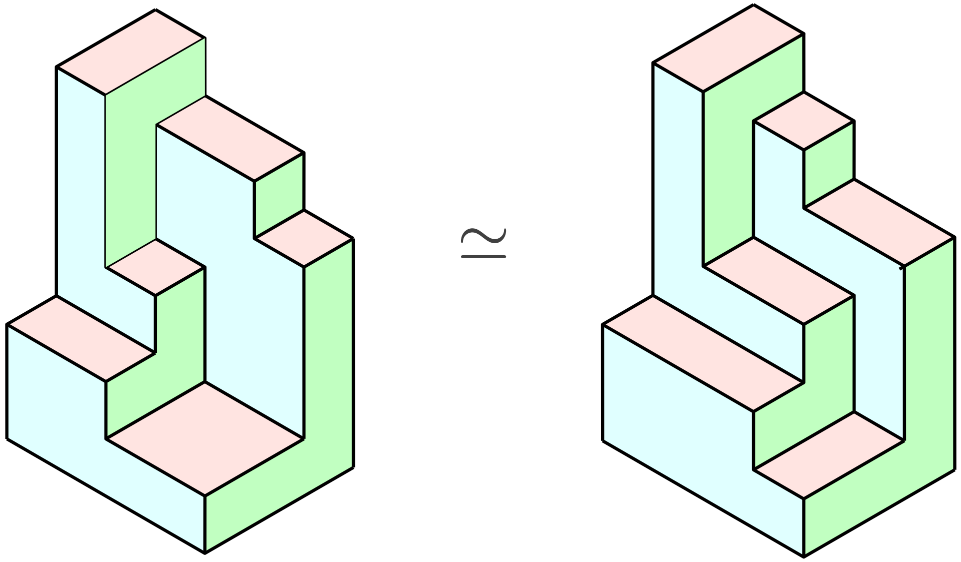
[MacMahon'1896]

$\prod_{i\geq 1}(1-q^i)^{-i}$

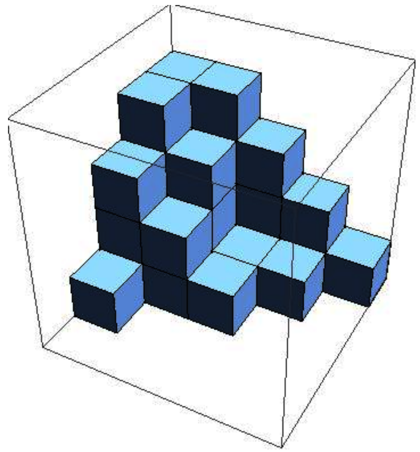
$p_n = \#$ combinatorial types of corner polyhedra of size n

where size $= \#$ flats -3

Enumeration of corner polyhedra



$\neq$ counting plane partitions by volume



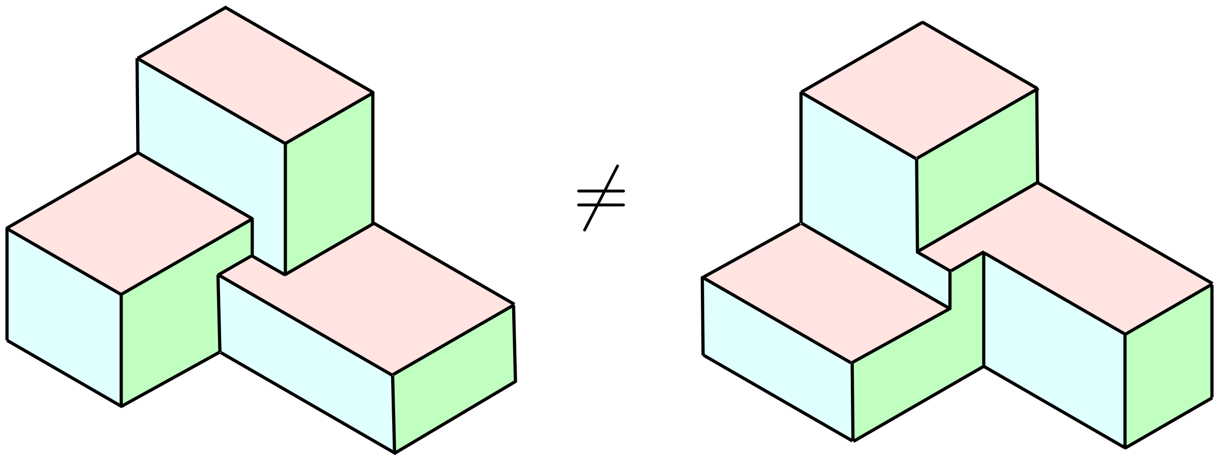
[MacMahon'1896]

$\prod_{i\geq 1}(1-q^i)^{-i}$

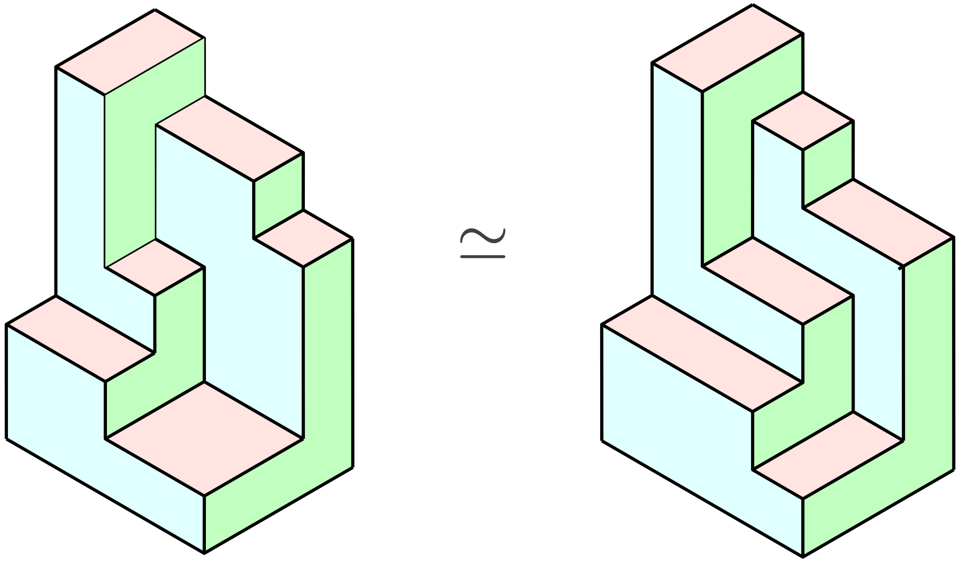
$p_n = \#$ combinatorial types of corner polyhedra of size n

where size $= \#$ flats -3

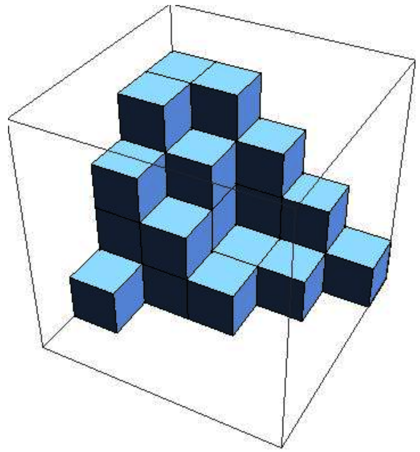
Rk: $p_n > c_n$ for $n \geq 9$



Enumeration of corner polyhedra



$\neq$ counting plane partitions by volume



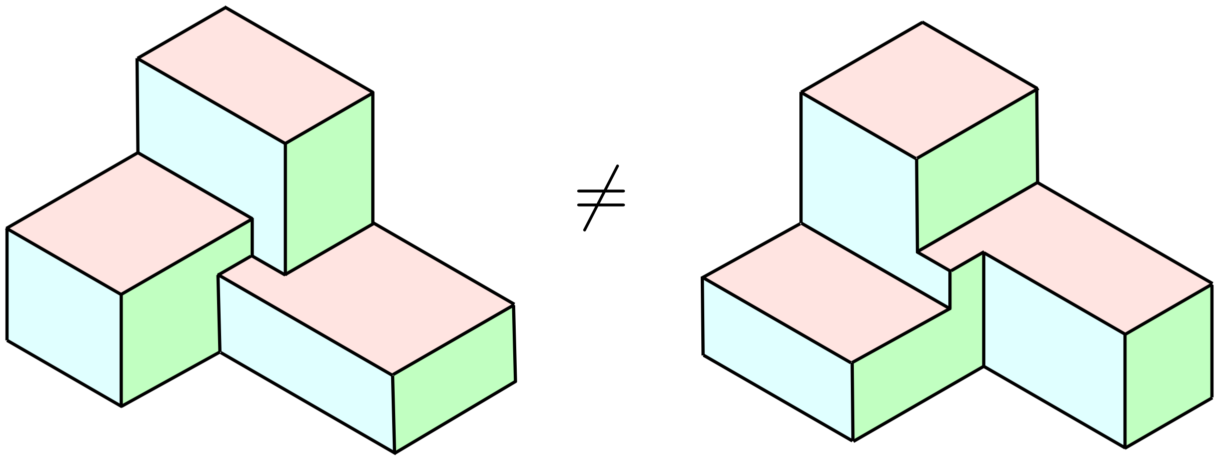
[MacMahon'1896]

$\prod_{i\geq 1}(1-q^i)^{-i}$

$p_n = \#$ combinatorial types of corner polyhedra of size n

where size $= \#$ flats -3

Rk: $p_n > c_n$ for $n \geq 9$

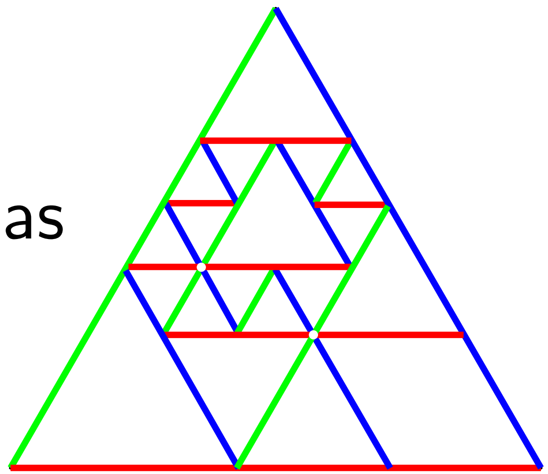


Q:
- exact counting: formula? recurrence?
- asymptotic estimate?

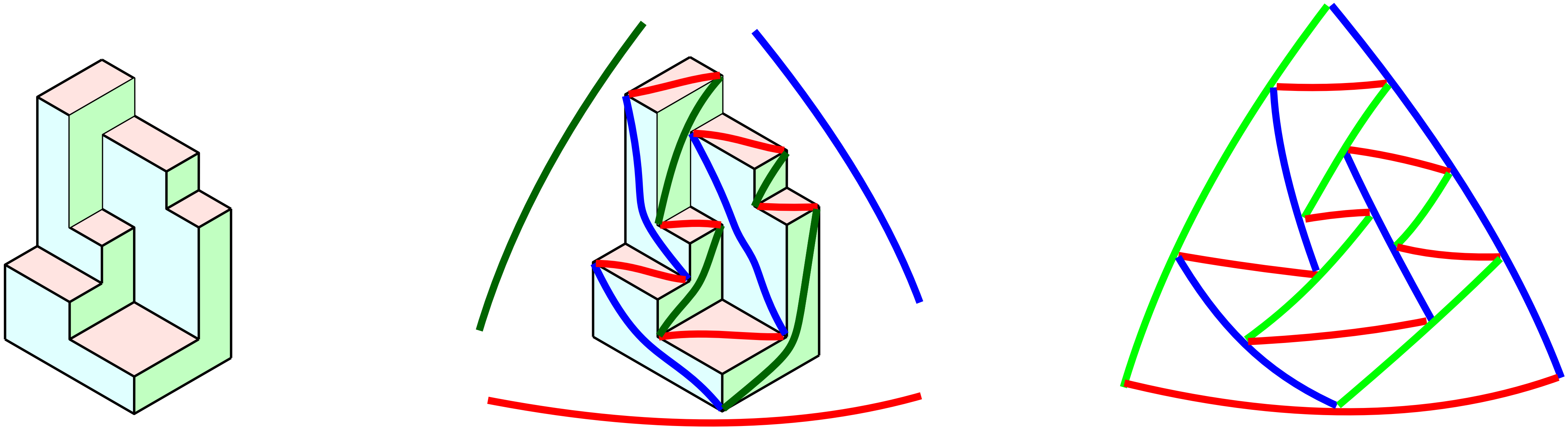
Relation to some tricolored contact-systems

[Gonçalves'19]

every **corner triangulation** has a unique
tricolored segment-contact representation as

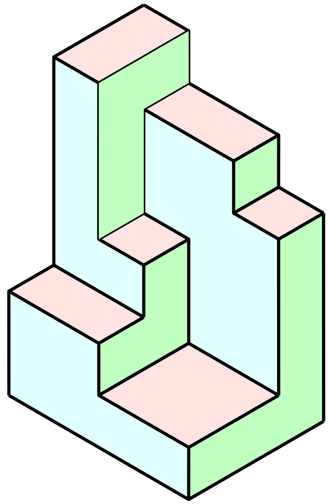


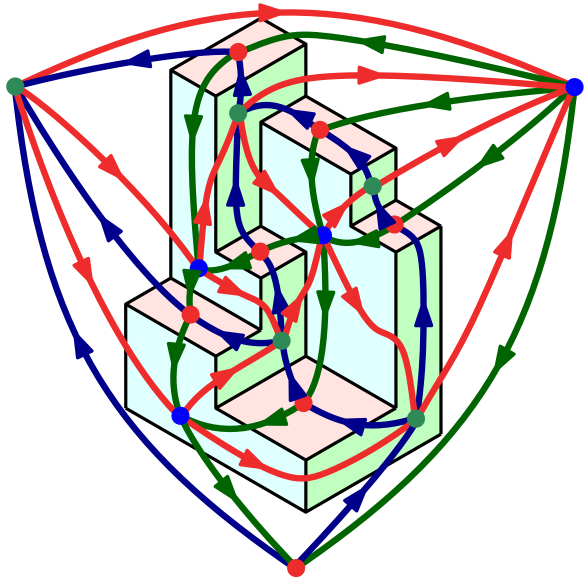
Corner polyhedra (types) can be encoded bijectively by such a
topological tricolored contact-system of (smooth) curves

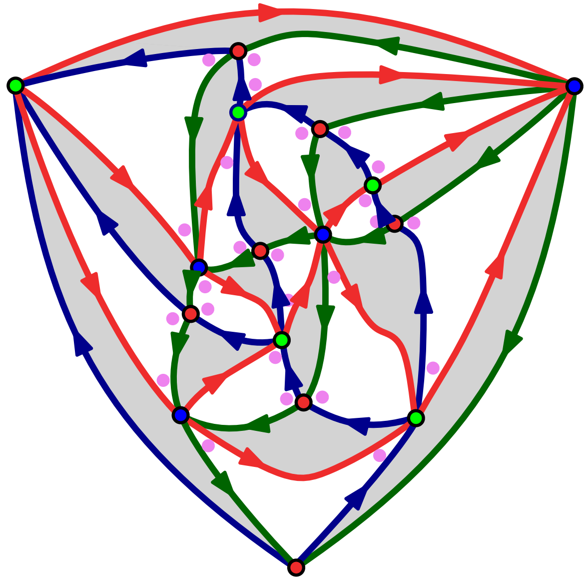


Encoding by orientations

[Eppstein-Mumford'09]



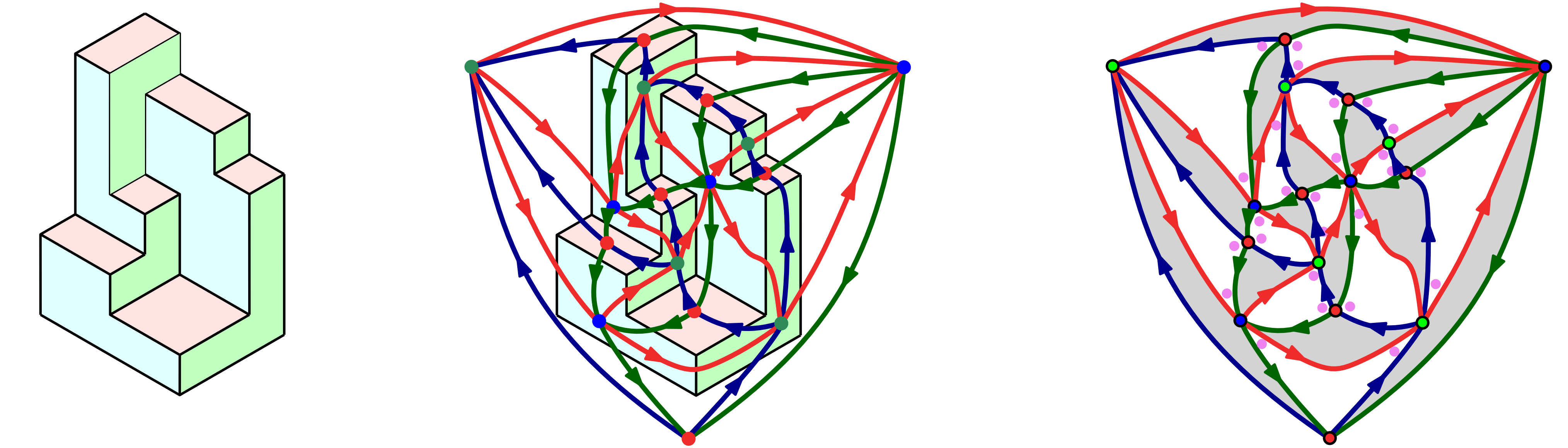




polyhedral orientation

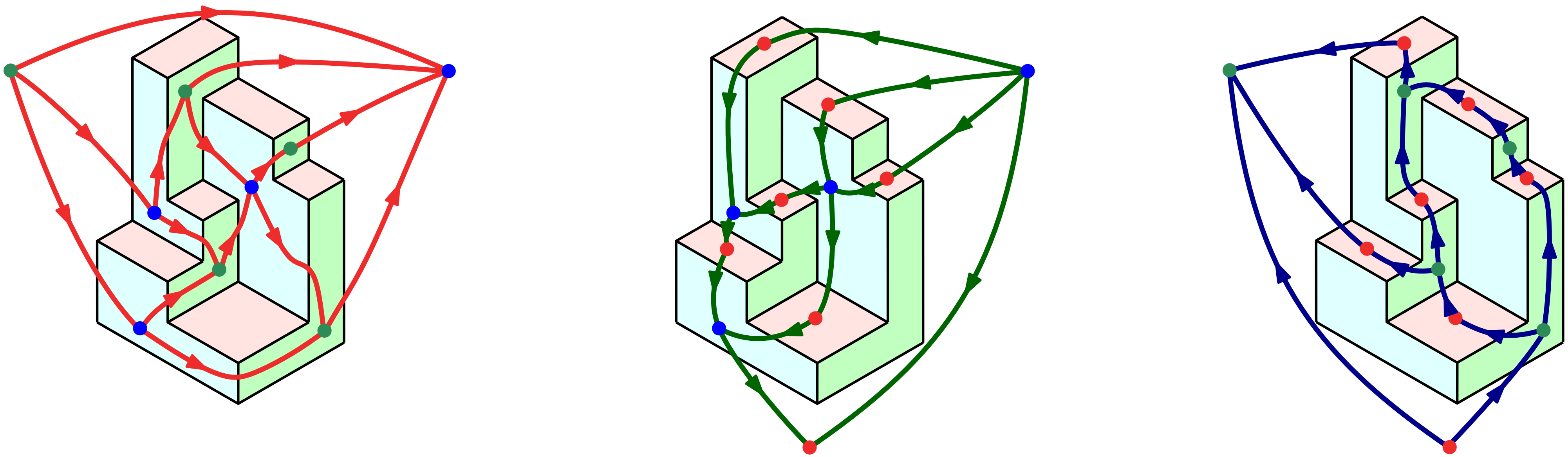
Encoding by orientations

[Eppstein-Mumford'09]



polyhedral orientation

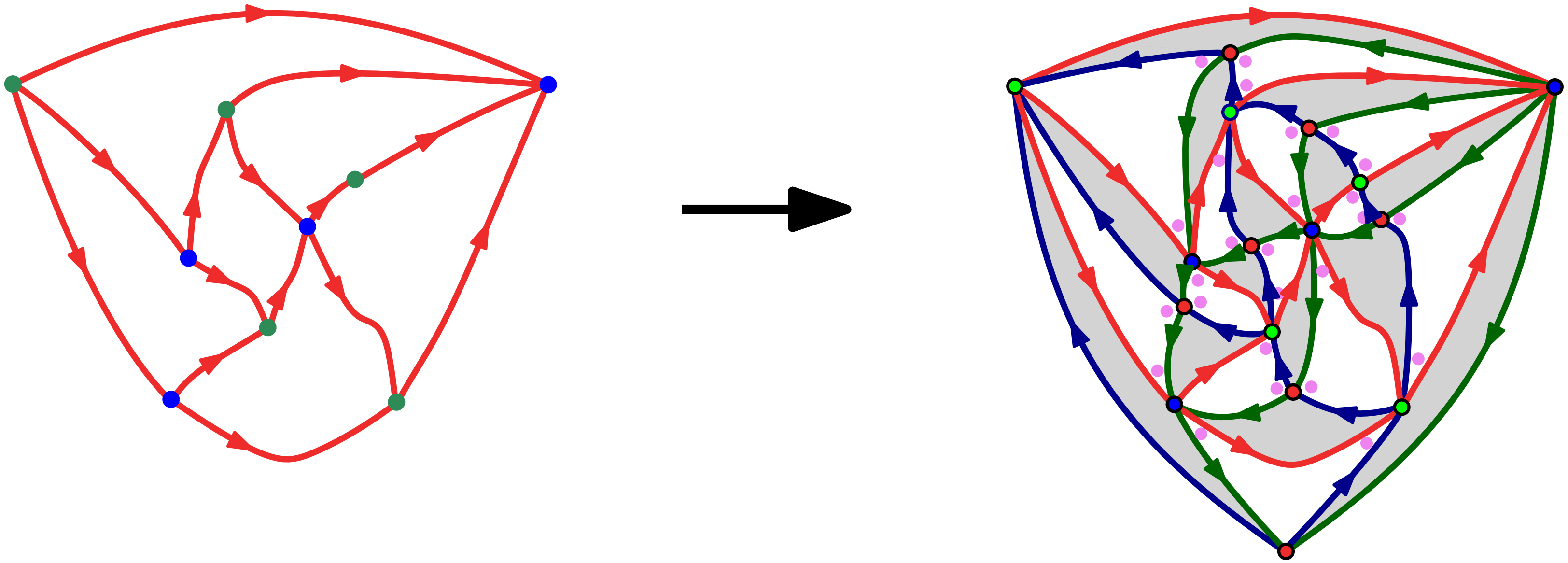
⇒ 3 plane bipolar orientations



One bipolar orientation is sufficient

[F,Narmanli,Schaeffer'23]

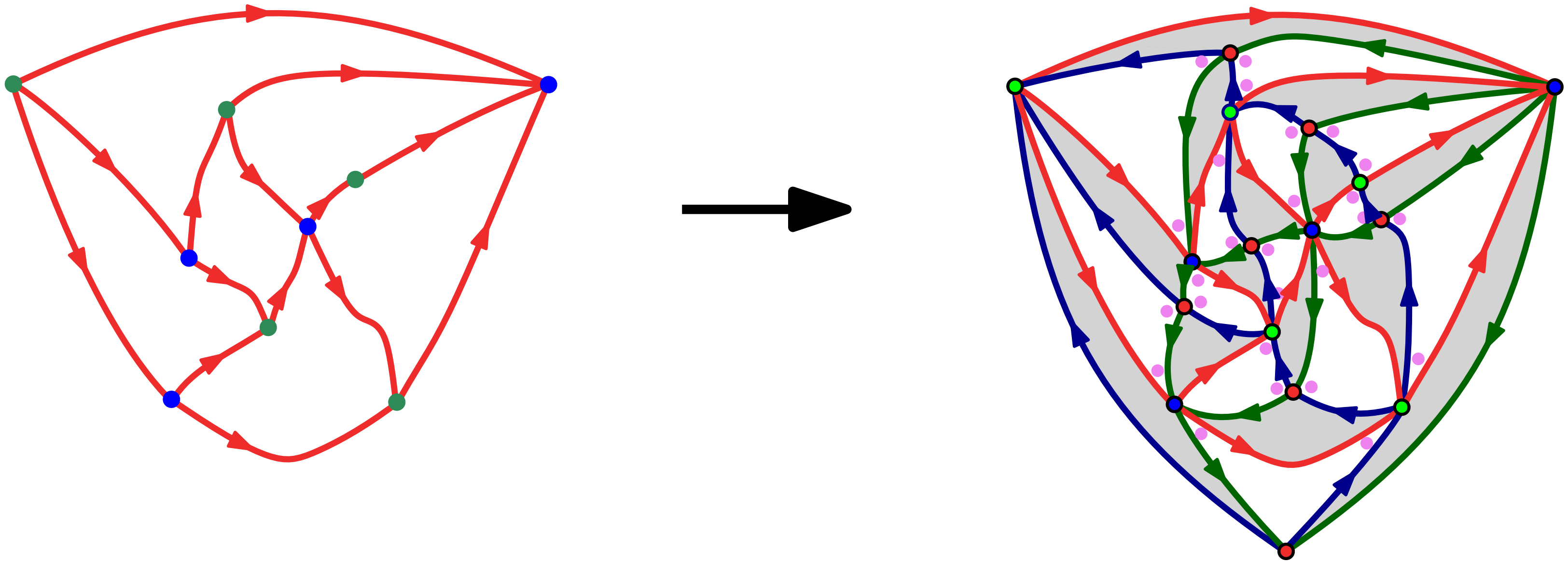
Polyhedral orientation can be reconstructed from red bipolar orientation

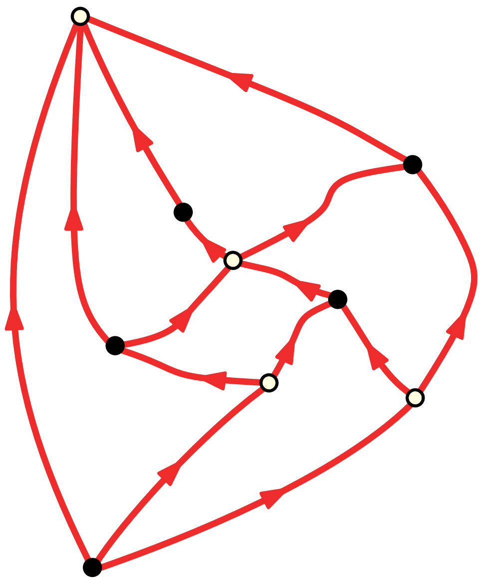


One bipolar orientation is sufficient

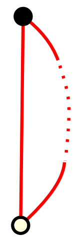
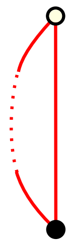
[F,Narmanli,Schaeffer'23]

Polyhedral orientation can be reconstructed from red bipolar orientation





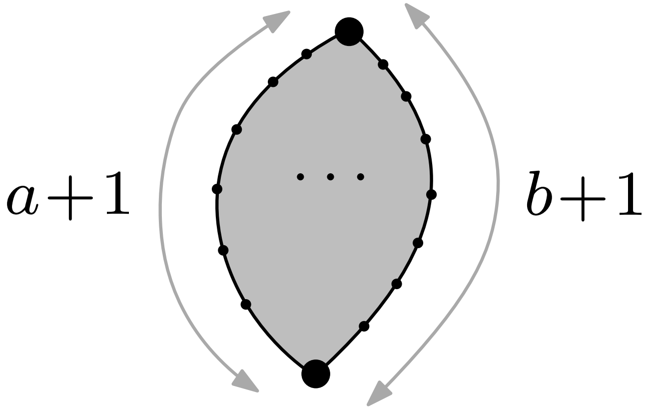
Characterization:

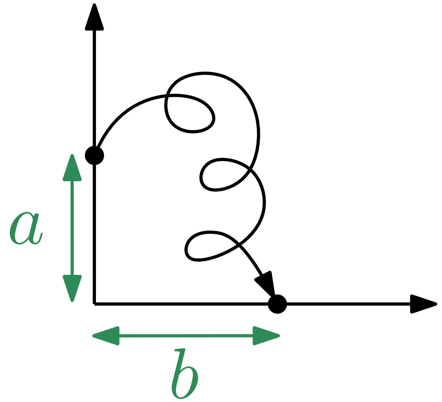
- bipartite
- avoids  and 

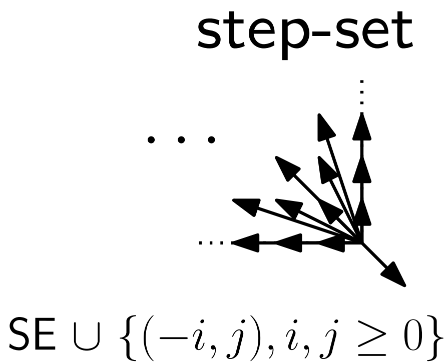
Encoding bipolar orientations by quadrant walks

[Kenyon, Miller, Sheffield, Wilson'15]

Plane bipolar orientations ⟷ "Tandem walks" in the quadrant





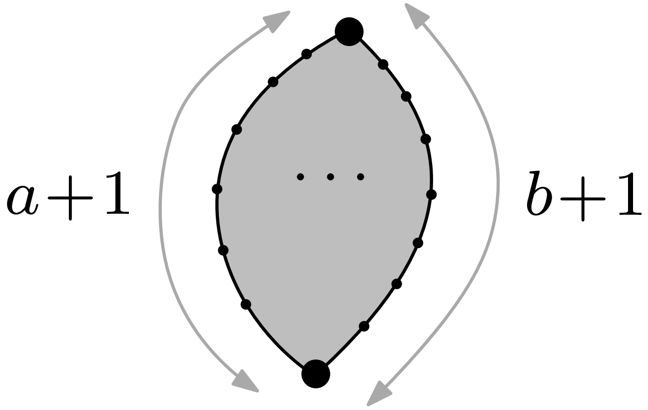


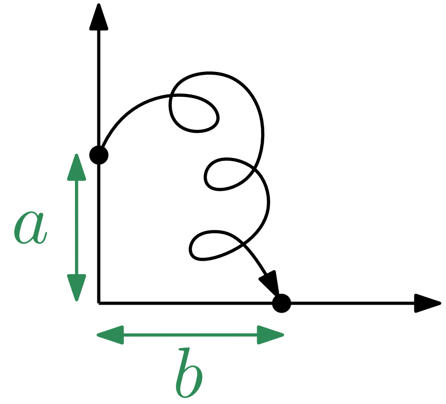
n edges ⟷ length $n-1$

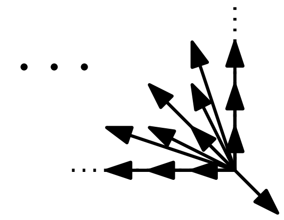
Encoding bipolar orientations by quadrant walks

[Kenyon, Miller, Sheffield, Wilson'15]

Plane bipolar orientations ⟷ "Tandem walks" in the quadrant

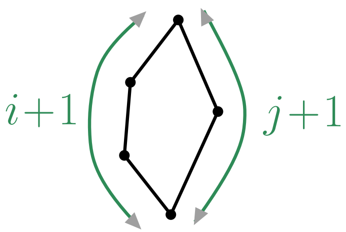






n edges ⟷ length $n-1$

face

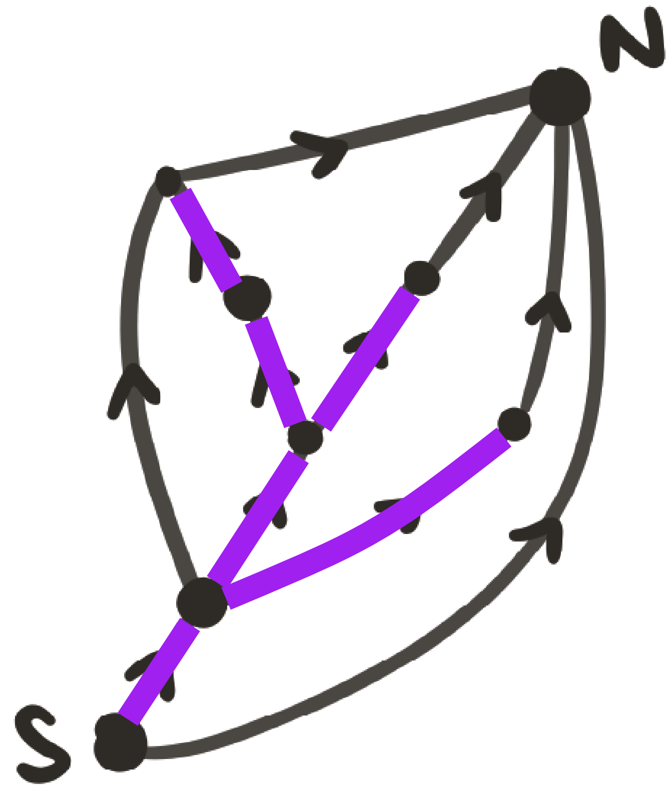


⟷ face-step $(-i,j)$

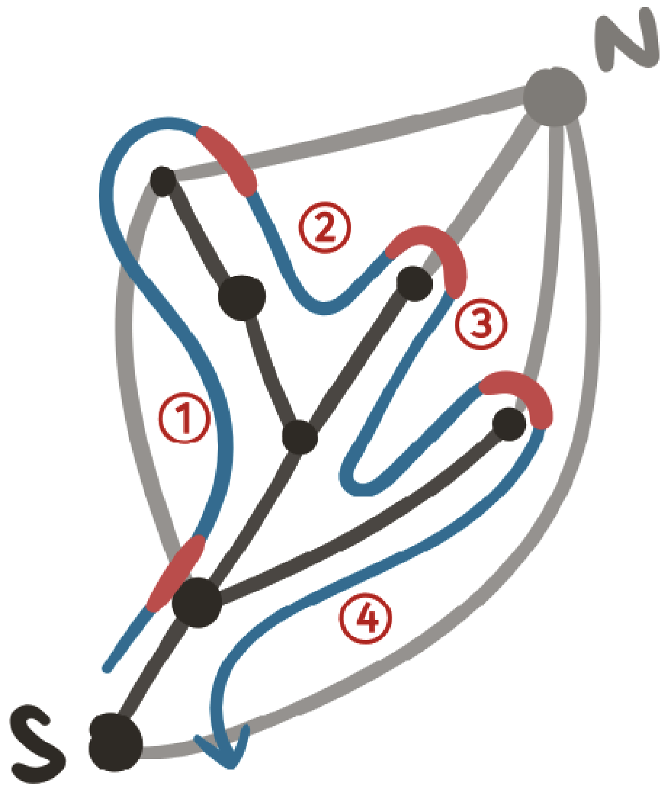
non-pole vertex ⟷ SE step

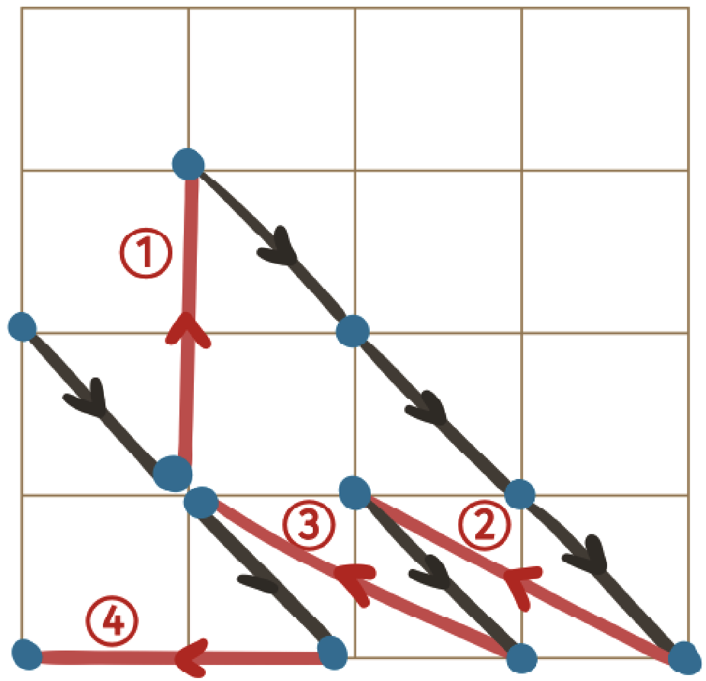
KMSW bijection

From bipolar orientation to tandem walk

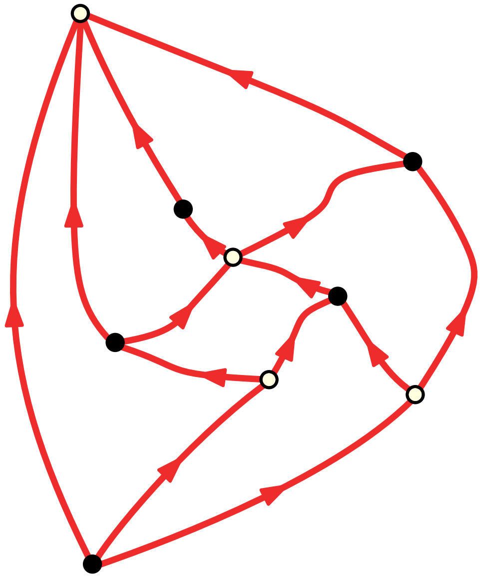


tree of rightmost incoming edges

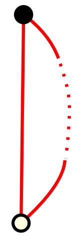
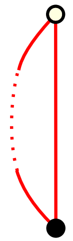




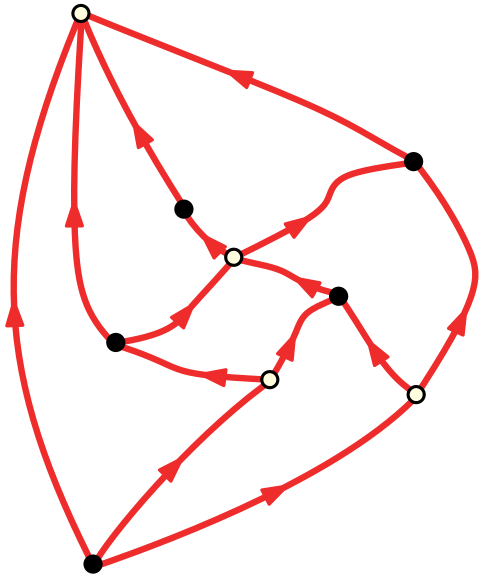
Specialization to the red bipolar orientations



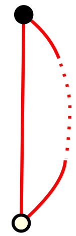
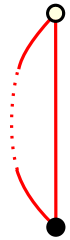
Characterization:

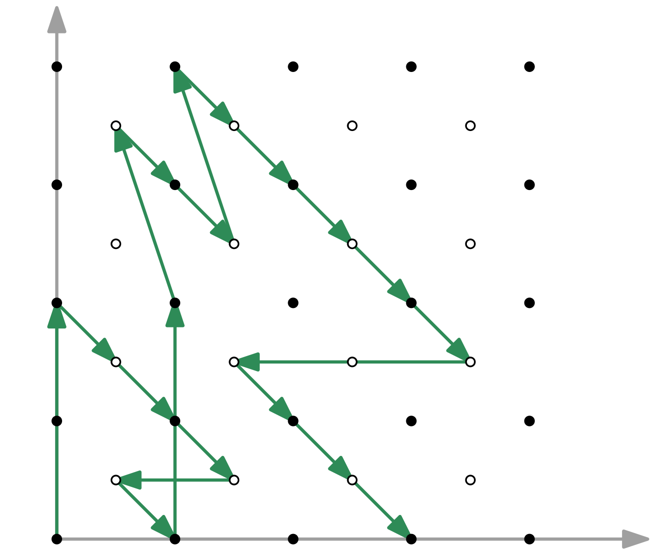
- bipartite
- avoids  and 

Specialization to the red bipolar orientations

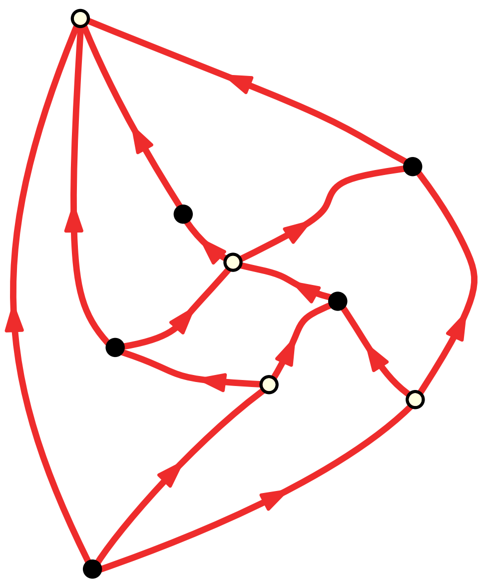


Characterization:

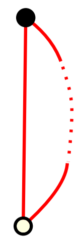
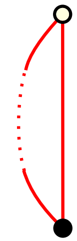
- bipartite
- avoids  and 

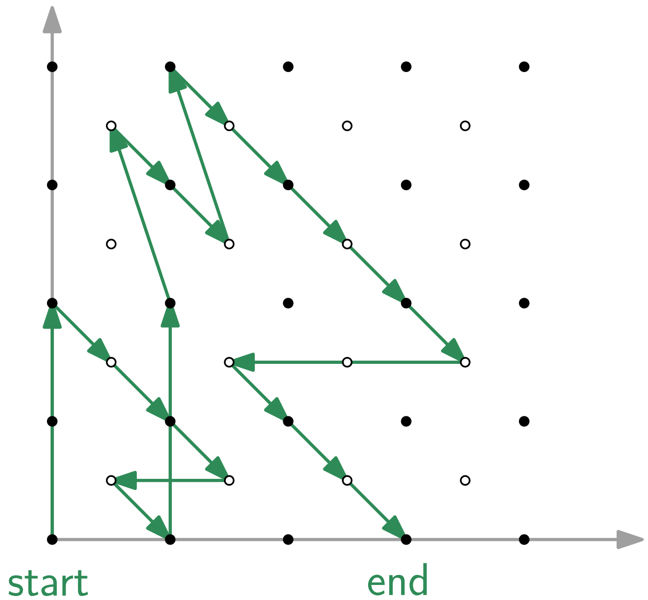


Specialization to the red bipolar orientations



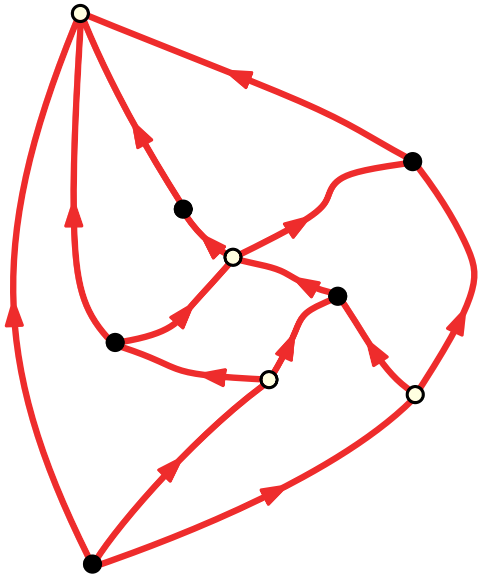
Characterization:

- bipartite
- avoids  and 

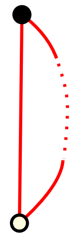
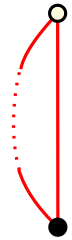


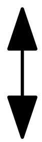
- starts at 0, ends on x-axis

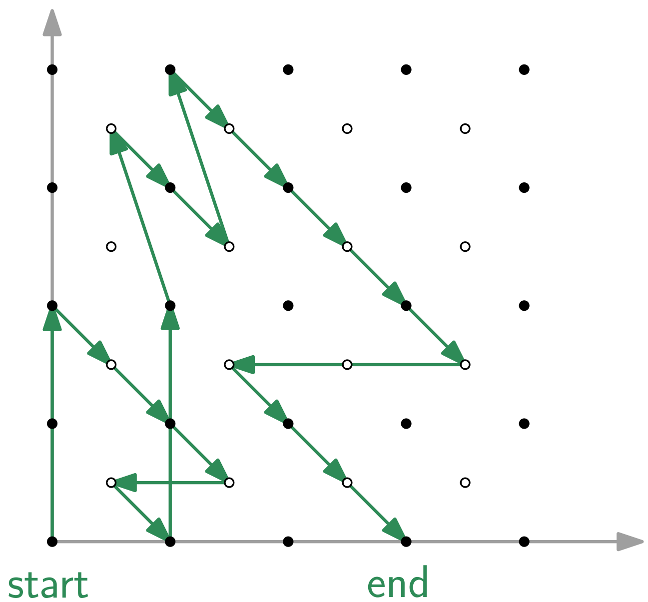
Specialization to the red bipolar orientations



Characterization:

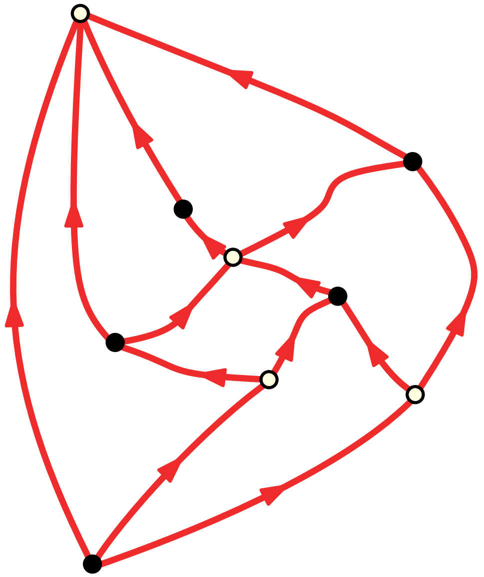
- bipartite
- avoids  and 



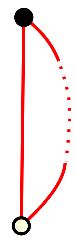
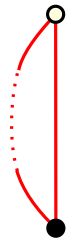


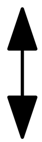
- starts at 0, ends on x-axis
- visits only points with $x + y$ even

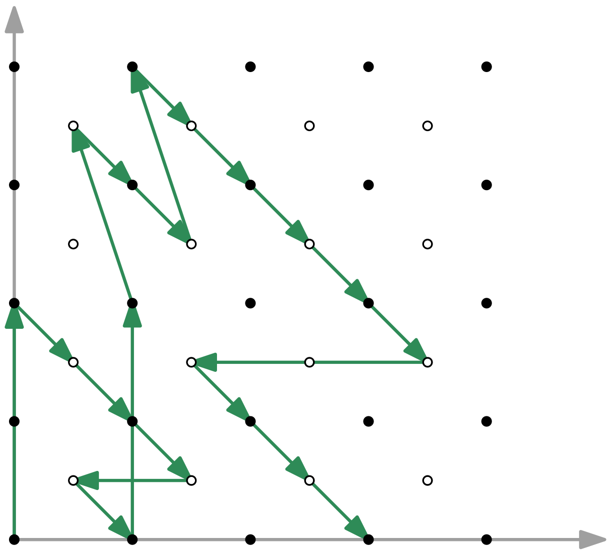
Specialization to the red bipolar orientations



Characterization:

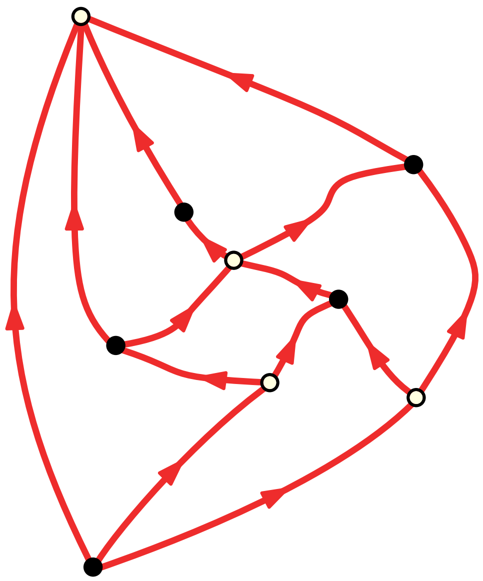
- bipartite
- avoids  and 



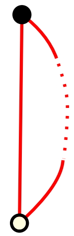
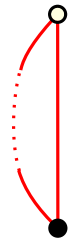


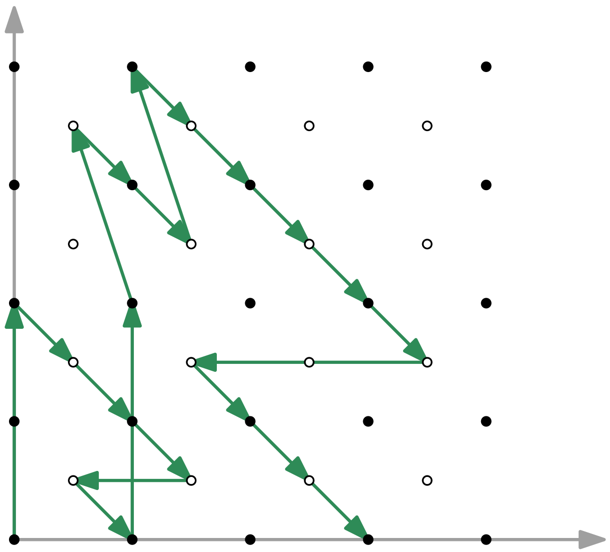
- starts at 0, ends on x-axis
- visits only points with $x+y$ even
- no horizontal step starting from ●
- no vertical step starting from ○

Specialization to the red bipolar orientations



Characterization:

- bipartite
- avoids  and 



- starts at 0, ends on x-axis
- visits only points with $x + y$ even
- no horizontal step starting from ●
- no vertical step starting from ○

(bimodal effect)

Exact counting: recurrence

By **last step removal**, obtain **recurrence** to compute p_n

($p_n = \sum_{i \geq 0} a_n(i,0)$, with recurrence on $a_n(i,j)$)

$$\sum_{n\geq 1} p_n t^n = t^3 + 3\,t^5 + 4\,t^6 + 15\,t^7 + 39\,t^8 + \mathbf{122}\,t^9 + 375\,t^{10} + 1212\,t^{11} + \cdots$$

Exact counting: recurrence

By **last step removal**, obtain **recurrence** to compute p_n

($p_n = \sum_{i\geq 0} a_n(i,0)$, with recurrence on $a_n(i,j)$)

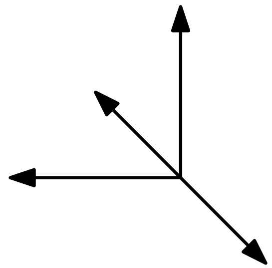
$$\sum_{n\geq 1} p_n t^n = t^3 + 3\,t^5 + 4\,t^6 + 15\,t^7 + 39\,t^8 + \mathbf{122}\,t^9 + 375\,t^{10} + 1212\,t^{11} + \cdots$$

Similarly, can obtain recurrence for $p_{a,b,c} = \#$ corner polyhedra with

a blue flats, b red flats, c green flats

$$\sum_{a,b,c\geq 1} p_{a,b,c} u^a v^b w^c = uvw + (u^2v^2w + uv^2w^2 + u^2vw^2) + 4u^2v^2w^2$$
$$+(u^3v^3w + 4u^3v^2w^2 + 4u^2v^3w^2 + u^3vw^3 + 4u^2v^2w^3 + uv^3w^3) + \cdots$$

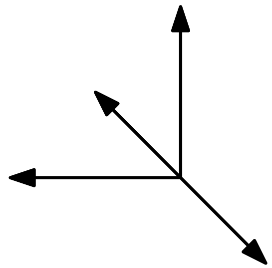
Asymptotic bounds for excursions in quadrant

General method (saddle bound), e.g. for $\mathcal{S} =$ 

Let $S(x,y) = xy^{-1} + x^{-2} + x^{-1}y + y^2$

Asymptotic bounds for excursions in quadrant

General method (saddle bound), e.g. for $\mathcal{S} =$



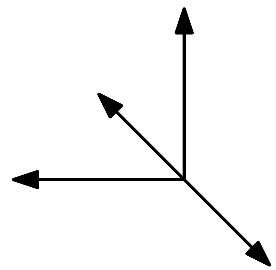
Let $S(x,y) = xy^{-1} + x^{-2} + x^{-1}y + y^2$

Let $a_n(i,j) = \#\mathcal{S}$-walks of length n in $\mathbb{Z}^2$ ending at (i,j)

Then $\forall x, y > 0$, $\quad \sum_{i,j\in\mathcal{Z}^2} a_n(i,j)x^i y^j = S(x,y)^n$

Asymptotic bounds for excursions in quadrant

General method (saddle bound), e.g. for $\mathcal{S} =$



Let $S(x,y) = xy^{-1} + x^{-2} + x^{-1}y + y^2$

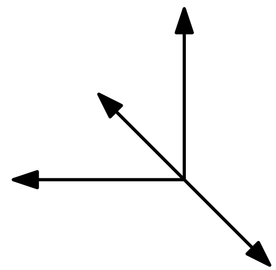
Let $a_n(i,j) = \#\mathcal{S}$-walks of length n in $\mathbb{Z}^2$ ending at (i,j)

Then $\forall x, y > 0$, $\quad \sum_{i,j \in \mathcal{Z}^2} a_n(i,j) x^i y^j = S(x,y)^n$

In particular $\quad a_n(0,0) \leq S(x,y)^n$

$$\leq \gamma^n \qquad \text{with } \gamma := \min_{x,y>0} S(x,y)$$

Asymptotic bounds for excursions in quadrant

General method (saddle bound), e.g. for $\mathcal{S} =$ 

Let $S(x,y) = xy^{-1} + x^{-2} + x^{-1}y + y^2$

Let $a_n(i,j) = \#\mathcal{S}$-walks of length n in $\mathbb{Z}^2$ ending at (i,j)

Then $\forall x, y > 0$, $\sum_{i,j \in \mathcal{Z}^2} a_n(i,j) x^i y^j = S(x,y)^n$

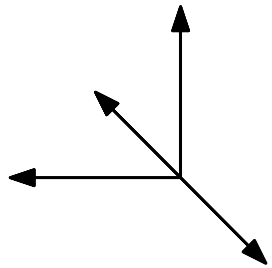
In particular $a_n(0,0) \leq S(x,y)^n$

$\leq \gamma^n$ with $\gamma := \min_{x,y>0} S(x,y)$

(here $\gamma = 2\sqrt{3}$)

Asymptotic bounds for excursions in quadrant

General method (saddle bound), e.g. for $\mathcal{S} =$



Let $S(x,y) = xy^{-1} + x^{-2} + x^{-1}y + y^2$

Let $a_n(i,j) = \#\mathcal{S}$-walks of length n in $\mathbb{Z}^2$ ending at (i,j)

Then $\forall x,y > 0$, $\quad \sum_{i,j\in\mathcal{Z}^2} a_n(i,j)x^i y^j = S(x,y)^n$

In particular $\quad a_n(0,0) \leq S(x,y)^n$

$\leq \gamma^n \qquad$ with $\gamma := \min_{x,y>0} S(x,y)$

(here $\gamma = 2\sqrt{3}$)

Rk: optimal $(x,y) \quad \leftrightarrow \quad (x,y)$-weighted random $\mathcal{S}$-walk has drift$= 0$

each step $s = (i,j) \in \mathcal{S}$ has proba $\quad \frac{x^i y^j}{S(x,y)}$

Asymptotic results for corner polyhedra

- **Growth rate**: $\lim_n (p_n)^{1/n} = 9/2$
- **Conjecture**: $p_n \sim c\,(9/2)^n\, n^{-\alpha}$ where $c > 0$

$$\alpha = 1 + \frac{\pi}{\arccos(9/16)} \approx 4.23$$

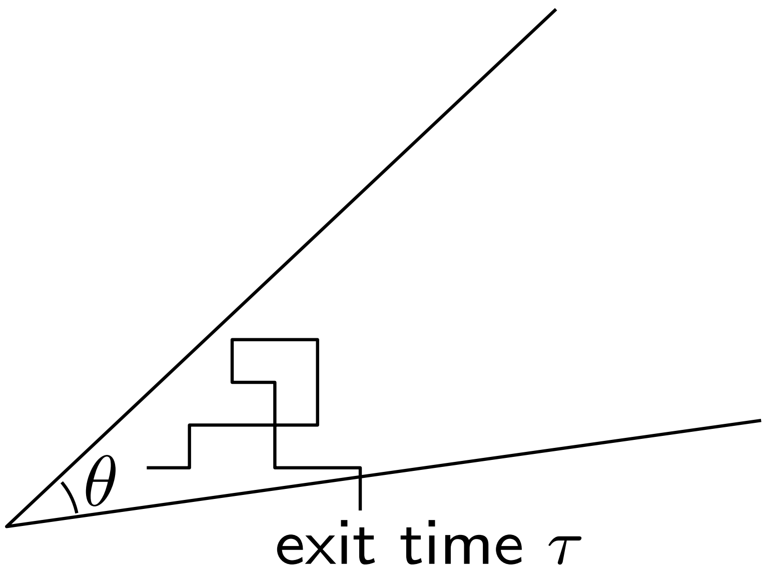
Asymptotic results for corner polyhedra

- **Growth rate**: $\lim_n (p_n)^{1/n} = 9/2$
- **Conjecture**: $p_n \sim c\,(9/2)^n\, n^{-\alpha}$ where $c > 0$

$$\alpha = 1 + \frac{\pi}{\arccos(9/16)} \approx 4.23$$

Explanation:

reduction to Denisov and Wachtel'15 "random walks in cones"



$$\mathbb{P}(\tau > n) \sim c'\; n^{-\frac{\pi}{2\theta}}$$

$$\mathbb{P}(\tau > n \;\&\; \text{excursion}) \sim c\; n^{-1-\frac{\pi}{\theta}}$$

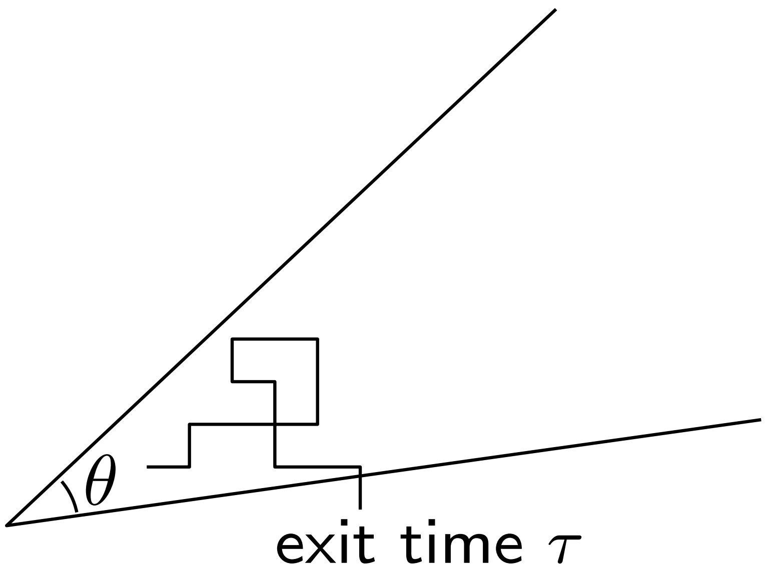
Asymptotic results for corner polyhedra

- **Growth rate**: $\lim_n (p_n)^{1/n} = 9/2$
- **Conjecture**: $p_n \sim c\,(9/2)^n\, n^{-\alpha}$ where $c > 0$

$$\alpha = 1 + \frac{\pi}{\arccos(9/16)} \approx 4.23$$

Explanation:

reduction to Denisov and Wachtel'15 "random walks in cones"



$$\mathbb{P}(\tau > n) \sim c'\ n^{-\frac{\pi}{2\theta}}$$

$$\mathbb{P}(\tau > n \ \&\ \text{excursion}) \sim c\ n^{-1-\frac{\pi}{\theta}}$$

(would need to be extended to bimodal setting)

Asymptotic results for corner polyhedra

- **Growth rate**: $\lim_n (p_n)^{1/n} = 9/2$
- **Conjecture**: $p_n \sim c\,(9/2)^n\, n^{-\alpha}$ where $c > 0$

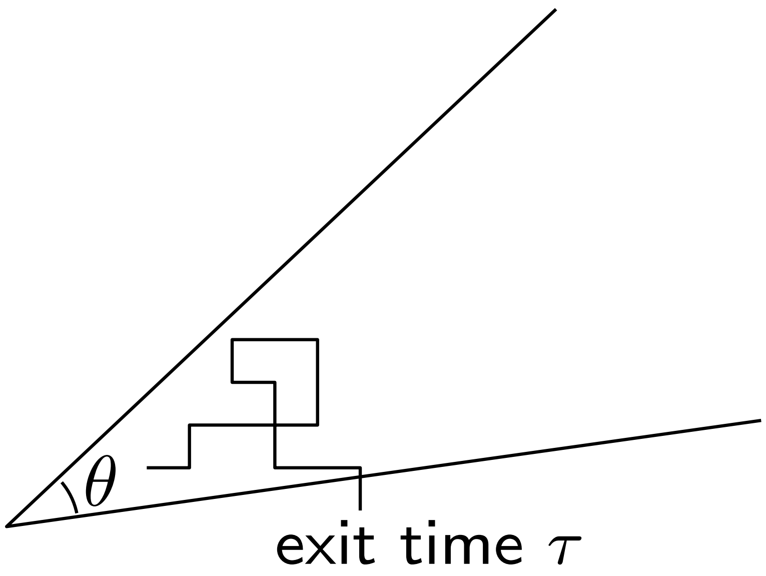
$$\alpha = 1 + \frac{\pi}{\arccos(9/16)} \approx 4.23$$

Rk: Conjecture would imply $\sum_n p_n z^n$ **not D-finite** (since $\alpha \notin \mathbb{Q}$)

criterion in **[Boston, Raschel, Salvy'14]**

Explanation:

reduction to Denisov and Wachtel'15 "random walks in cones"



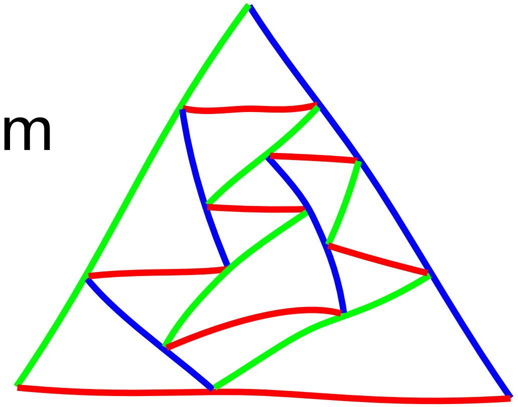
$\mathbb{P}(\tau > n) \sim c'\ n^{-\frac{\pi}{2\theta}}$

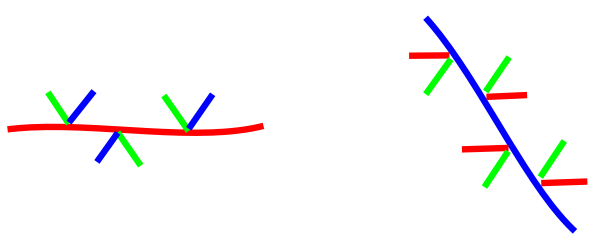
$\mathbb{P}(\tau > n \ \&\ \text{excursion}) \sim c\ n^{-1-\frac{\pi}{\theta}}$

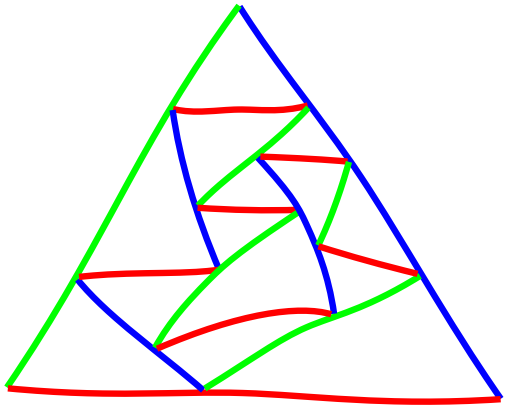
(would need to be extended to bimodal setting)

Back to tricolored contact-systems

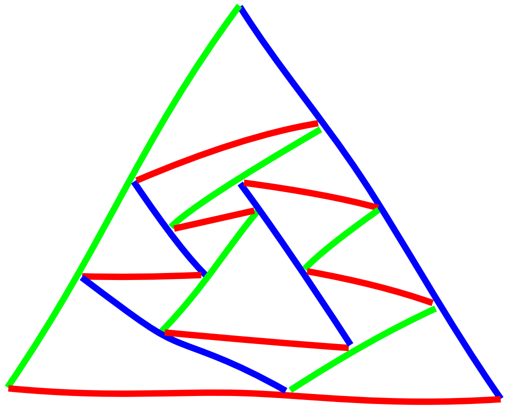
Contact-system
of curves



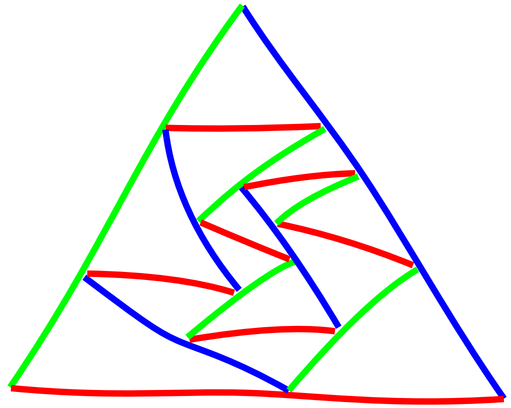




$\underset{\text{strong}}{\simeq}$



$\underset{\text{weak}}{\simeq}$

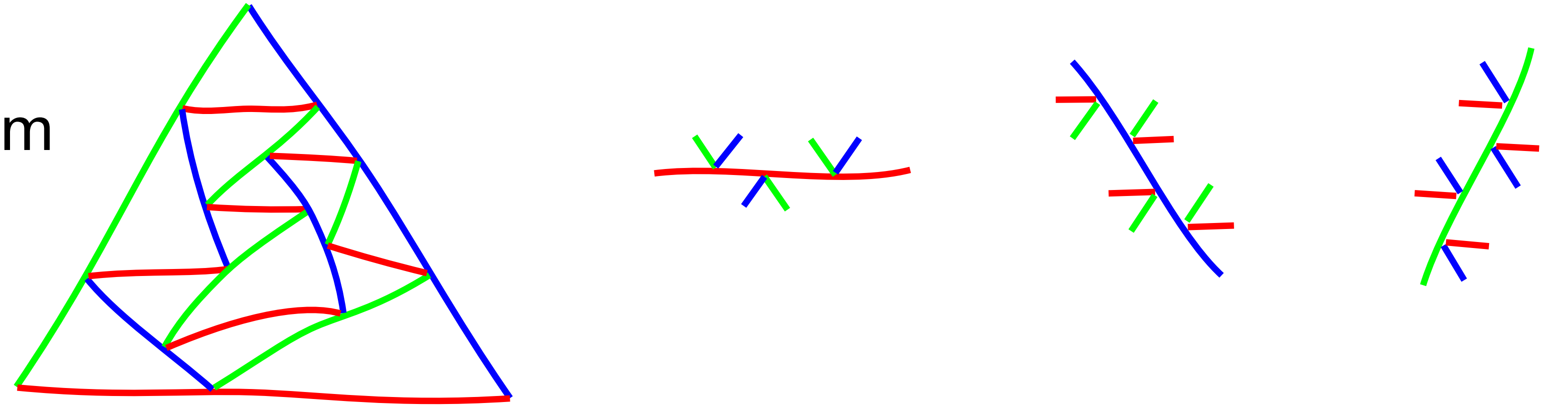


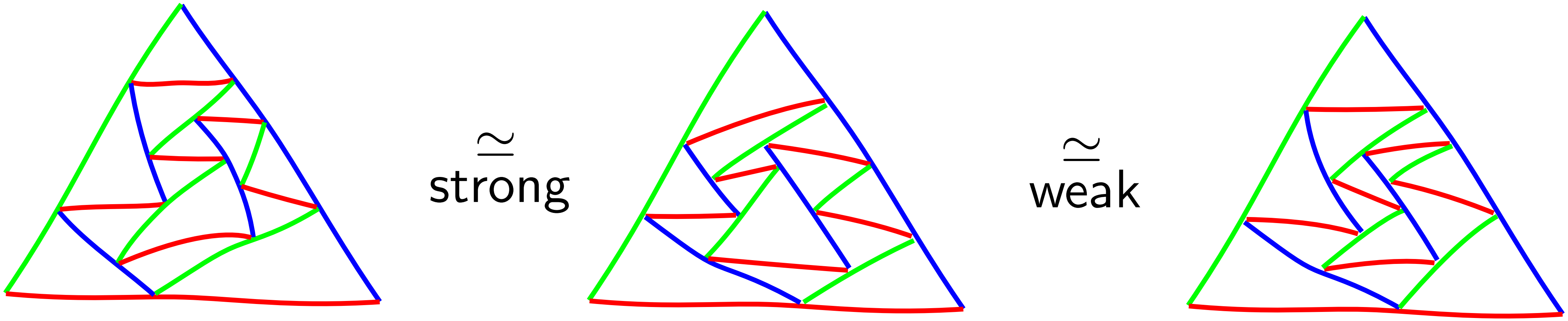
$w_n =$ # weak equivalence classes with $2n$ regions

$s_n =$ # strong equivalence classes with $2n$ regions

Back to tricolored contact-systems

Contact-system of curves



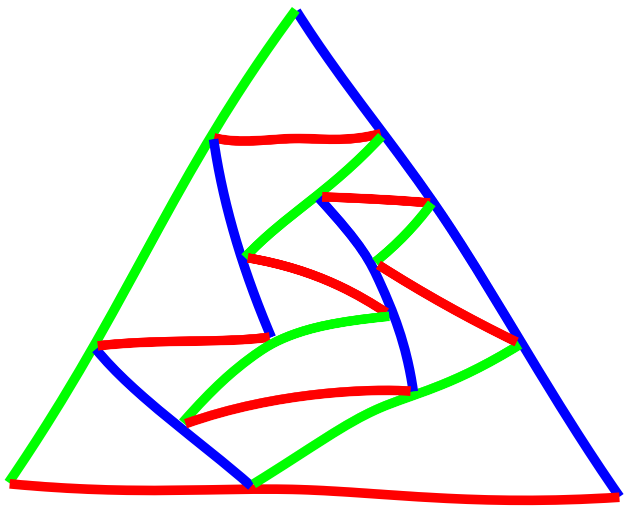


$w_n = $ # weak equivalence classes with $2n$ regions $= p_n$

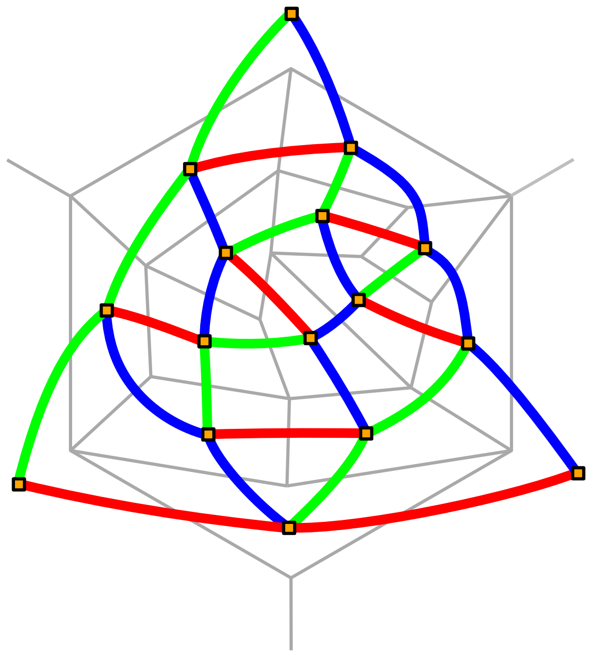
$s_n = $ # strong equivalence classes with $2n$ regions

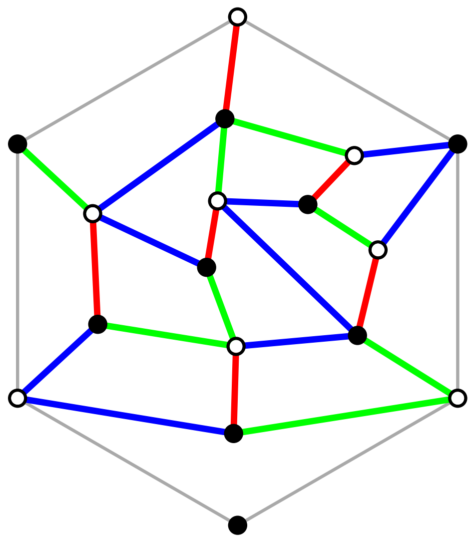
Strong tricolored systems

Duality:

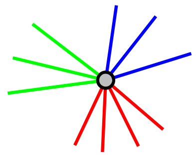


strong contact-system



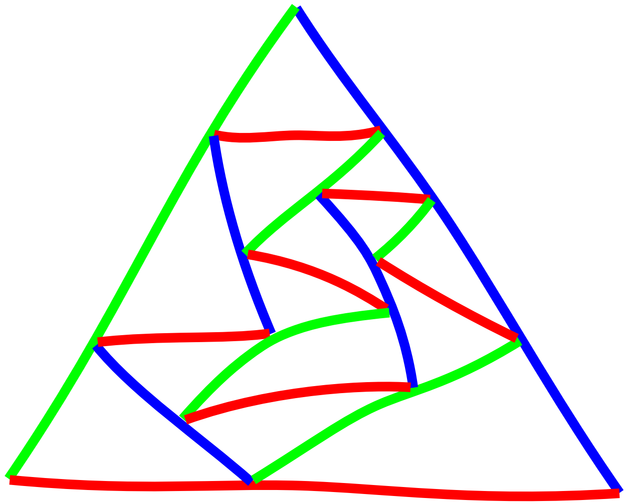


quadrangulation of hexagon
+ edge-tricoloration

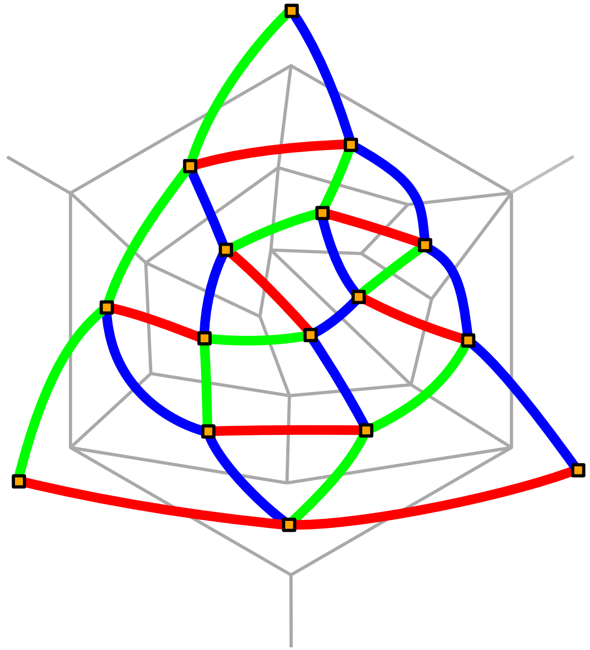
satisfying 

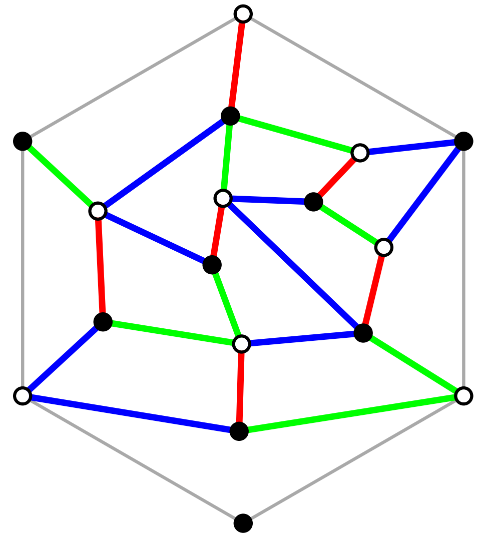
Strong tricolored systems

Duality:

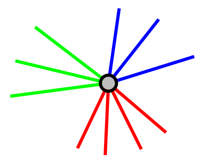


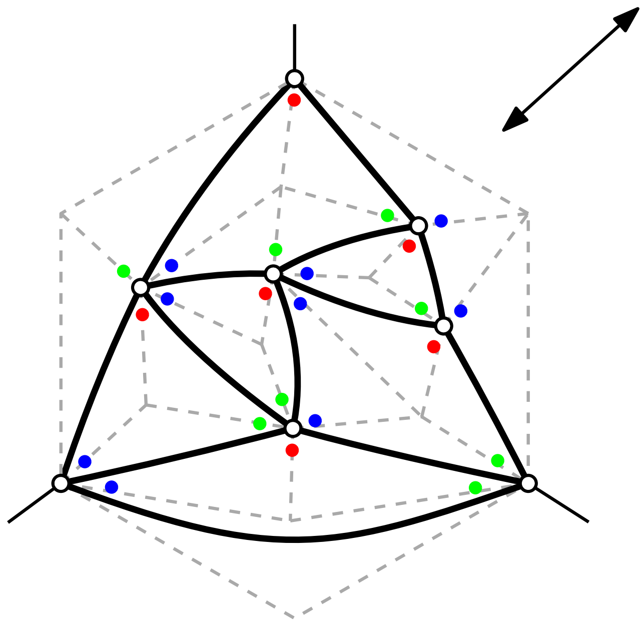
strong contact-system





quadrangulation of hexagon
+ edge-tricoloration

satisfying 

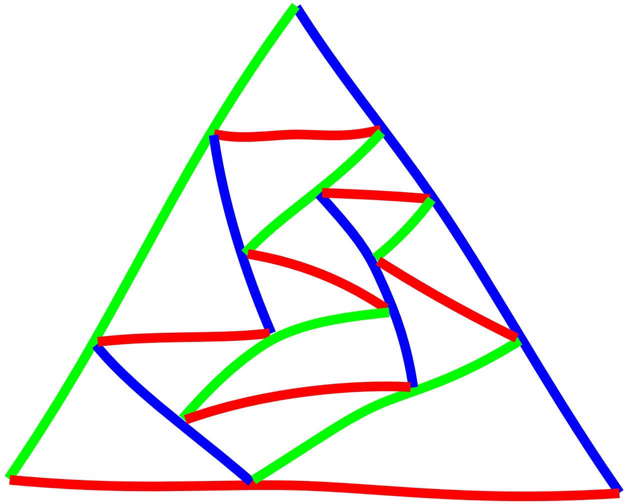


[Felsner'01]

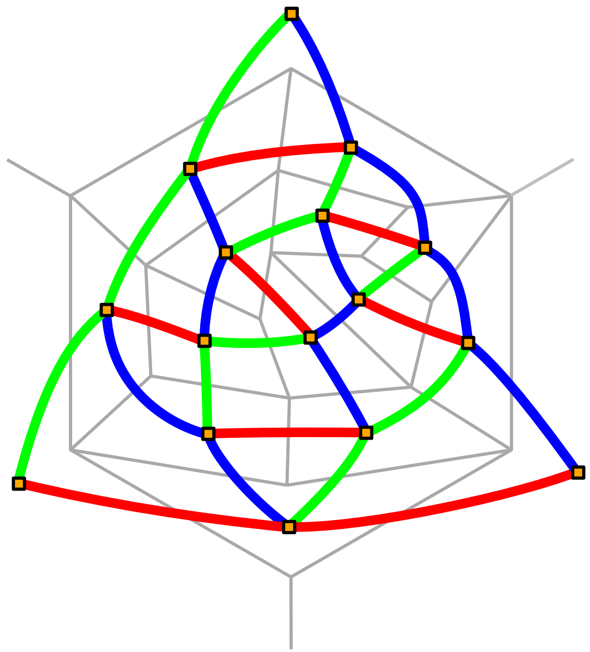
3-rooted 3-connected map + Schnyder labeling

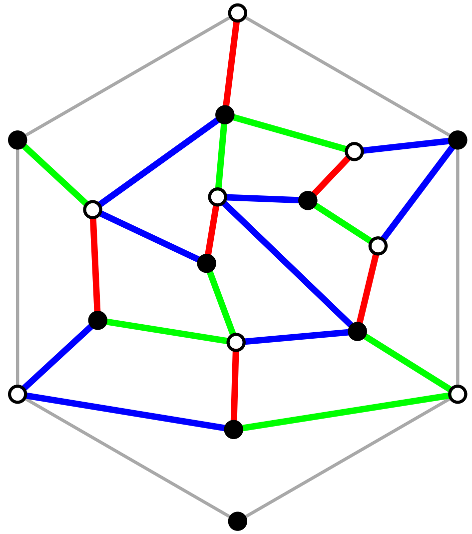
Strong tricolored systems

Duality:

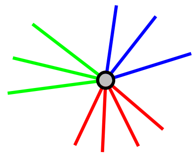


strong contact-system





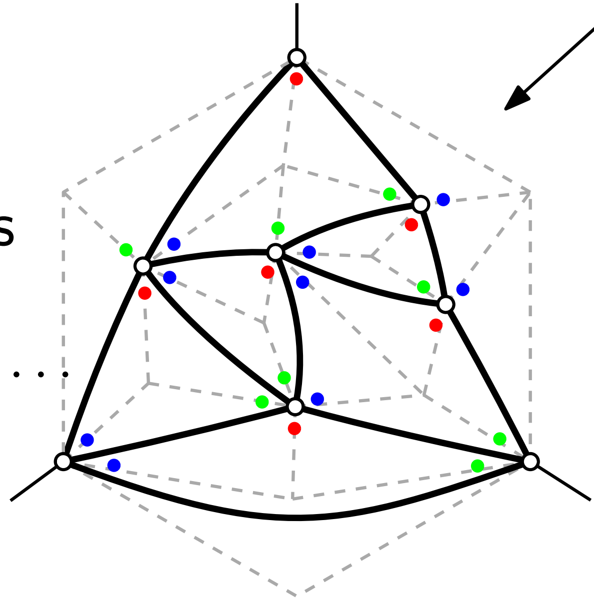
quadrangulation of hexagon
+ edge-tricoloration

satisfying 

Rk: 3-rooted 3-connected maps
have same counting series as
corner triangulations

$t^3 + 3t^5 + 4t^6 + 15t^7 + 39t^8 + 120t^9 + \cdots$

(possibly explained via "minimal" structures)

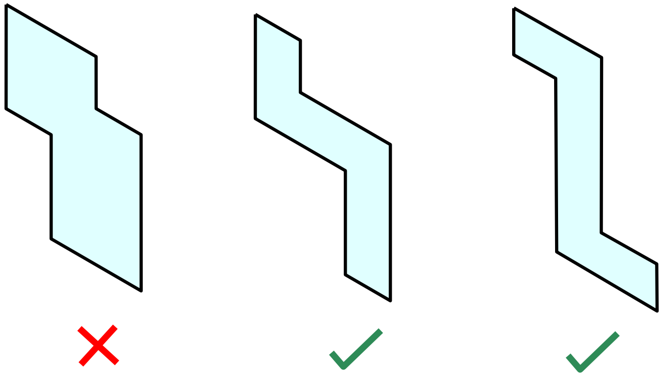


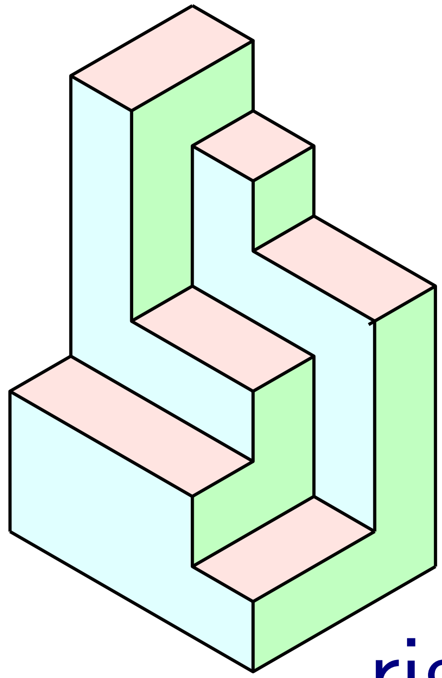
[Felsner'01]

3-rooted 3-connected map + Schnyder labeling

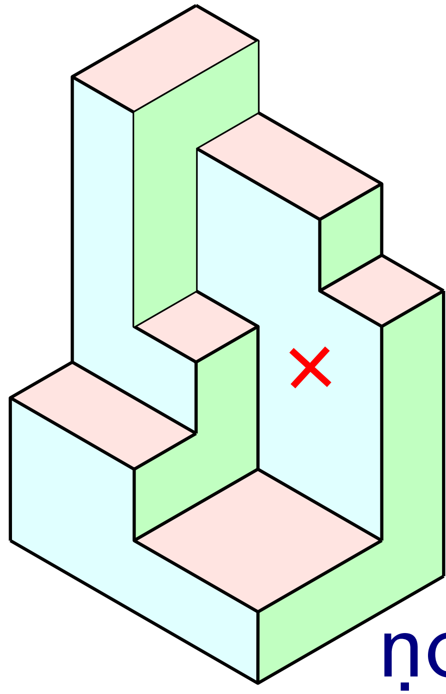
Corresponding corner polyhedra

Rigidity condition:
facets have "zig-zag" shape





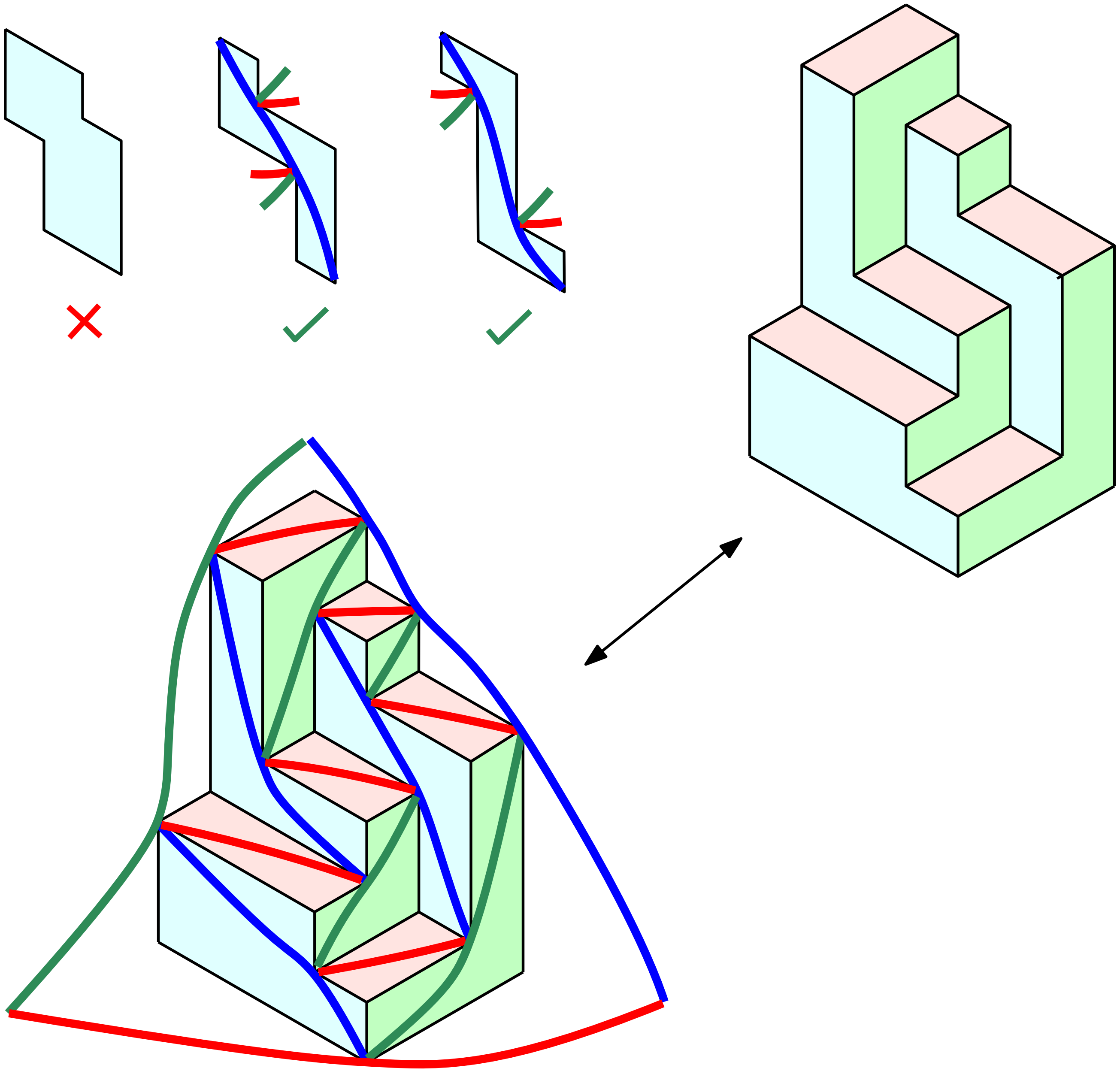
rigid



not
rigid

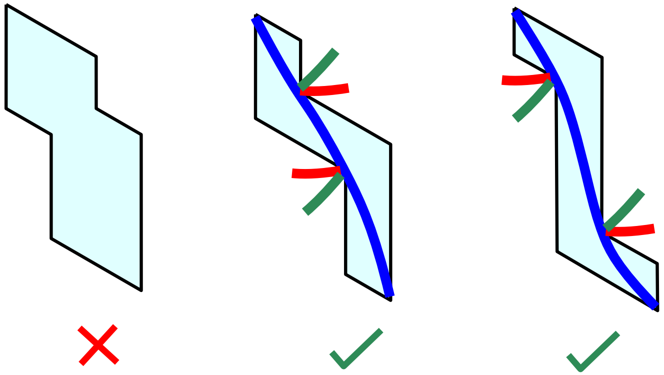
Corresponding corner polyhedra

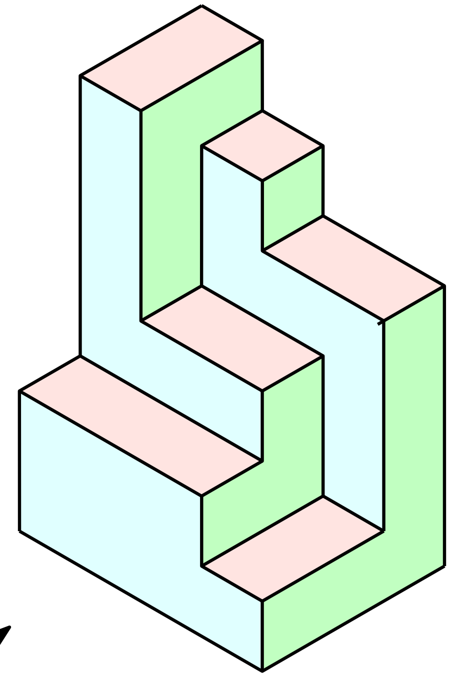
Rigidity condition:
facets have “zig-zag” shape

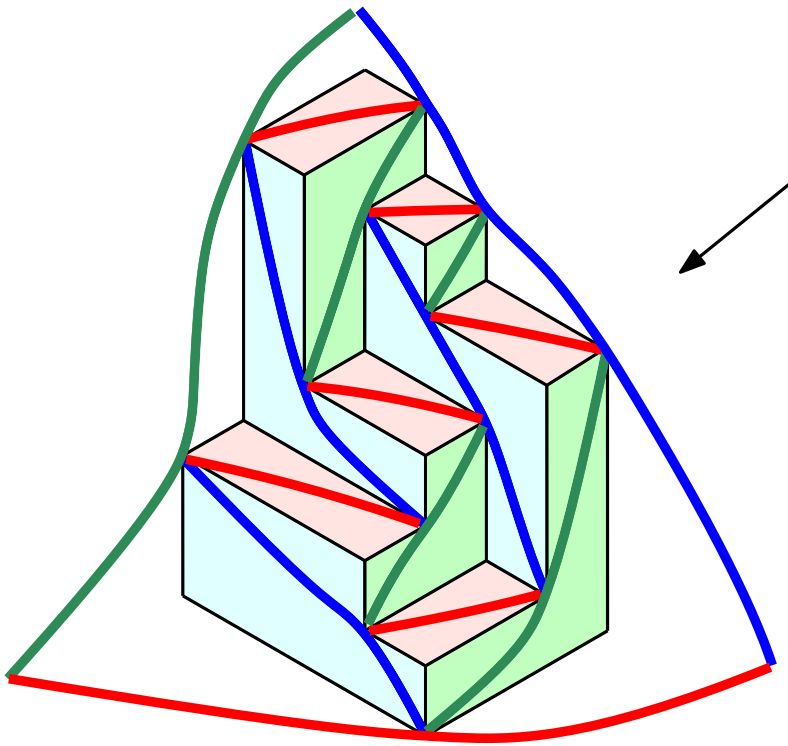


Corresponding corner polyhedra

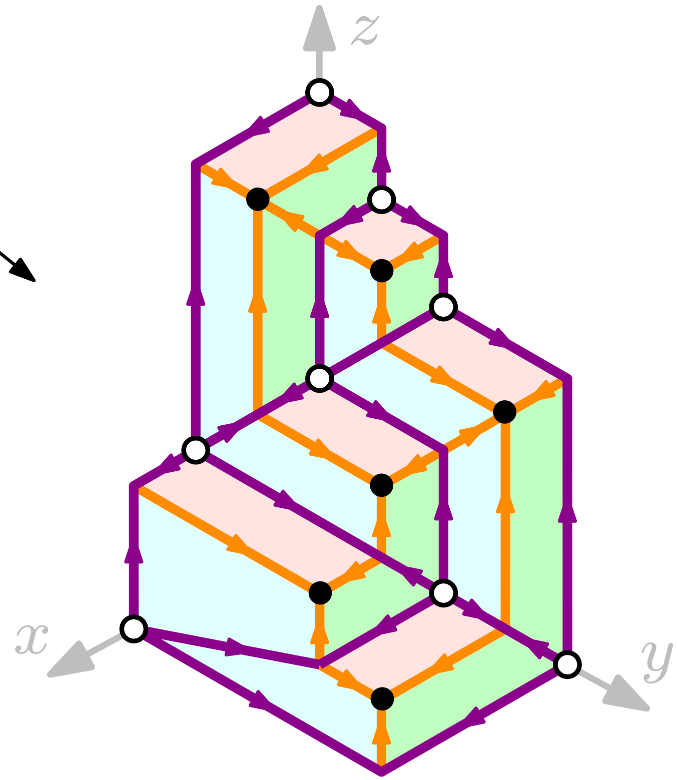
Rigidity condition:
facets have “zig-zag” shape



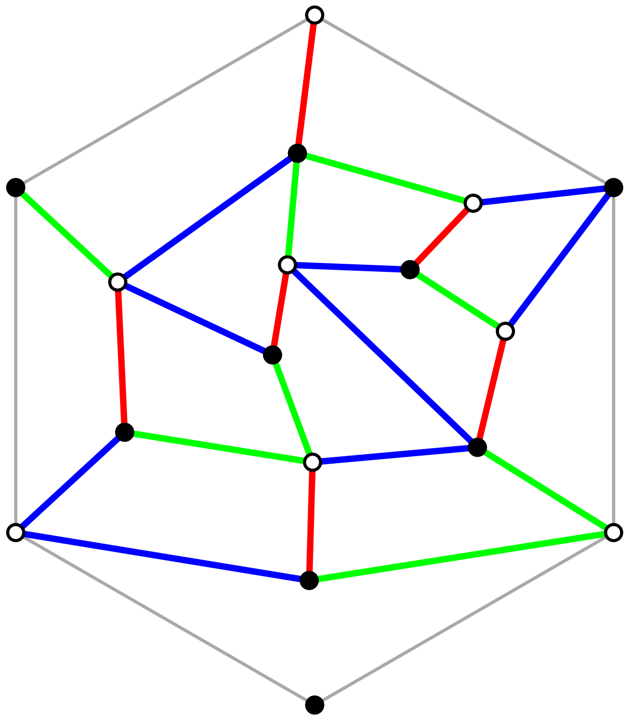


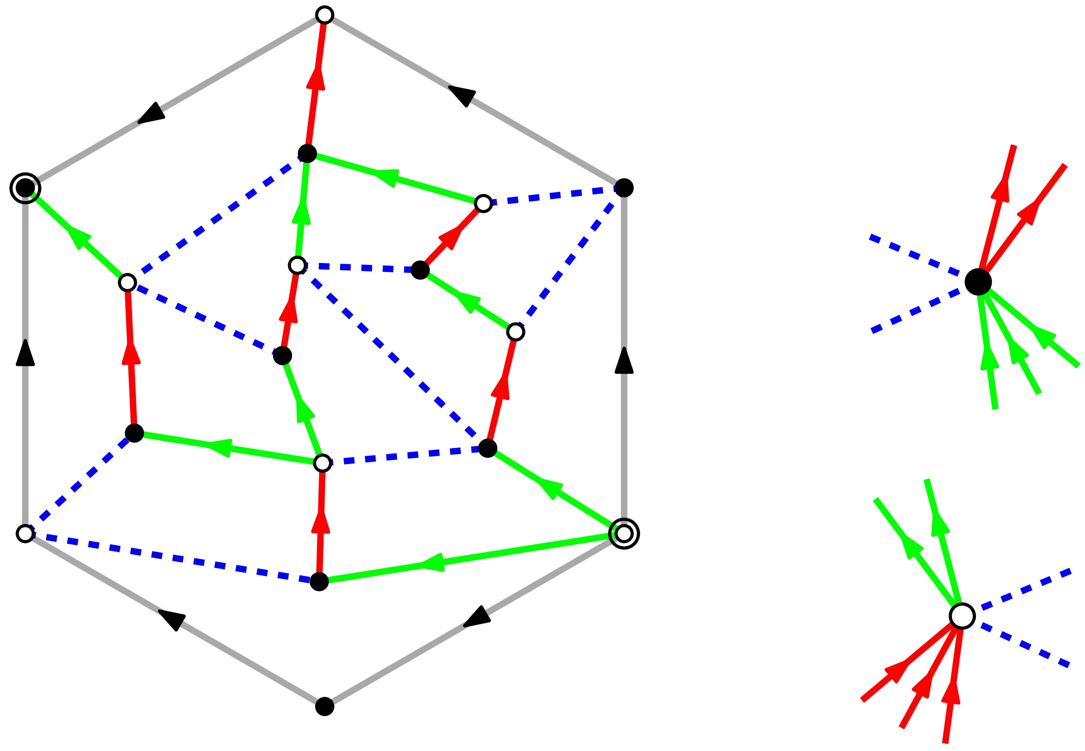


dominance drawing
[Miller’02,Felsner’03]



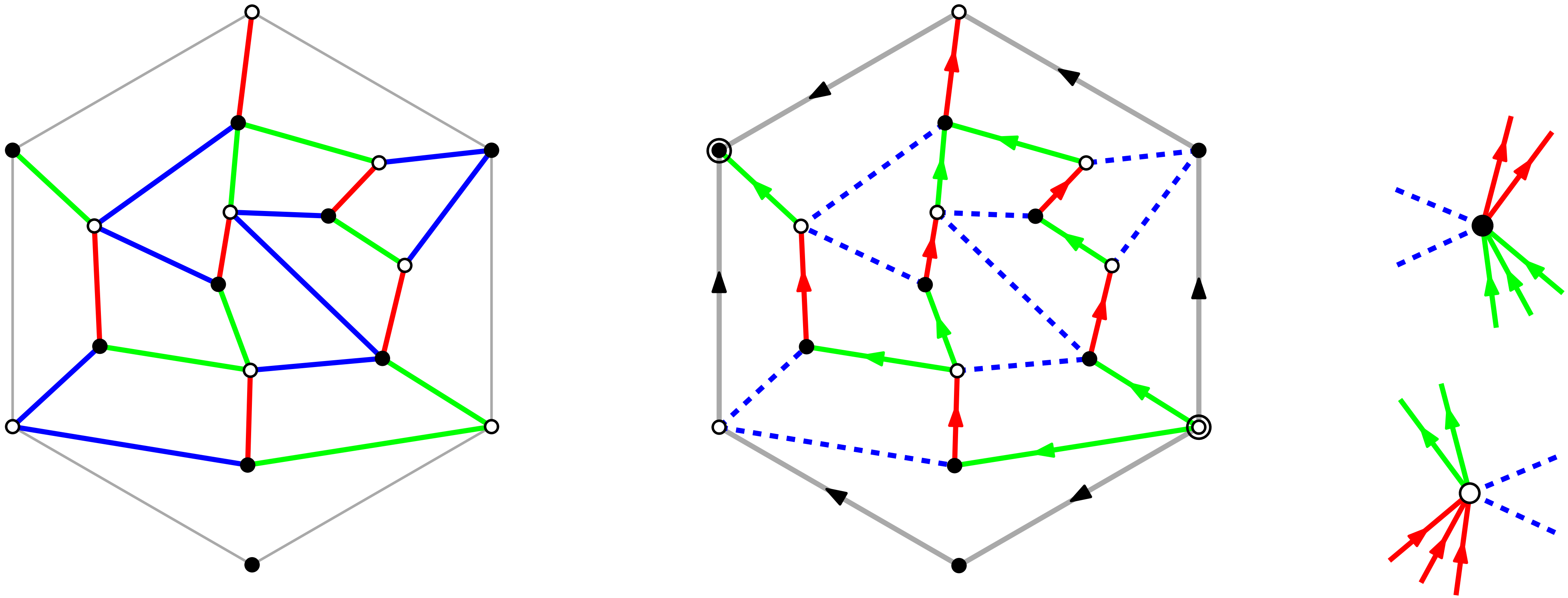
Corresponding plane bipolar orientations





bipartite bipolar orientation
+ transversal edges

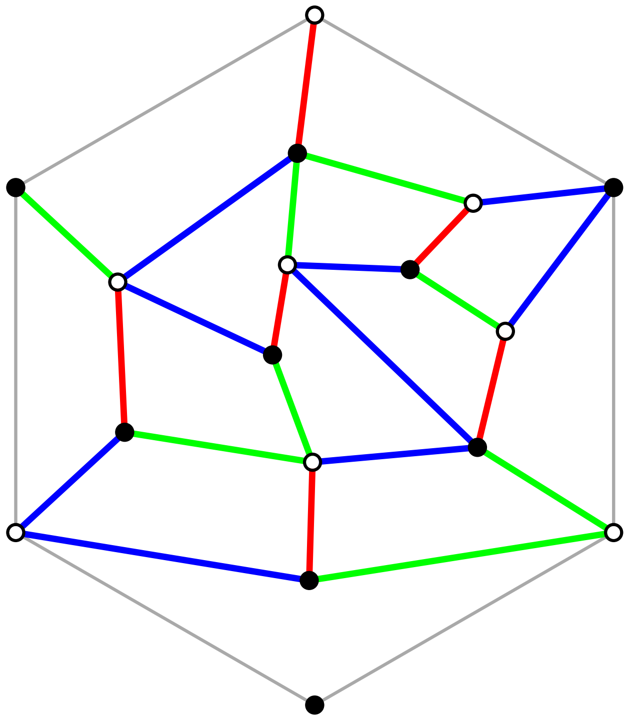
Corresponding plane bipolar orientations

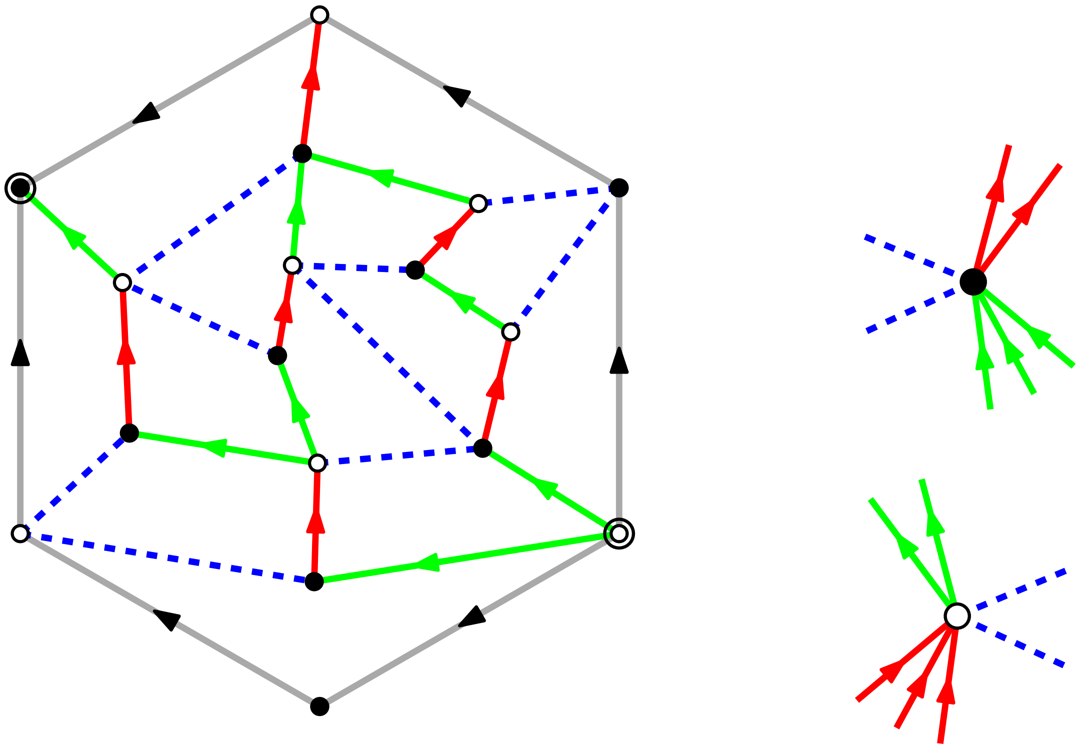


bipartite bipolar orientation
+ transversal edges

tandem walks have a bimodal condition + binomial weights

Corresponding plane bipolar orientations



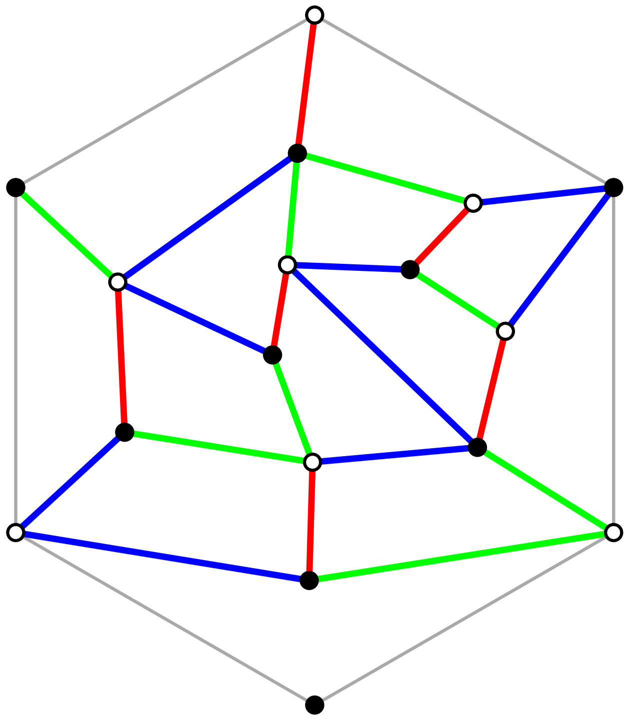


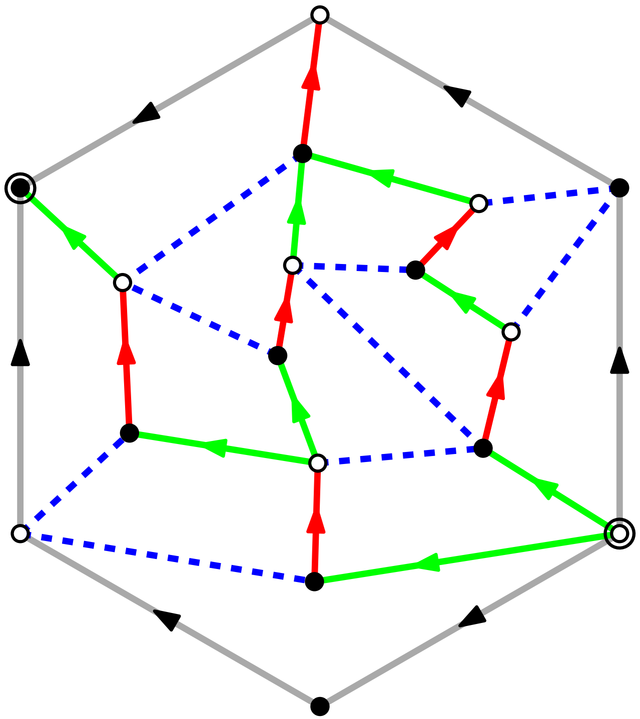
bipartite bipolar orientation
+ transversal edges

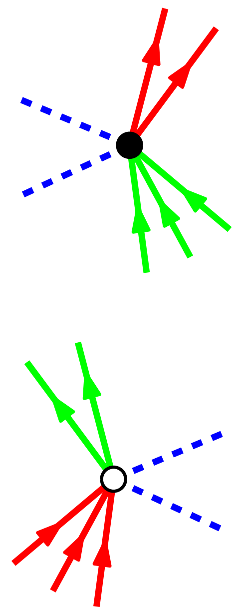
tandem walks have a bimodal condition + binomial weights

- recurrence for s_n
- **Conjecture:** $s_n \sim c\,(16/3)^n\, n^{-\alpha}$ for $c > 0$ and $\alpha = 1 + \frac{\pi}{\arccos(22/27)} \notin \mathbb{Q}$

Corresponding plane bipolar orientations







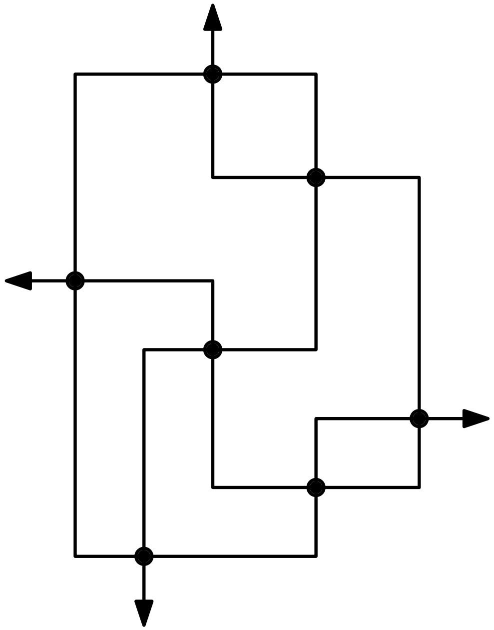
bipartite bipolar orientation
+ transversal edges

tandem walks have a bimodal condition + binomial weights

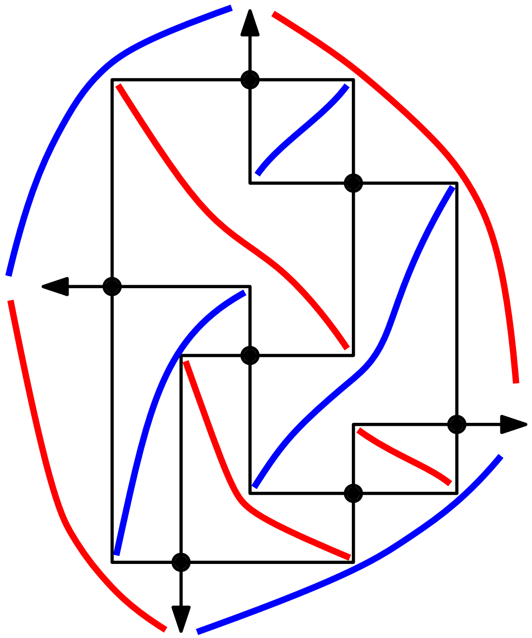
- recurrence for s_n
- **Conjecture:** $s_n \sim c\,(16/3)^n\, n^{-\alpha}$ for $c > 0$ and $\alpha = 1 + \frac{\pi}{\arccos(22/27)}$ $\notin \mathbb{Q}$
- triangulated case: $\leftrightarrow$ non-crossing pairs of Dyck walks

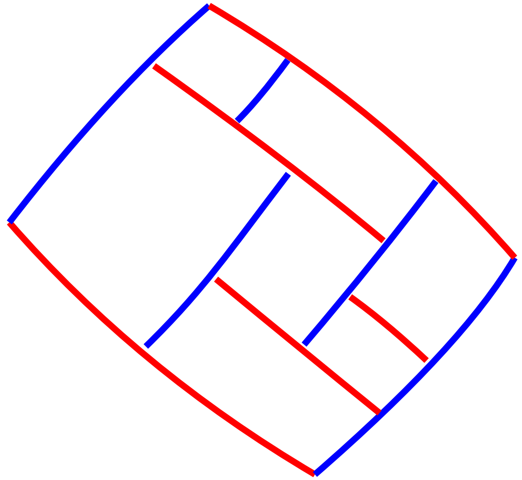
recover [Bernardi-Bonichon'09]

Similar models in 2d with 2 colors

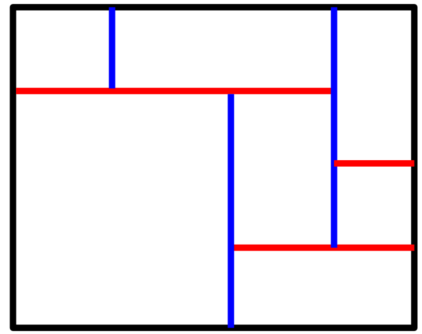


1-bent orthogonal drawing



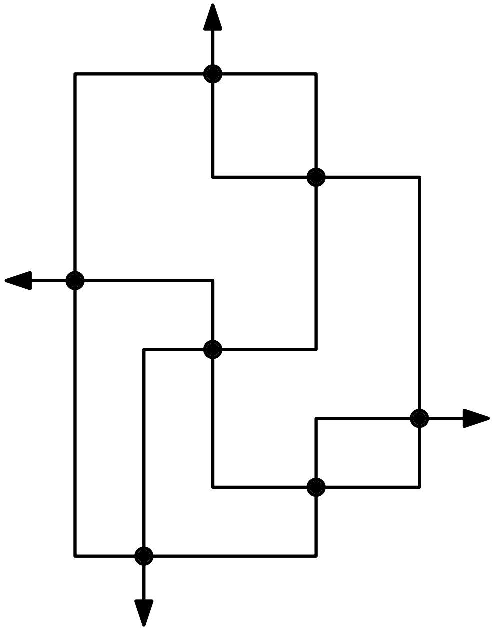


2-colored contact-system

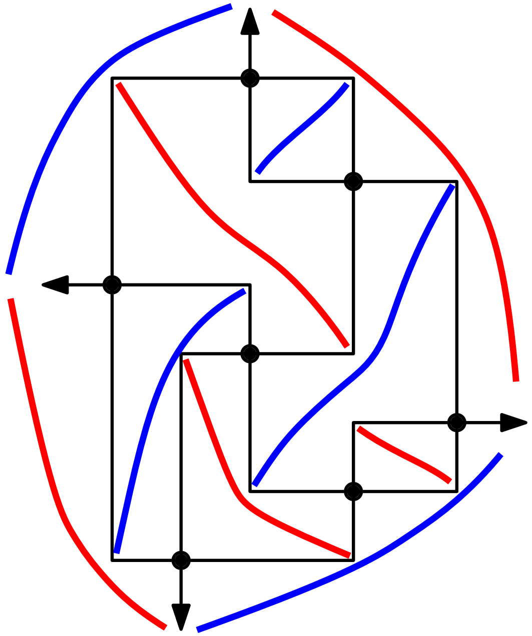


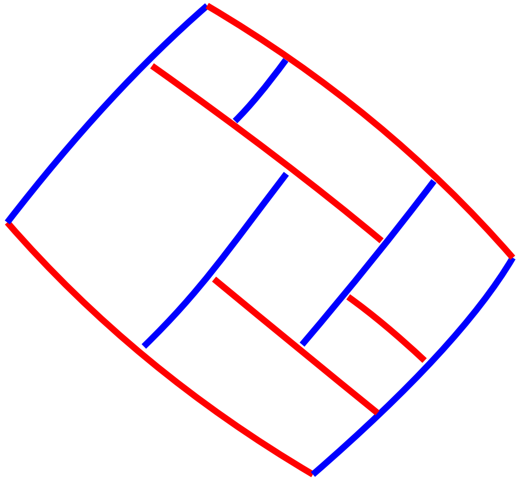
rectangulation

Similar models in 2d with 2 colors

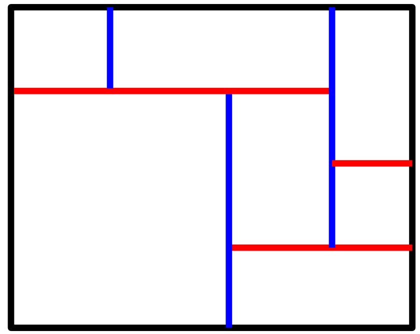


1-bent orthogonal drawing

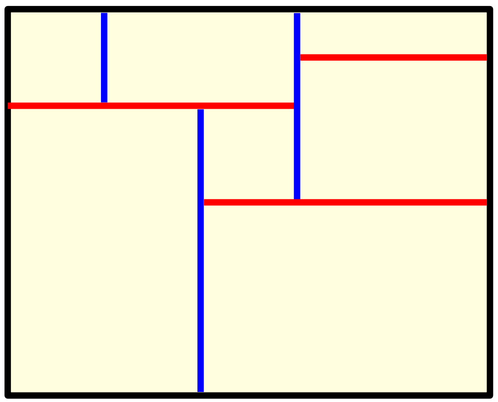




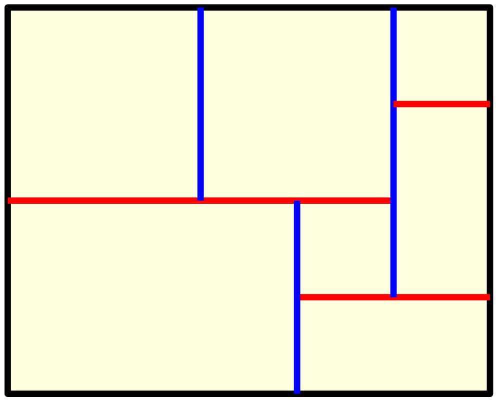
2-colored contact-system



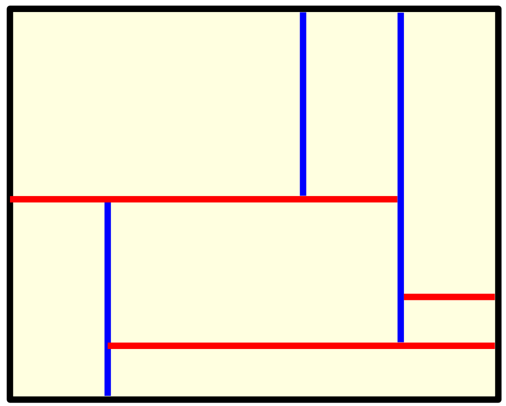
rectangulation



$\underset{\text{strong}}{\simeq}$



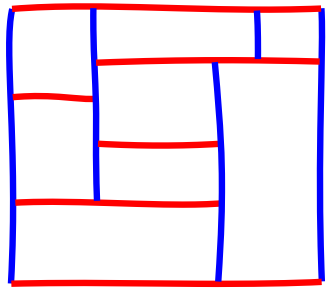
$\underset{\text{weak}}{\simeq}$

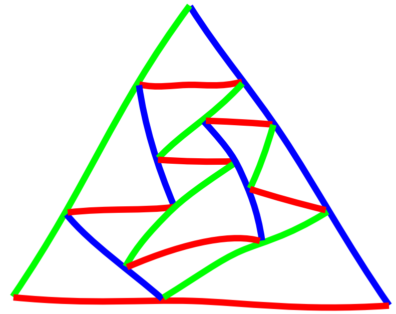


Summary on asymptotic enumeration

Asymptotic estimate $c\ \gamma^n\ n^{-\alpha}$

$\alpha = 1 + \frac{\pi}{\theta}$

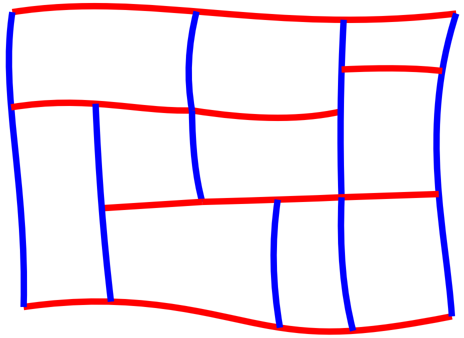


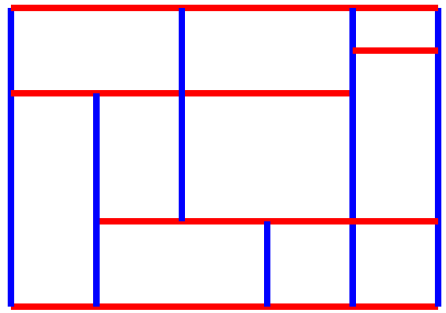


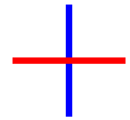
	weak	strong	weak	strong
	bipolar orientations	transversal structures	polyhedral orientations	(3c) Schnyder labelings
γ	8	$27/2$	$9/2$	$16/3$
$\cos(\theta)$	$1/2$	$7/8$	$9/16$ (*)	$22/27$ (*)
α	4	$\approx 7.21 \notin \mathbb{Q}$	$\approx 4.23 \notin \mathbb{Q}$	$\approx 6.08 \notin \mathbb{Q}$

(*) up to extending [Denisov-Wachtel] to bimodal setting

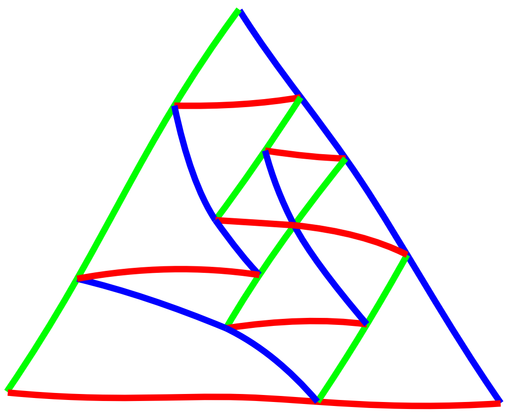
Extension to models with degeneracies

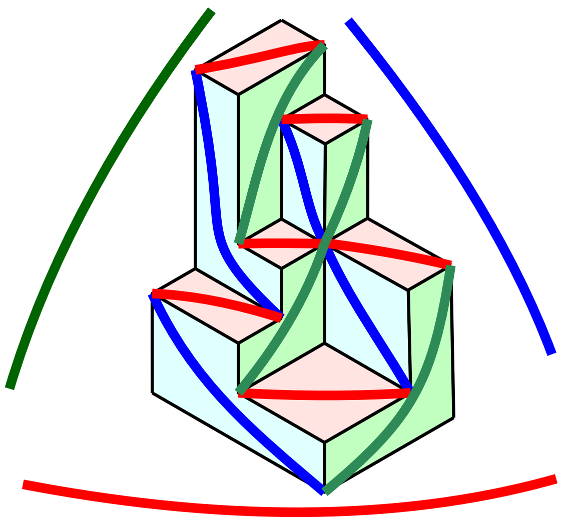


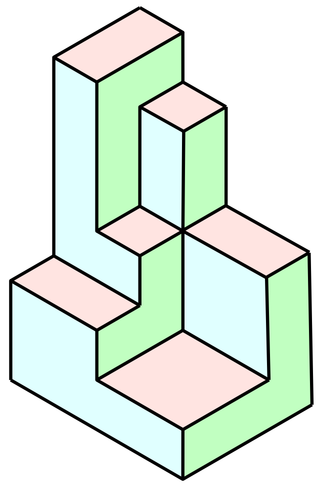


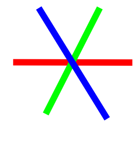
weight v per 

also counted in [Conant, Michaels'12]

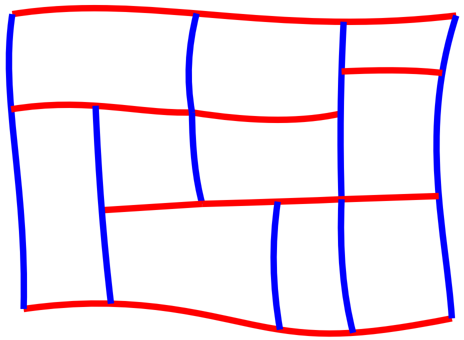


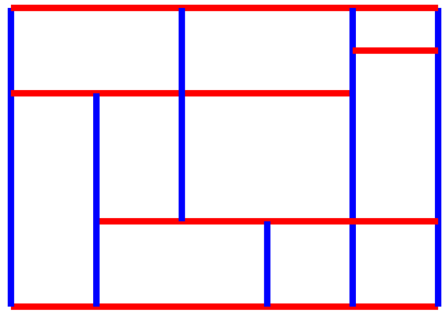


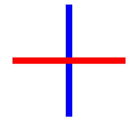


weight v per 

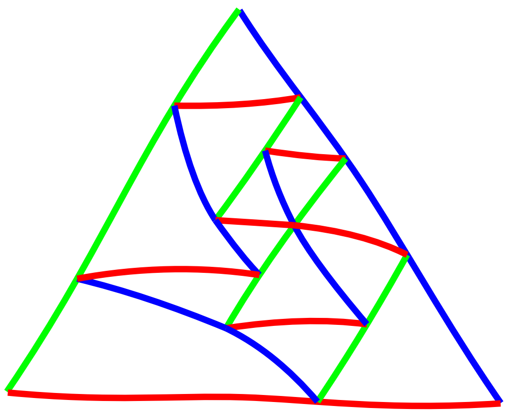
Extension to models with degeneracies

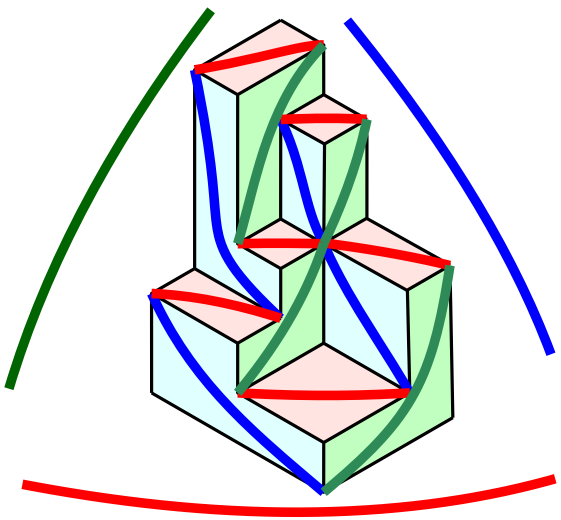


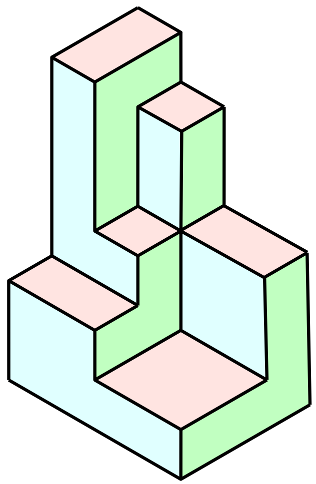


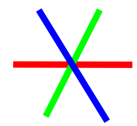
weight v per 

also counted in [Conant, Michaels'12]

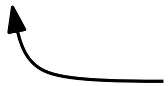






weight v per 

Asymptotic exponent $\alpha(v)$ computable $\alpha(v) \to \infty$ as $v \to \infty$



regular grid behaviour

Some open questions

- Combinatorial explanation of growth rates $9/2$ (resp. $16/3$)
  (would be convenient for entropic encoding)

- Counting (types of) simple orthogonal polyhedra (and subfamilies)

- Models of 2d (or 3d) permutations in bijection to corner polyhedra