Bijections for Baxter families

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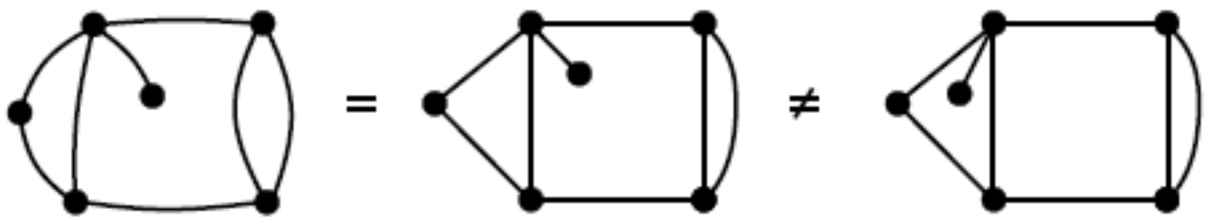
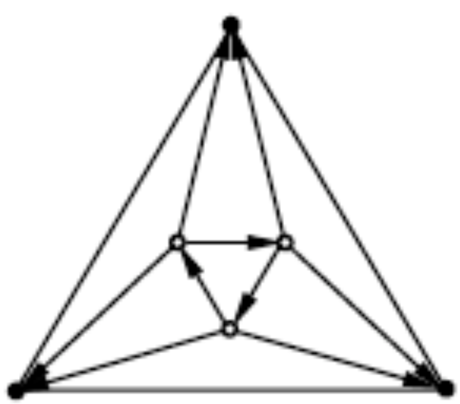
Part 1: decomposition of oriented maps

Oriented maps

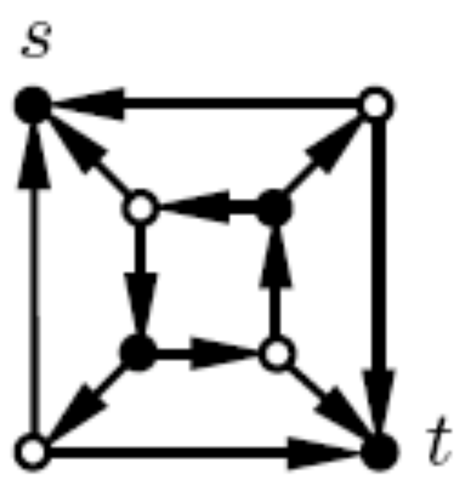
- Planar map = planar graph embedded in the plane



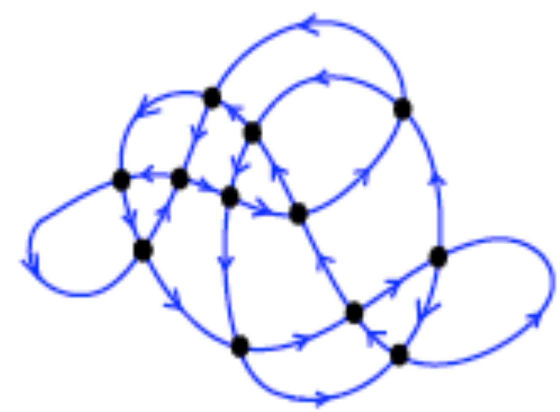
- Oriented map = planar map + orientation of the edges
- Families of oriented maps:



3-orientations



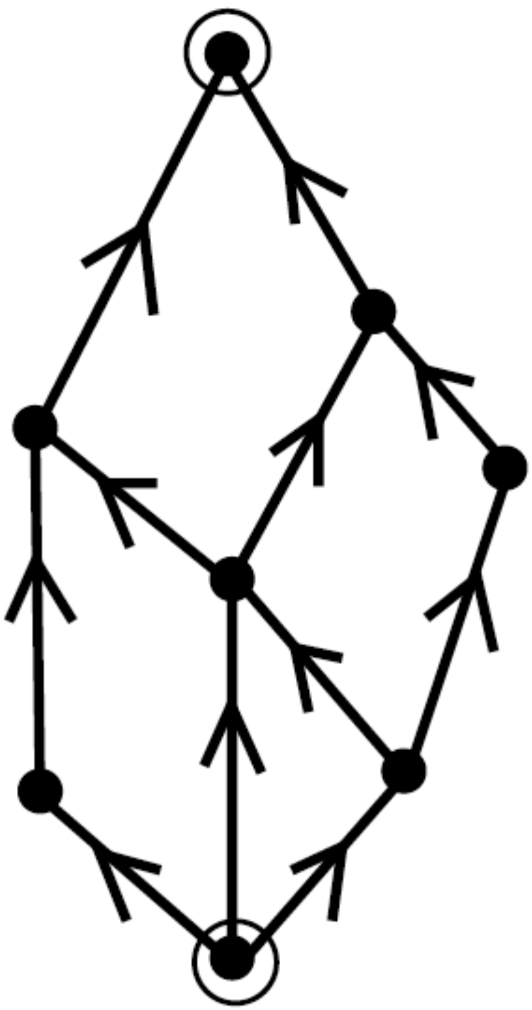
2-orientations



Eulerian orientations

Plane bipolar orientations

- Bipolar orientation = acyclic orientation with unique source and unique sink

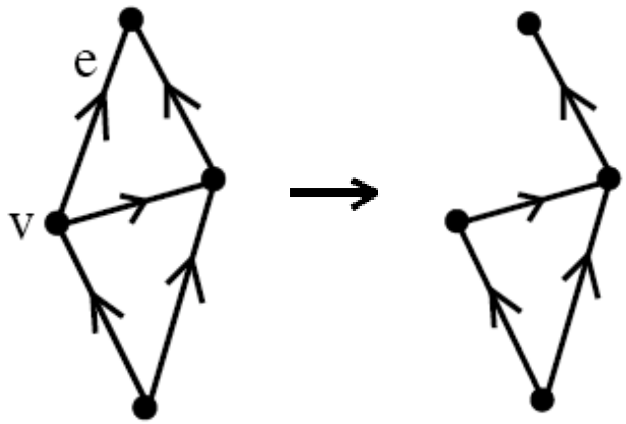


- Plane bipolar orientation :
  - bipolar orientation on a planar map
  - the source and the sink are incident to the outer face

Decomposing plane bipolar orientations

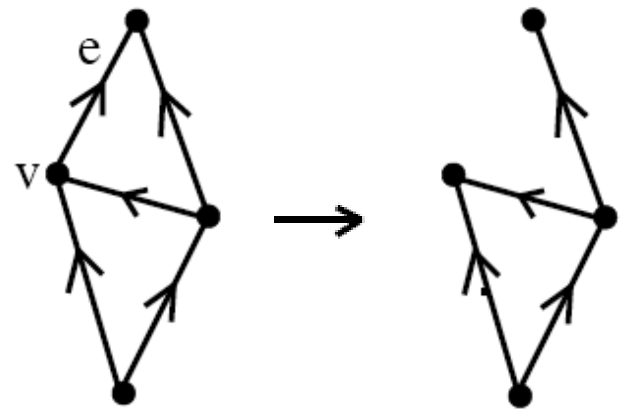
- Root-edge e = top-edge of left outer boundary
  root-vertex v = origin of root-edge
- Tutte's method, delete the root-edge. Then two cases:

1. still bipolar



Outdeg(v) > 1

2. not bipolar anymore



Outdeg(v) = 1

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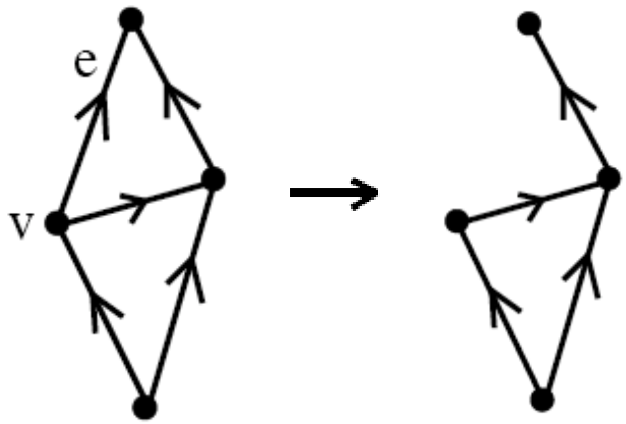

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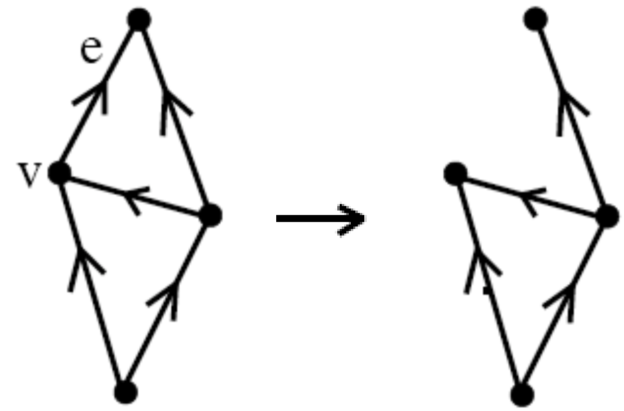
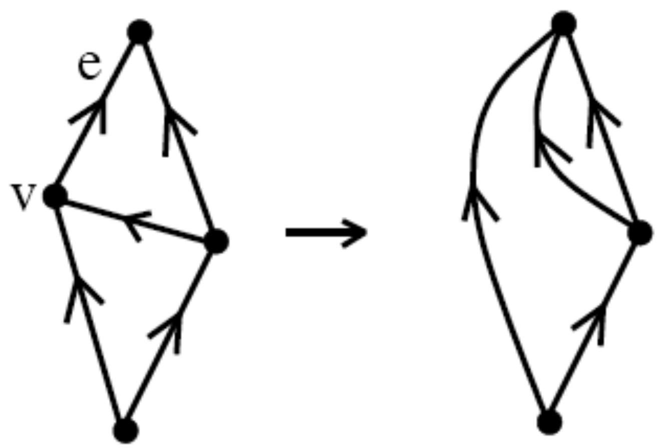


Outdeg(v) = 1

- In case 2., rather contract the root-edge to remain bipolar

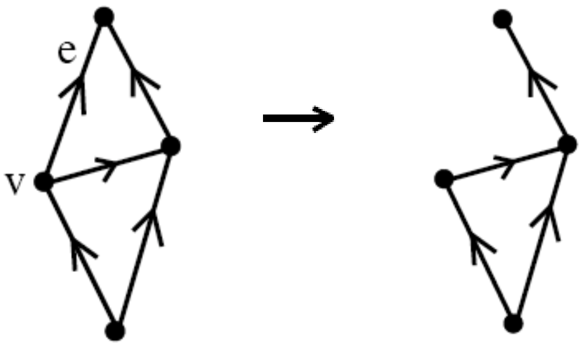


Generating tree

- Parent of a plane bipolar orientation:

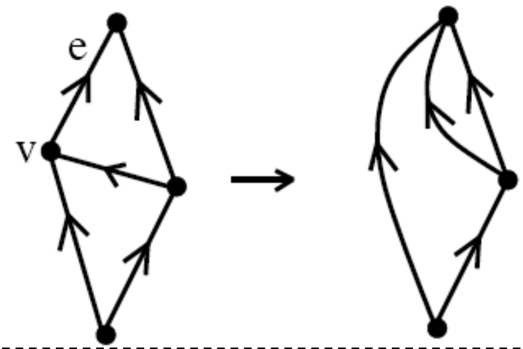
Delete root-edge

if Outdeg(Root-vertex) > 1

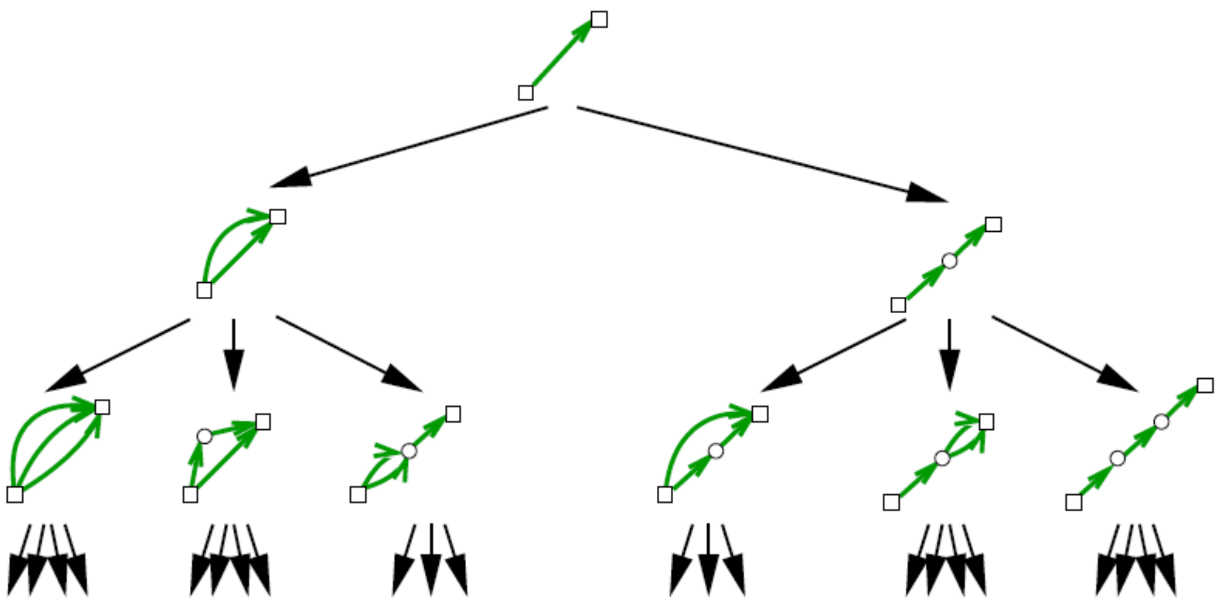


Contract root-edge

if Outdeg(Root-vertex) = 1



- Yields a generating tree for plane bipolar orientations:

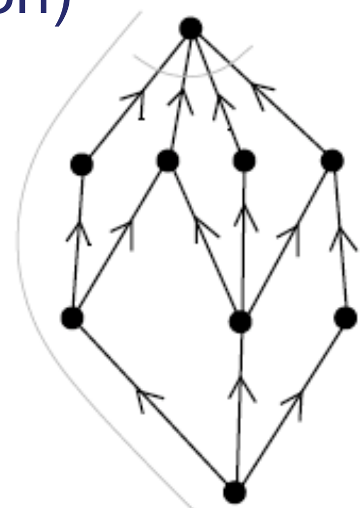


Characterizing the children

- Two types of children (whether parent is obtained by deletion or contraction)

j = length(left boundary)

j=3 here



i = indegree(sink)

i=4 here

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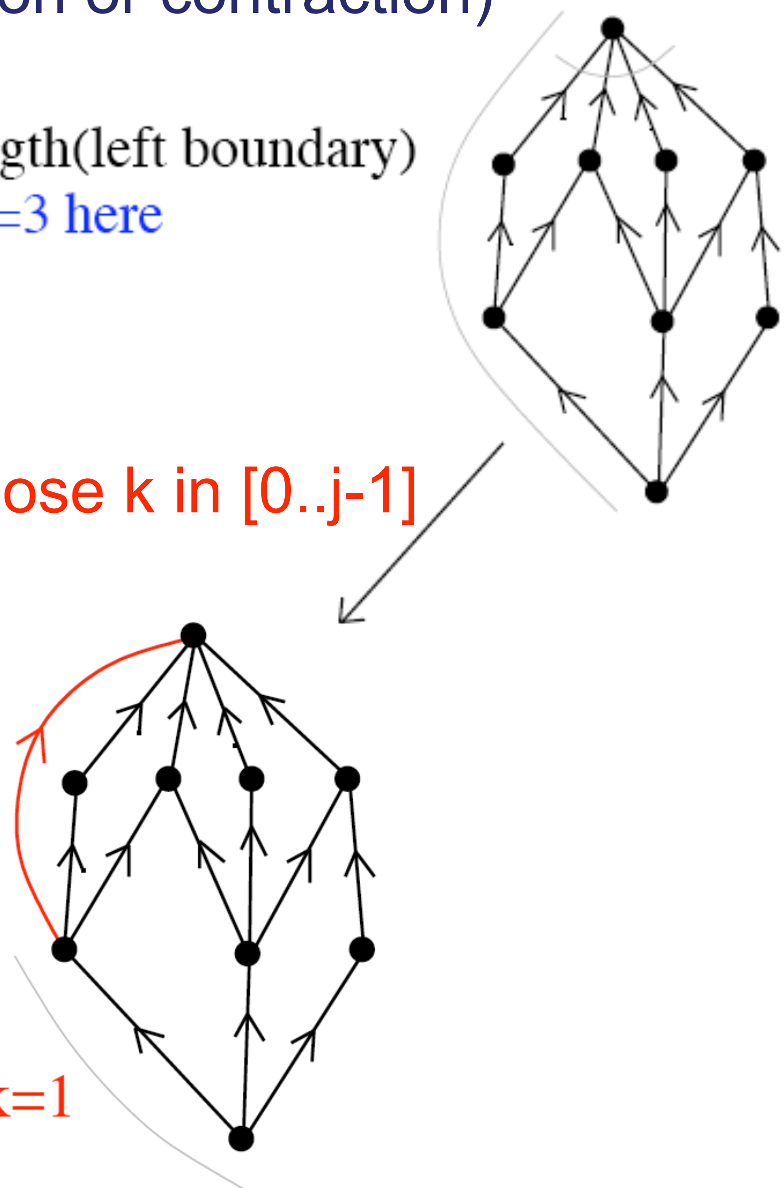

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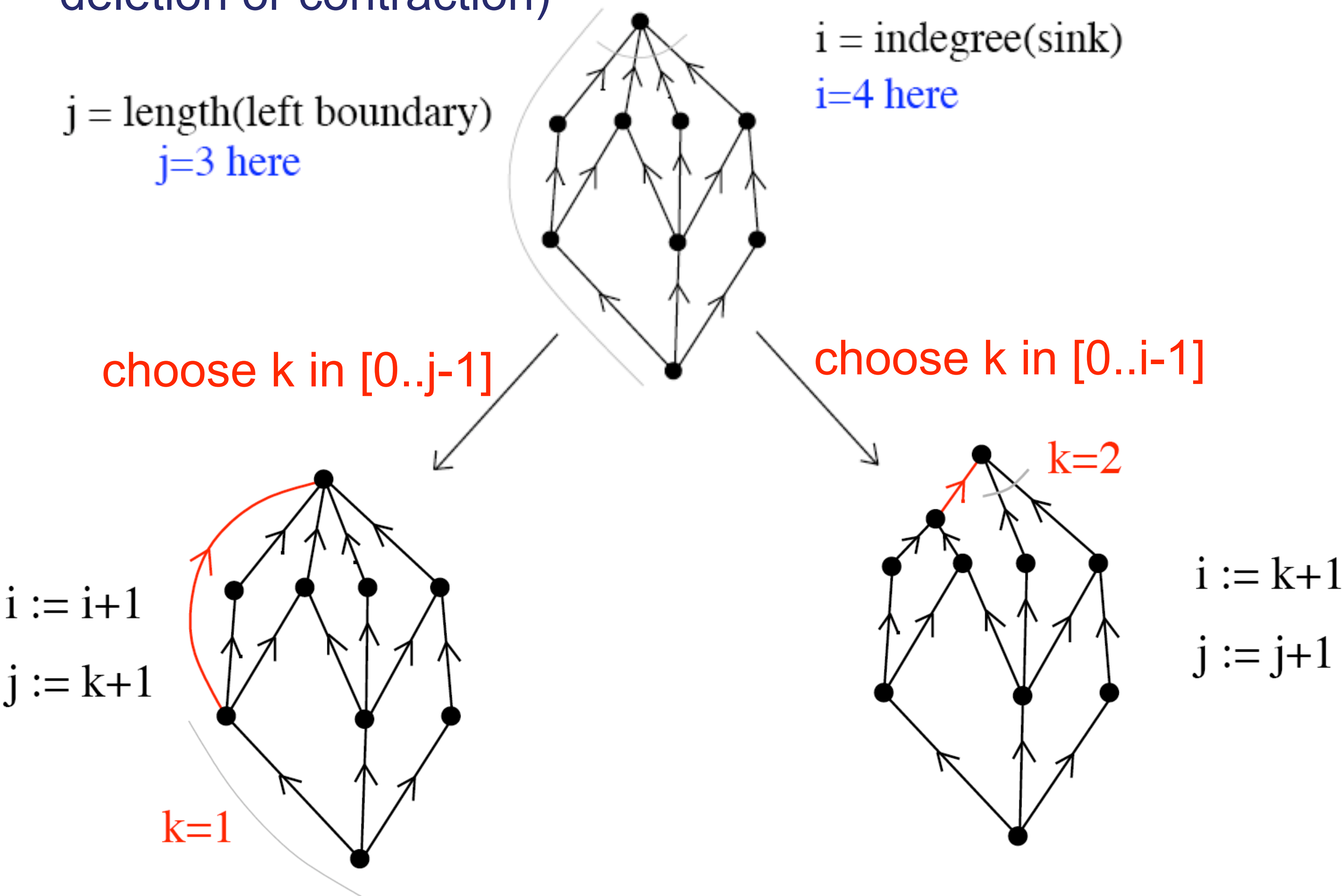
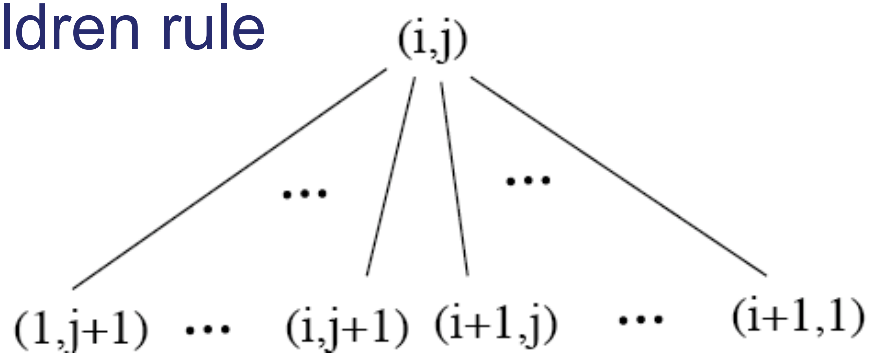
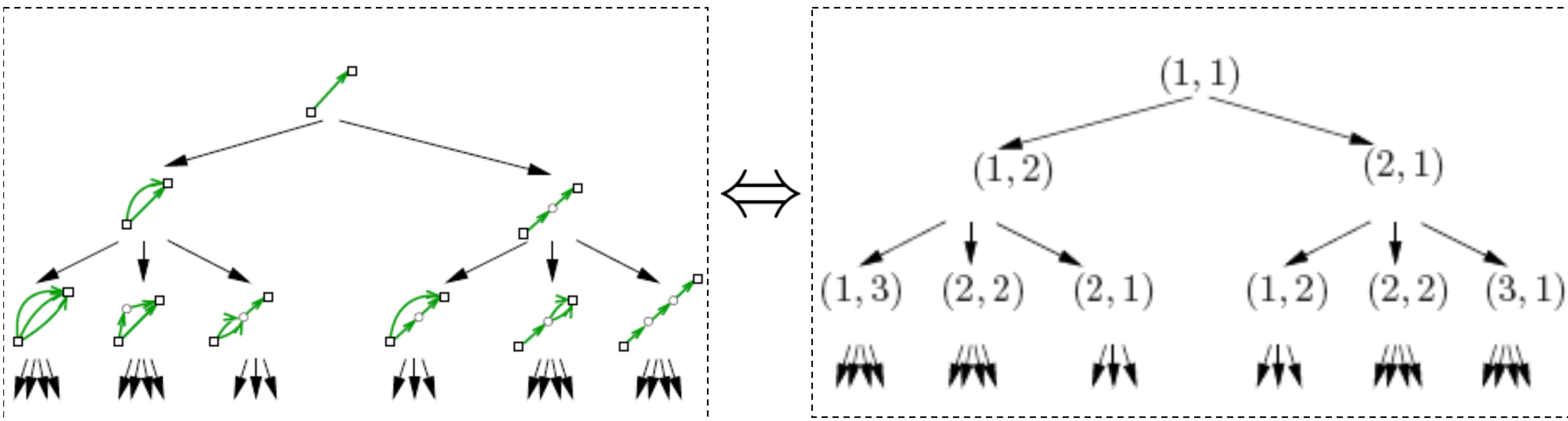


Summary

Theorem [reformulate Baxter’01]: The generating tree of plane bipolar orientations is isomorphic to the generating tree T with root (1,1) and children rule

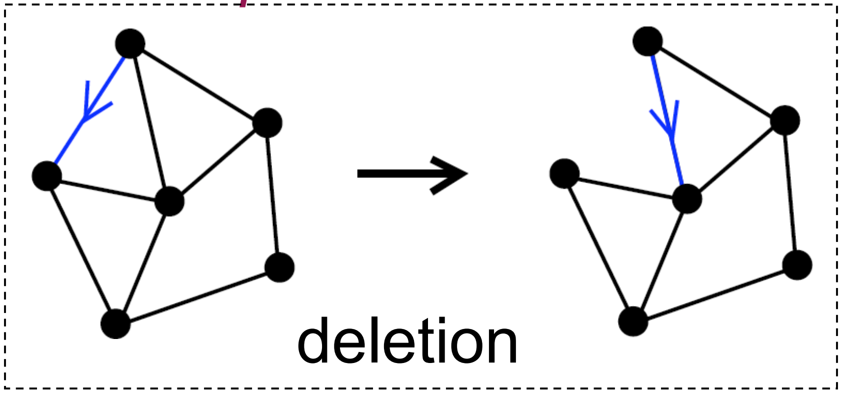


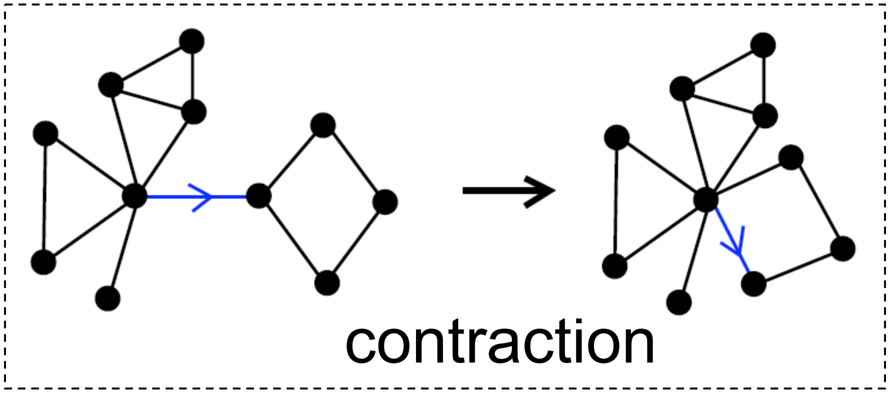
First levels:



Similar approach for unoriented maps ?

- The *parent* of a rooted map, two cases:





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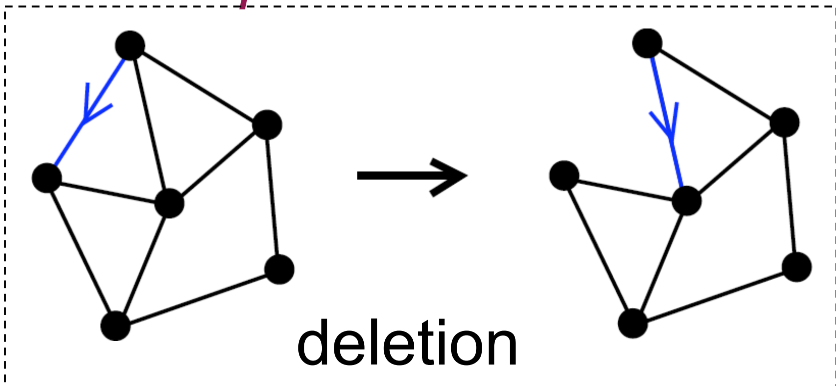
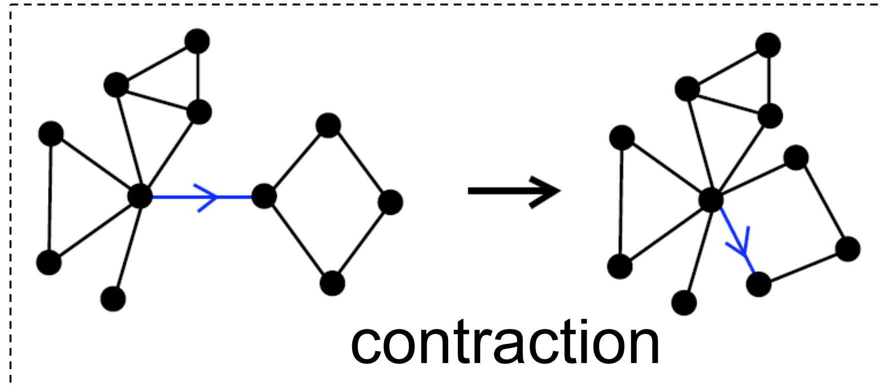
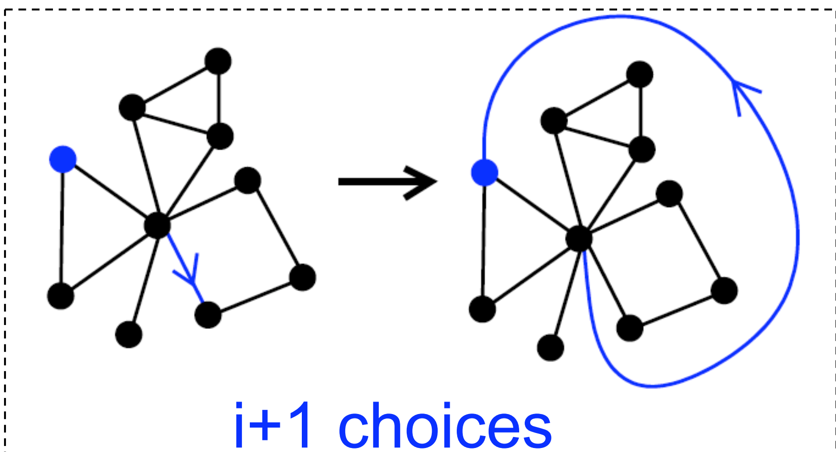
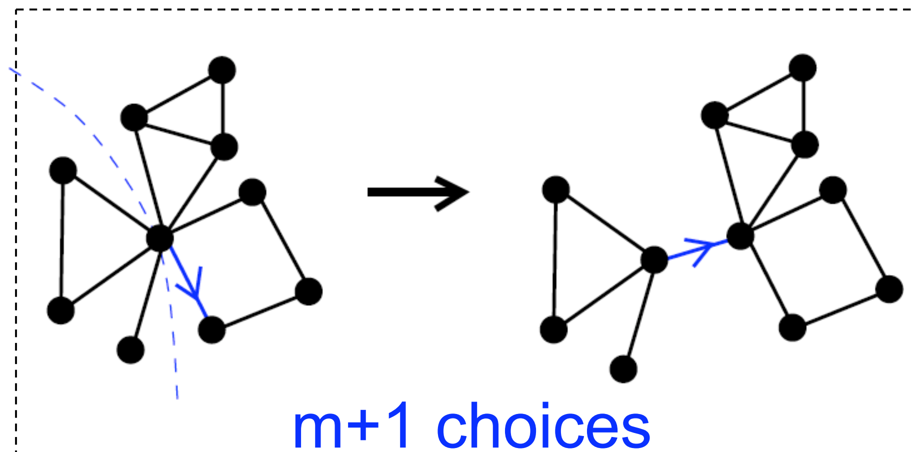




- Number of children of a map ? Needs two parameters

i = degree(outer face)   *m* = # blocks at root-vertex





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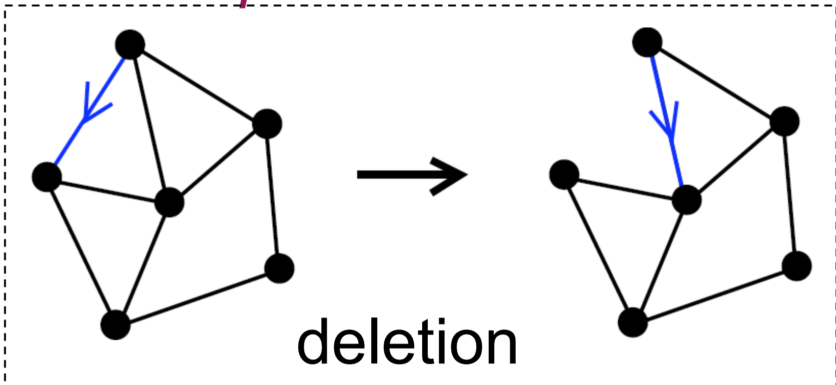

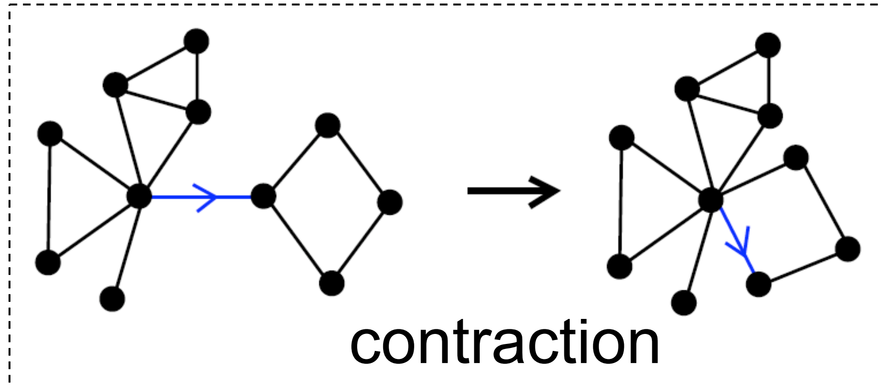

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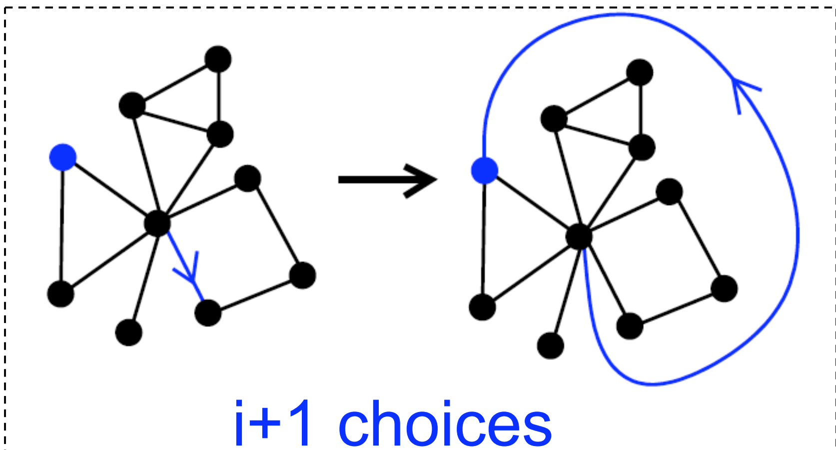
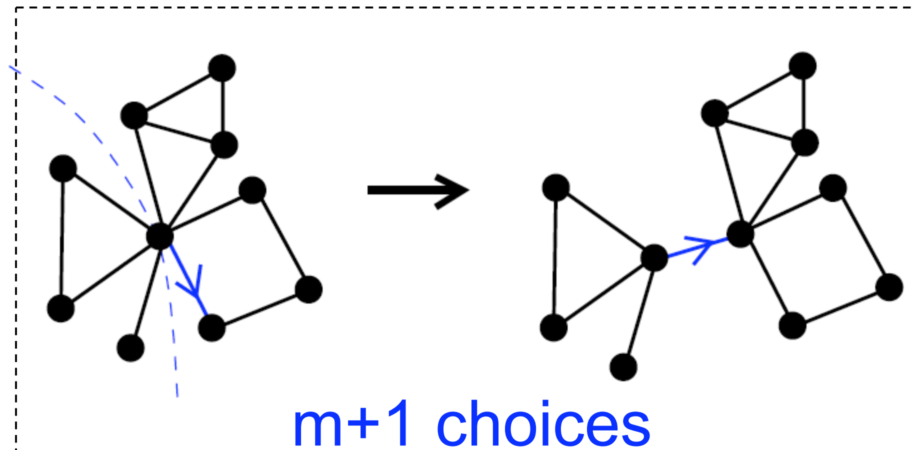




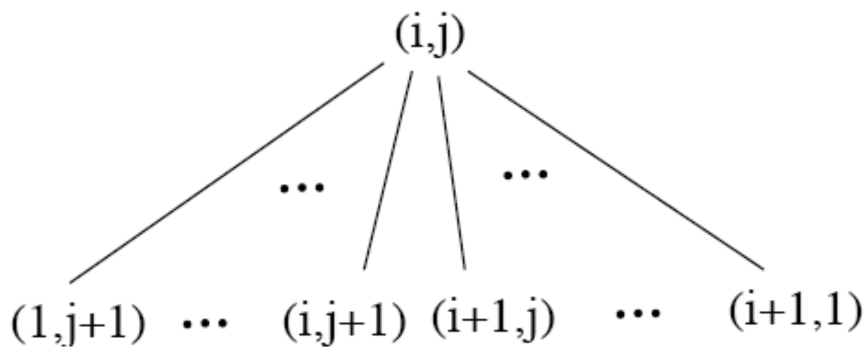
- Needs to maintain outer degrees $d_1,\ldots,d_m$ of the blocks incident to the root-vertex.

  Unbounded number of catalytic parameters !

Part 2: Baxter families

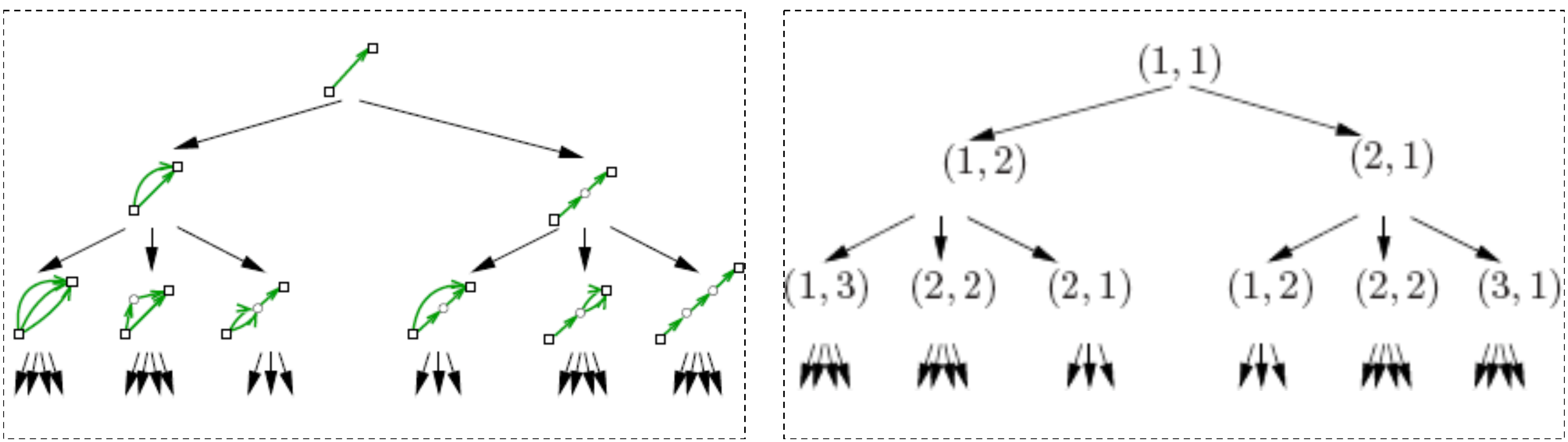
Definition of Baxter families

Def: Any combinatorial family with generating tree isomorphic to the generating tree T with root (1,1) and children rule



is called a Baxter family

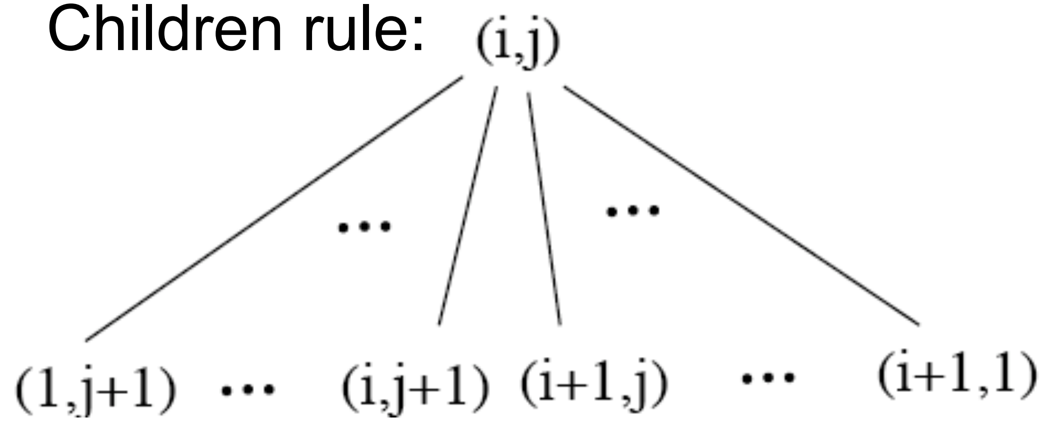
Example: plane bipolar orientations form a Baxter family



Parallel with Catalan families

- Baxter families: two catalytic parameters

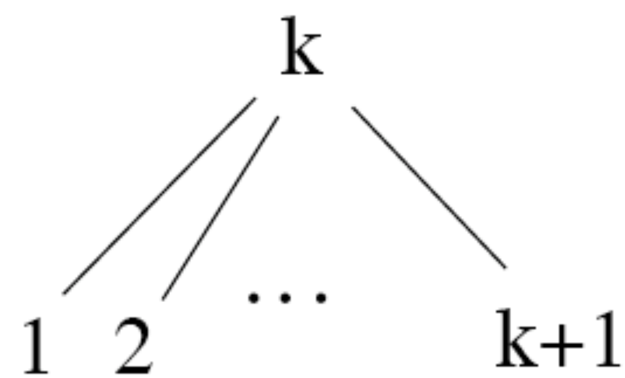
Children rule:



- Catalan families: one catalytic parameter

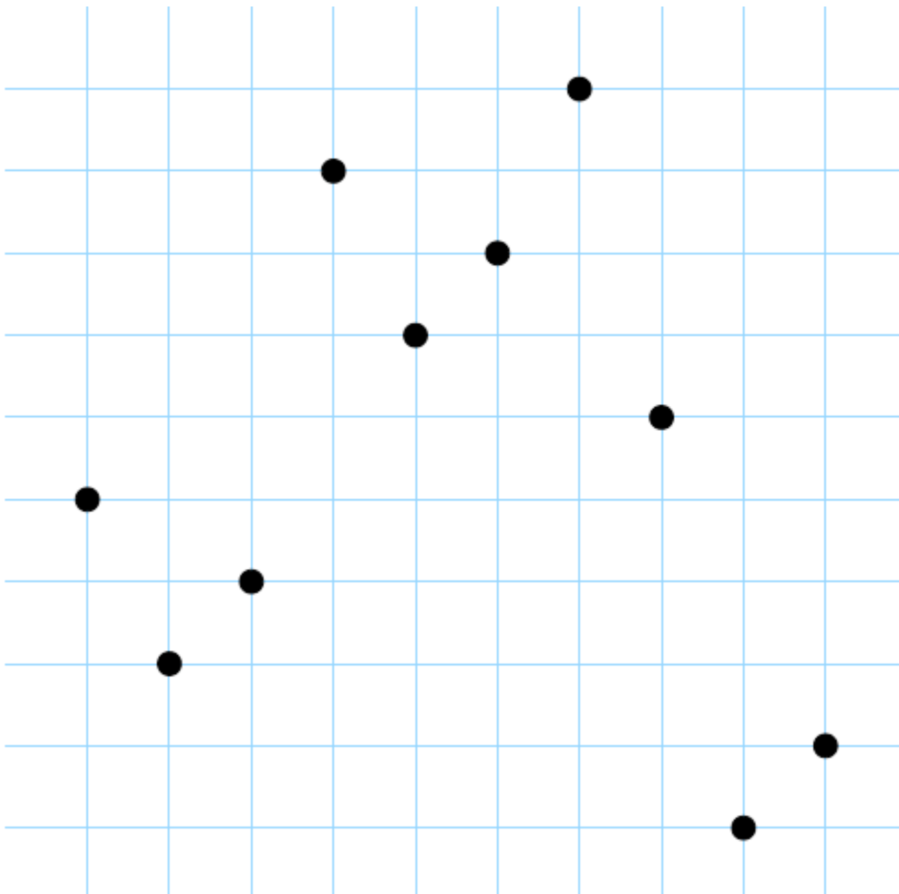
Children rule:

Dyck paths



Other Baxter families: Baxter permutations

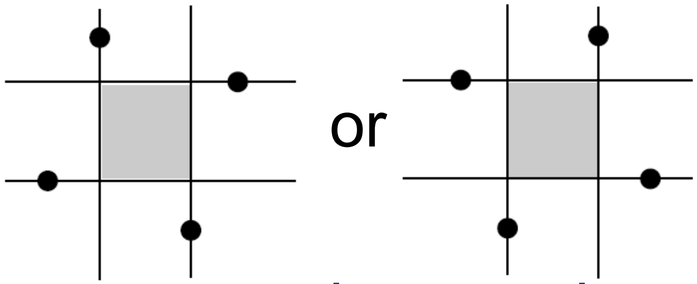
- We adopt the diagram-representation of a permutation



$$\pi = 5\,3\,4\,9\,7\,8\,10\,6\,1\,2$$

Other Baxter families: Baxter permutations

- **Def:** Whenever there are 4 points in position



then the dashed square is not empty.

(i.e., no pattern $2\,5\,\bar{3}\,1\,4$ nor $4\,1\,\bar{3}\,5\,2$)

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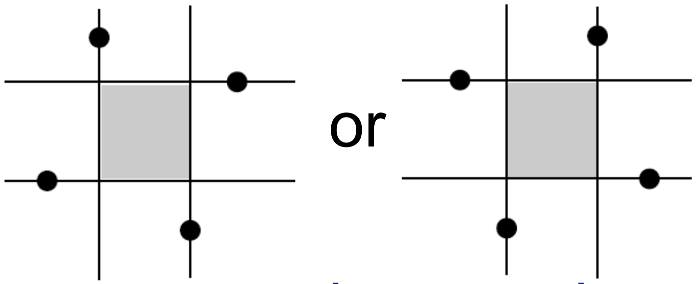

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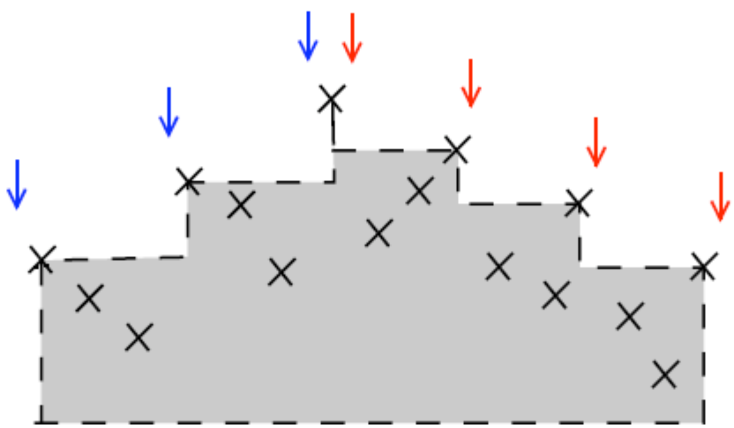
- **Inductive construction:** at each step, insert *n* either

  - just before a left-to-right maximum (among i of them)
  - just after a right-to-left maximum (among j of them)

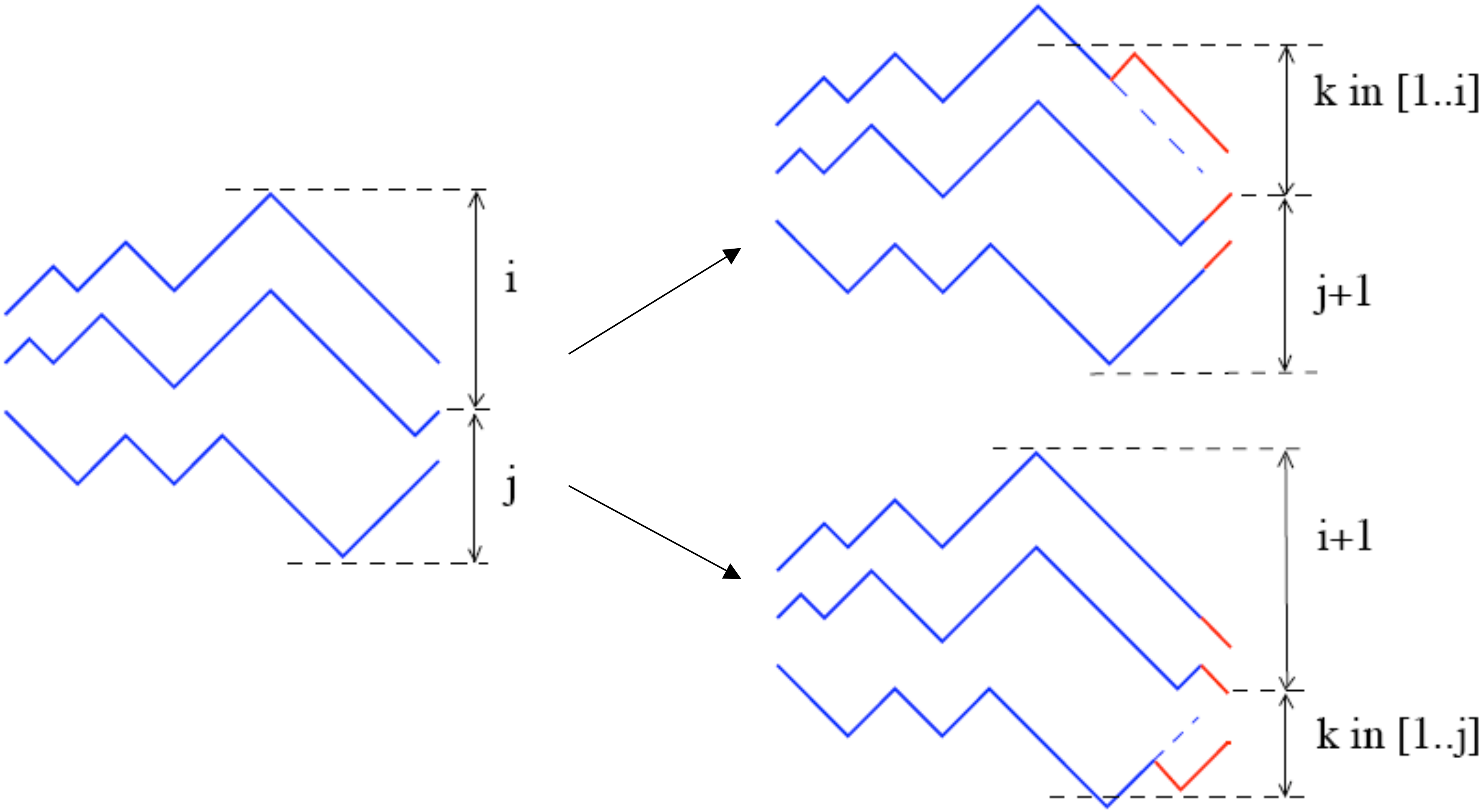


Blue insertion: i := k and j := j+1
for chosen k in [1..i]

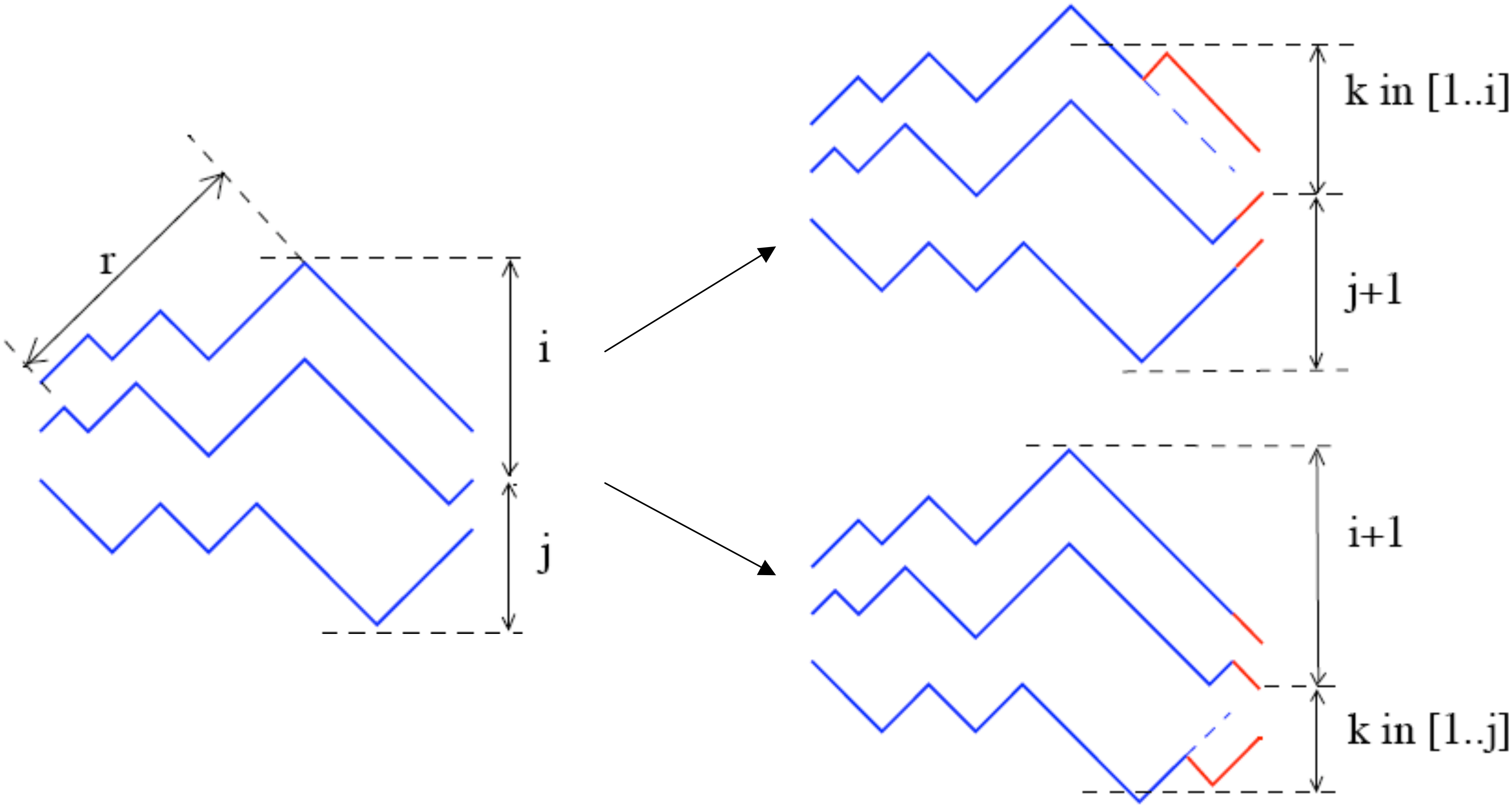
Red insertion: i := i+1 and j := k
for chosen k in [1..j]

⇨ Baxter permutations form a Baxter family

Other Baxter families: triples of paths



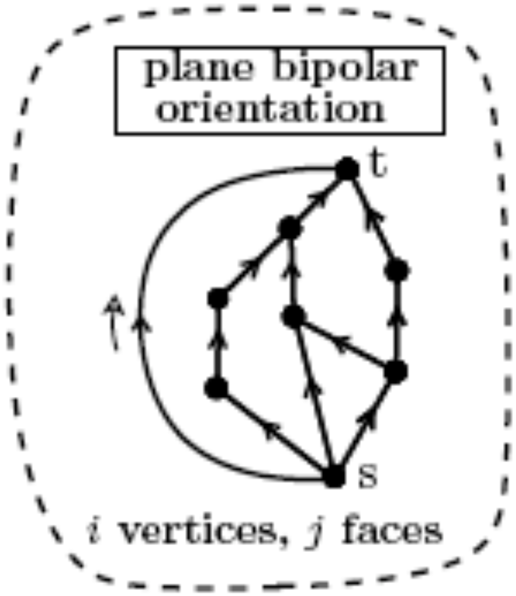
Other Baxter families: triples of paths



- Counting (by Gessel-Viennot's lemma):

$$q_n = \frac{1}{\binom{n+1}{1}\binom{n+1}{2}} \sum_{r=0}^{n-1} \binom{n+1}{r}\binom{n+1}{r+1}\binom{n+1}{r+2}$$

A bijective scheme [F, Poulalhon, Schaeffer'07]



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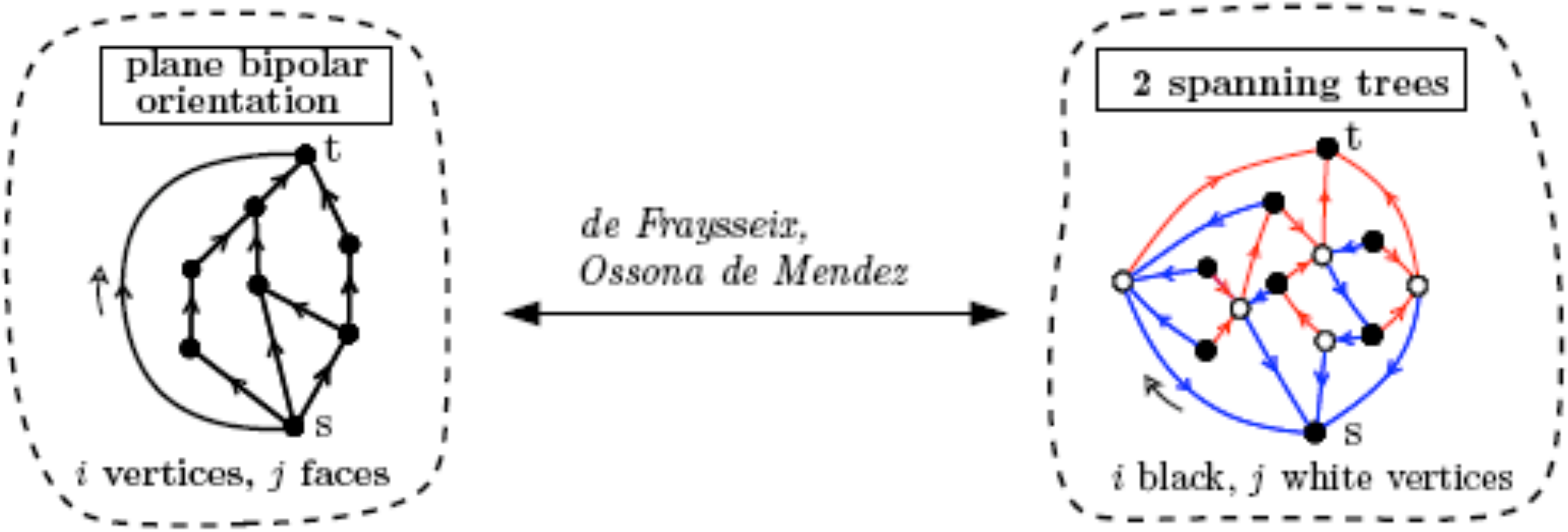

A bijective scheme

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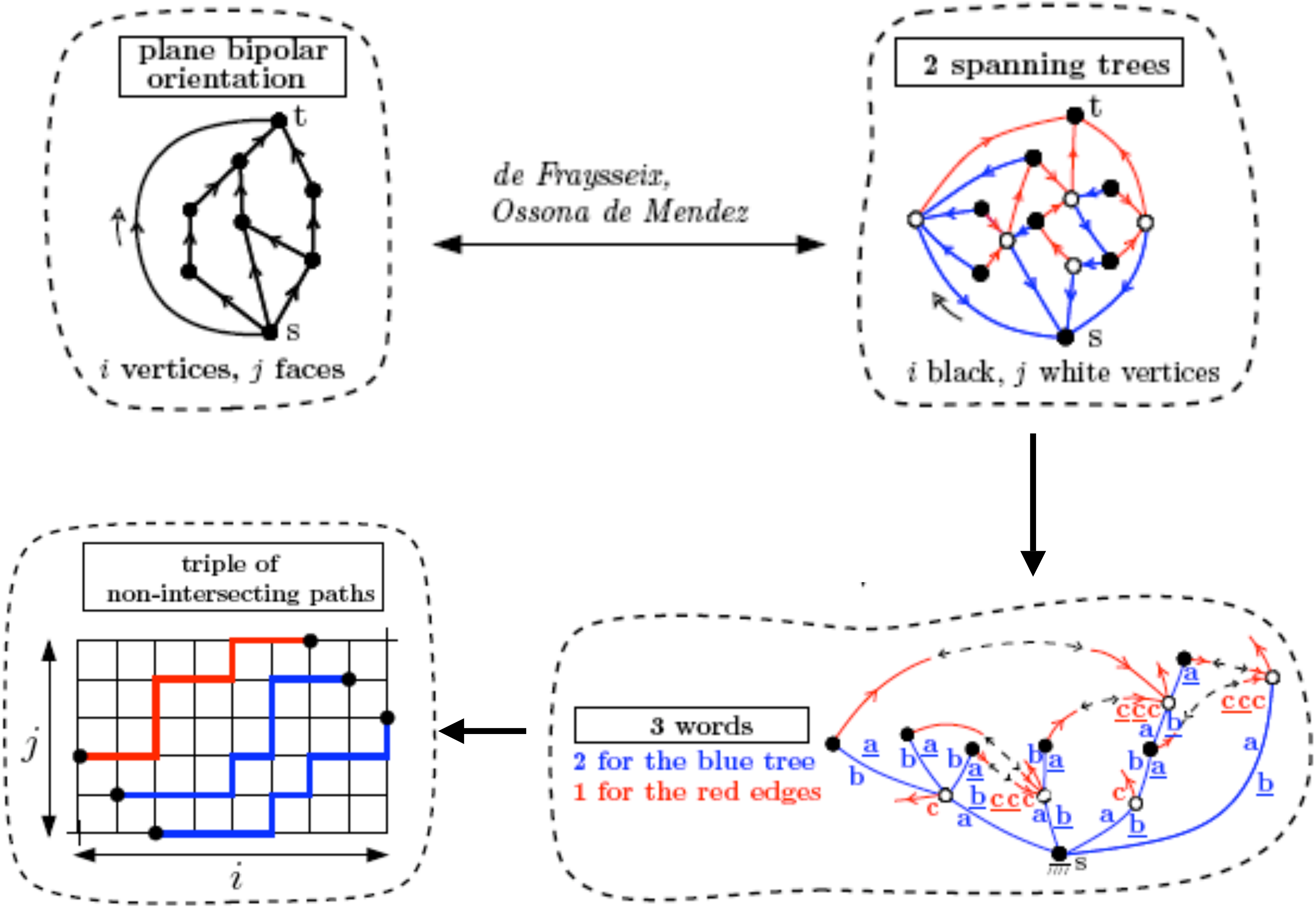
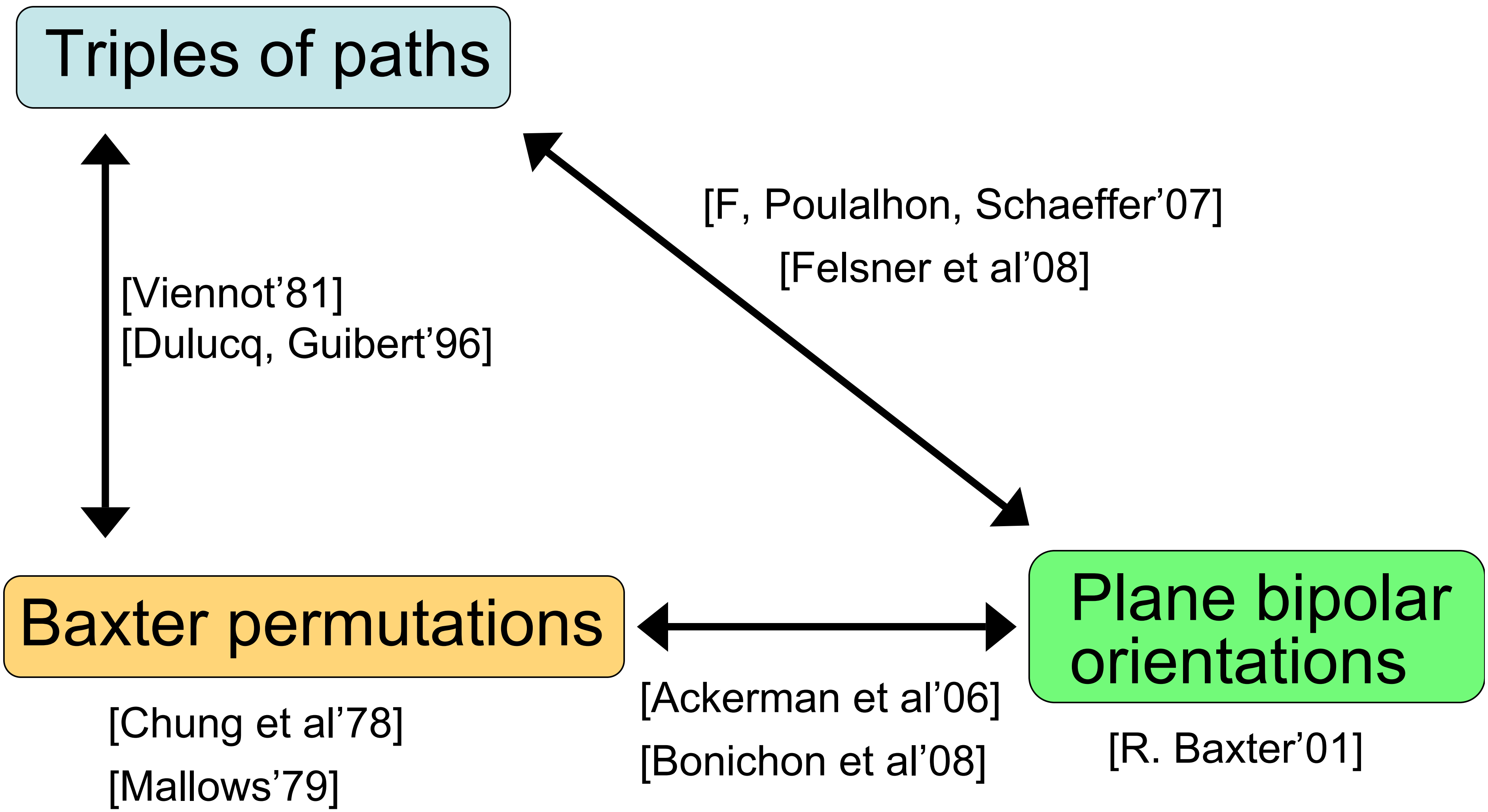


Bijective links and bibliography



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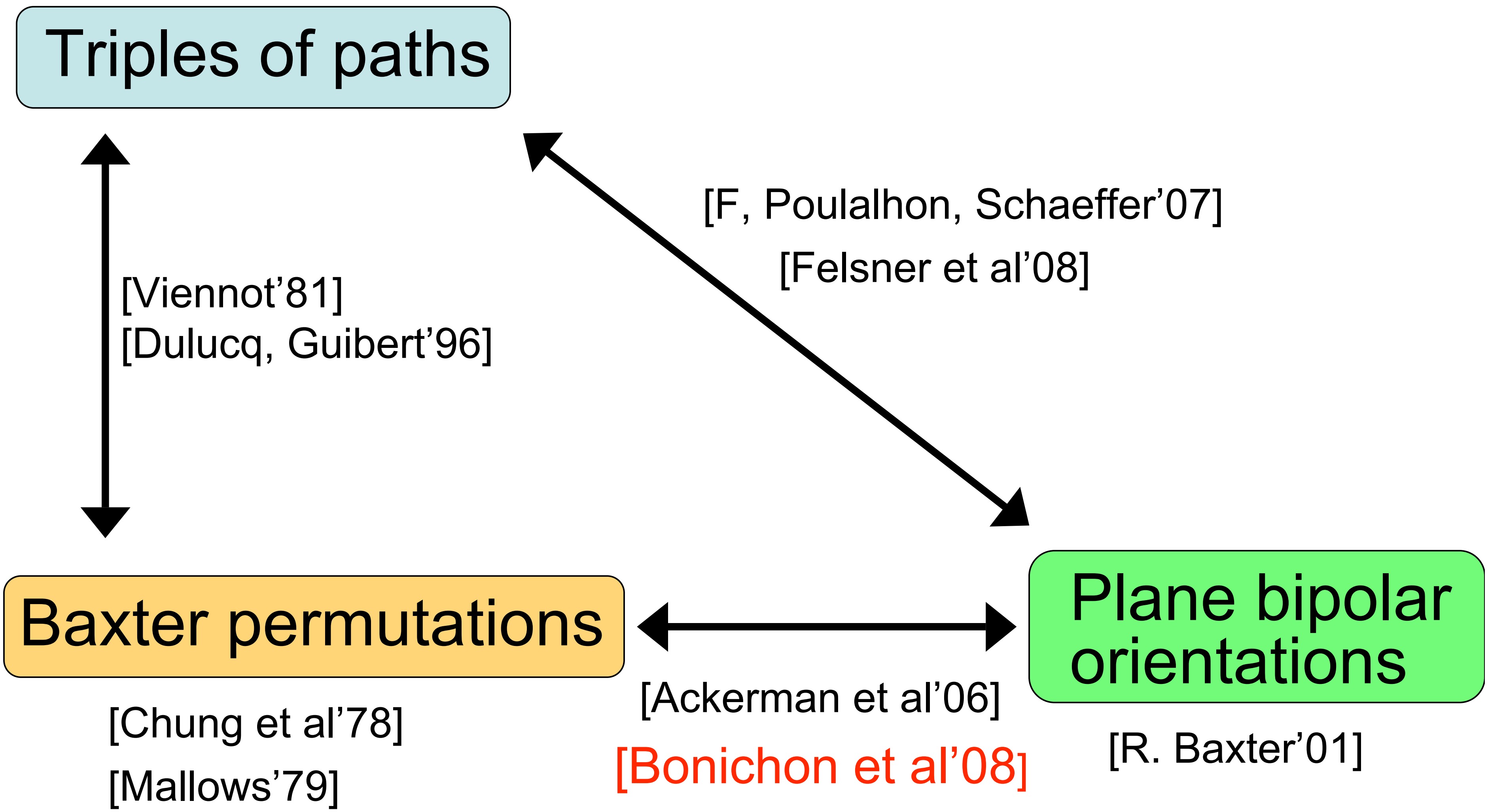
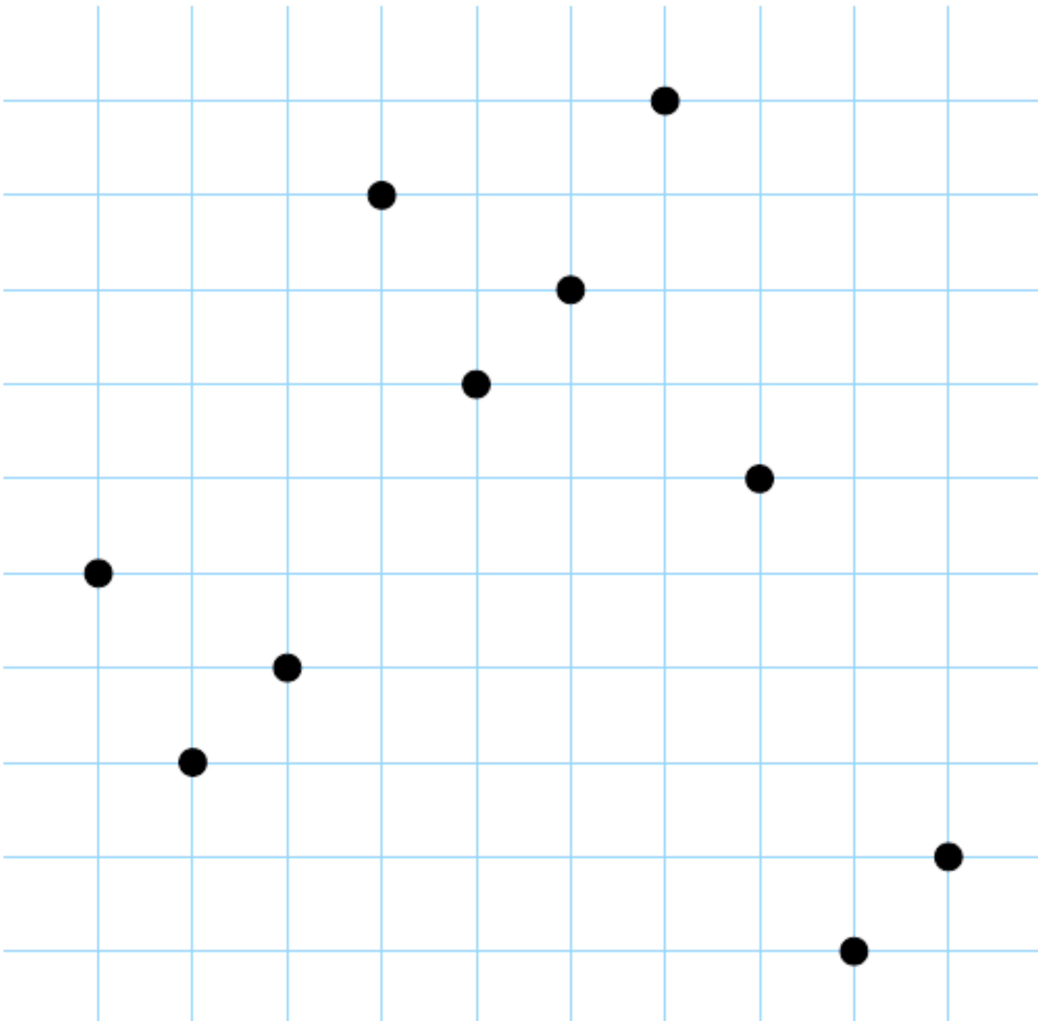


Part 3: maps and permutations

Baxter permutation -> plane bipolar orientation

(hint: #ascents is distributed like #vertices)



$$\pi = 5\,3\,4\,9\,7\,8\,10\,6\,1\,2$$

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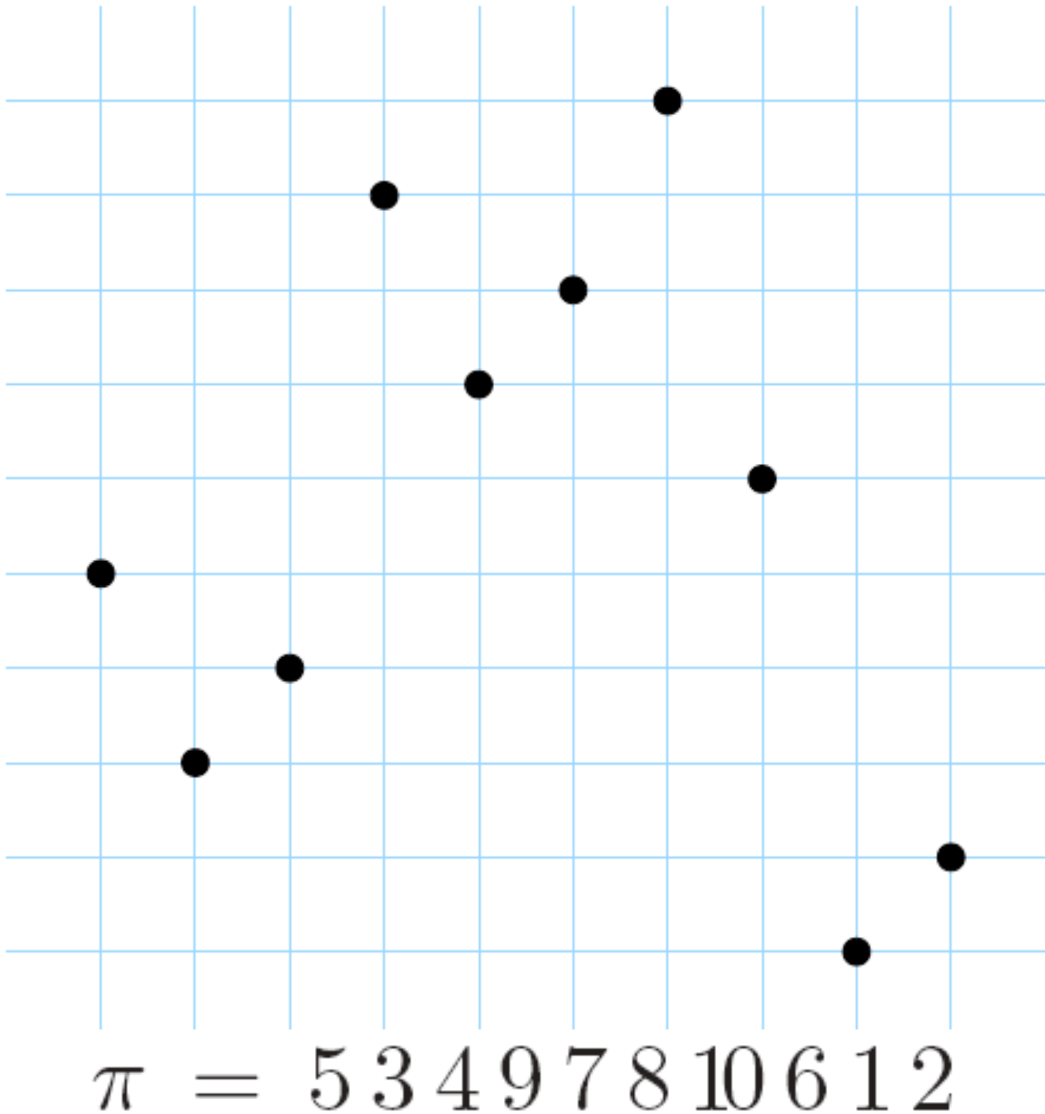
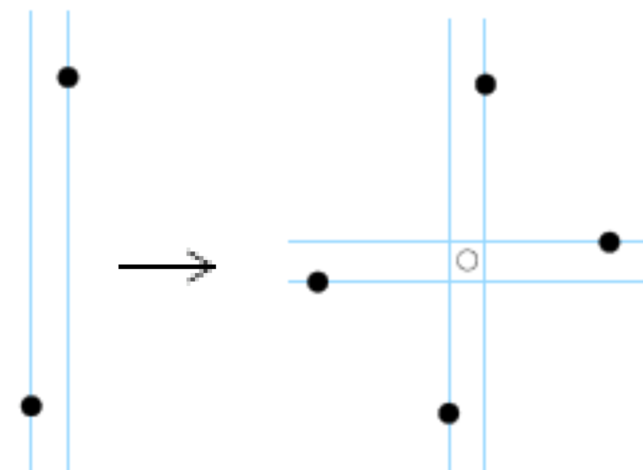


- Ascents of π are in 1-to-1 correspondence with ascents of π^{-1}

- Place a white vertex at the intersection



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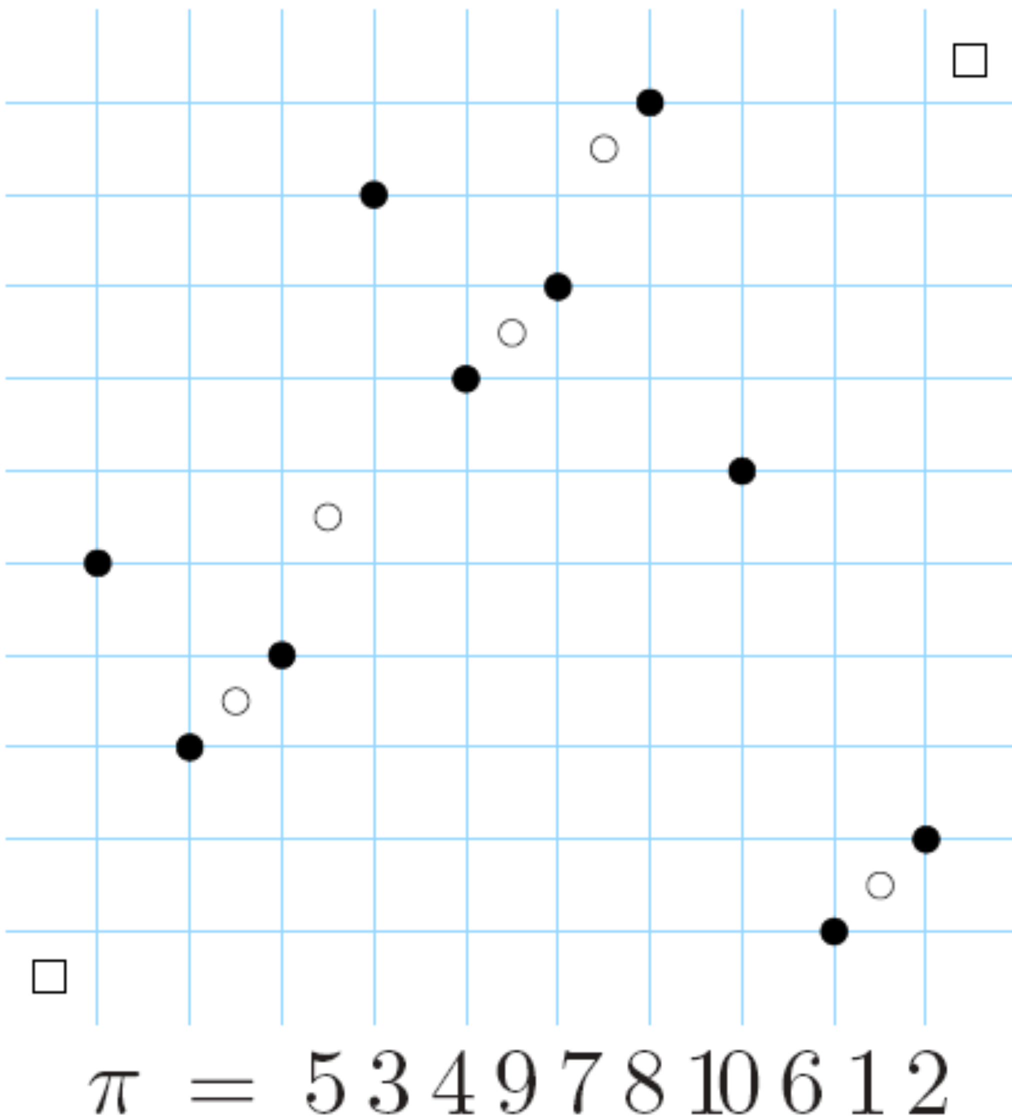

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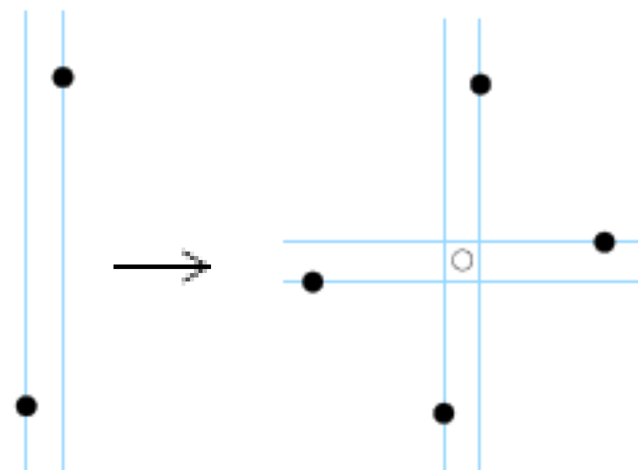

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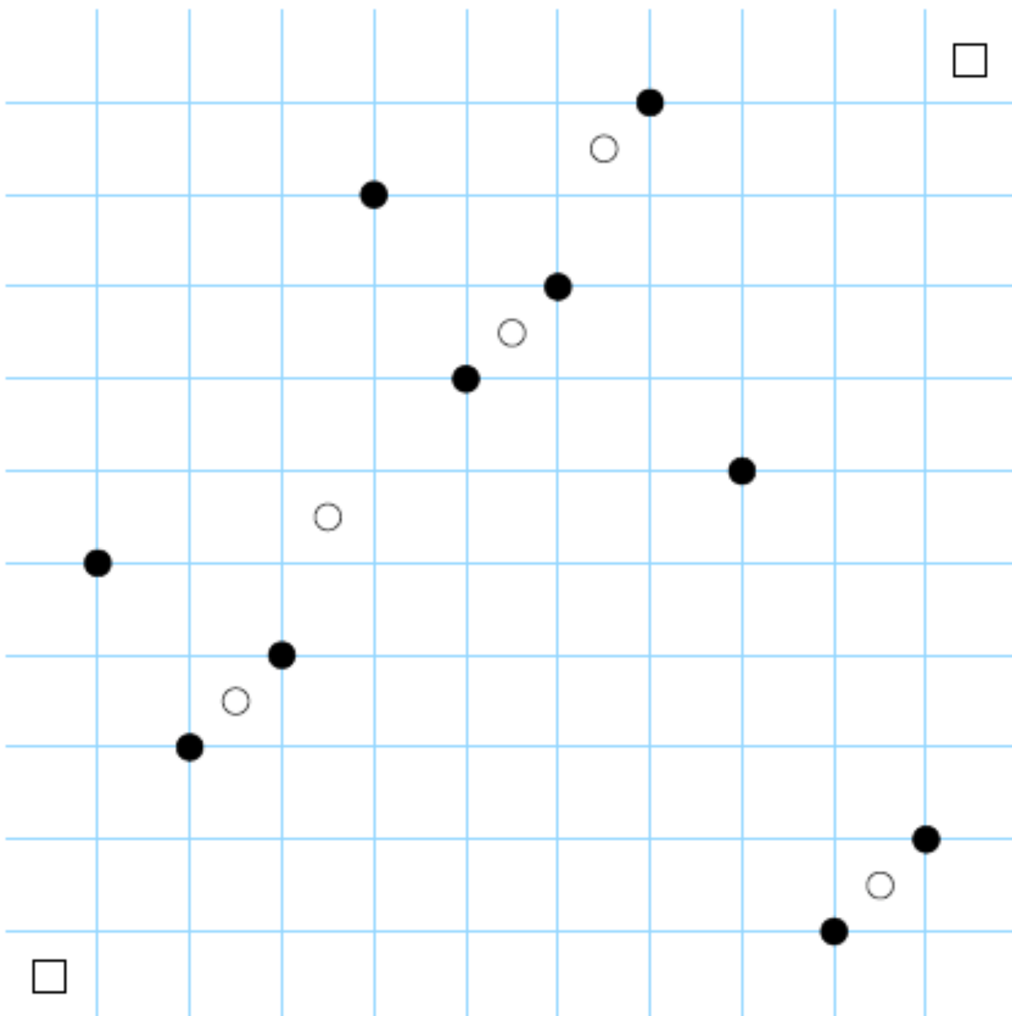
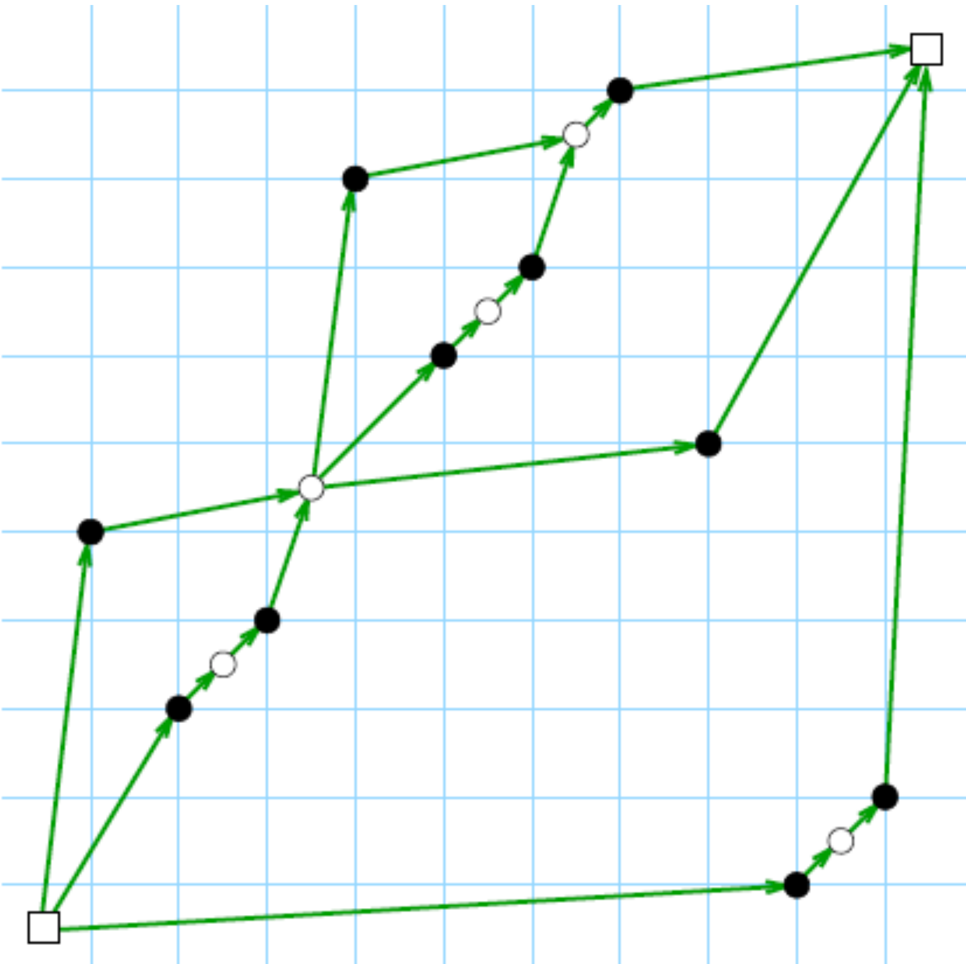


$\pi = 5\,3\,4\,9\,7\,8\,10\,6\,1\,2$

Dominance drawing:
draw segment (x,y) -> (x',y')
whenever x<x' and y<y'

Baxter permutation -> plane bipolar orientation



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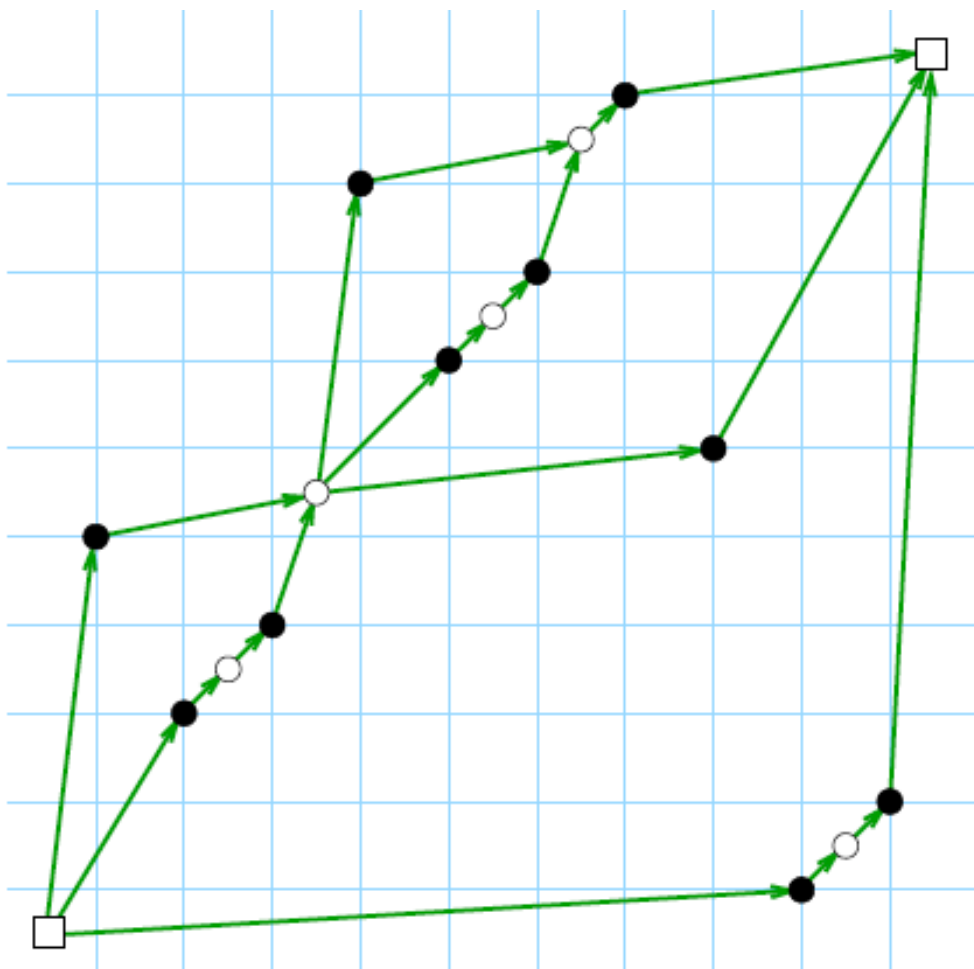

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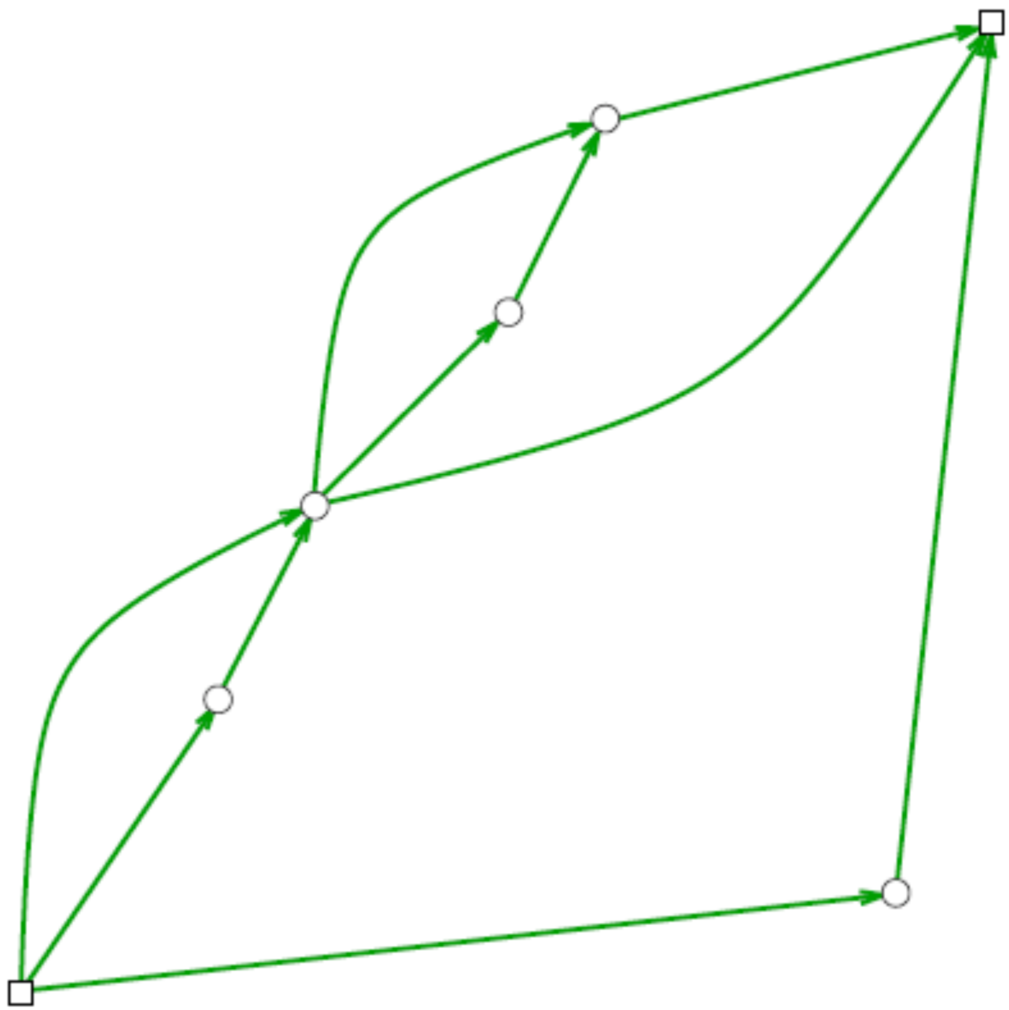
Erase the black vertices
(all have degree 2)

Baxter permutation -> plane bipolar orientation



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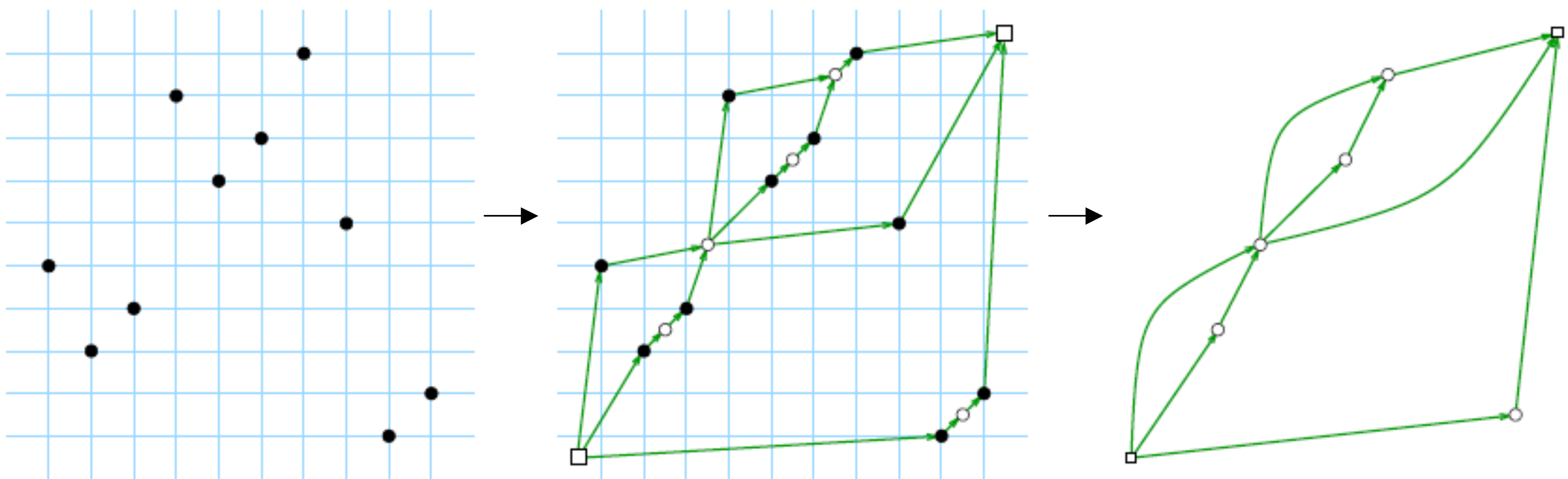
Baxter permutation -> plane bipolar orientation

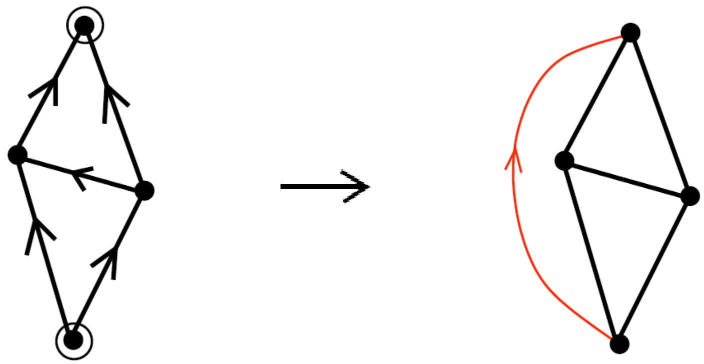
Theorem [Bonichon, Bousquet-Mélou, F'08]:
The mapping is the canonical bijection (implements the isomorphism between generating trees)



- The mapping respect many symmetries
- Specializations to map families (non-separable, series-parallel maps)

Rooted non-separable maps

- Surjective mapping from plane bipolar orientations



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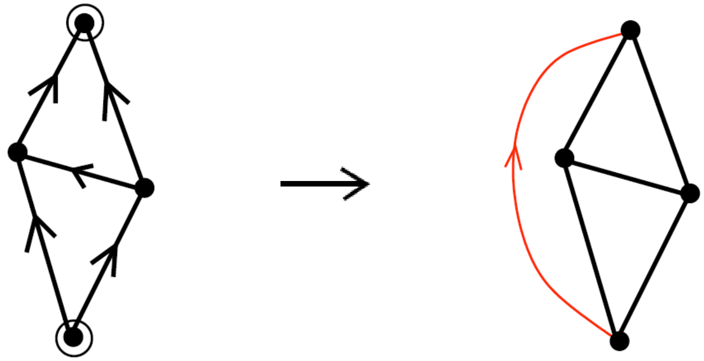
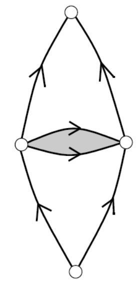


Rooted non-separable map

- Bijective if bipolar orientation avoids the pattern 

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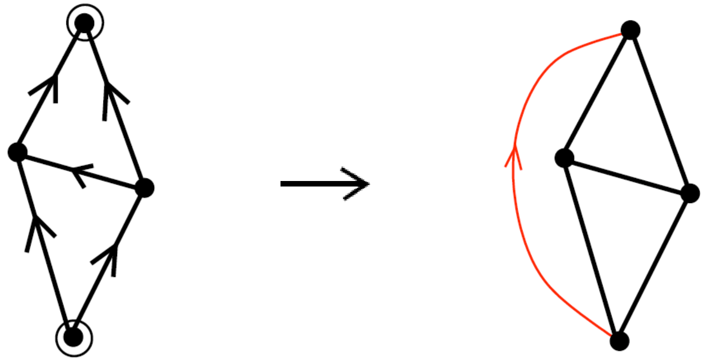

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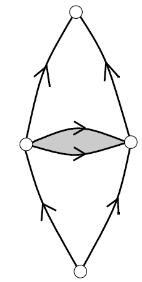
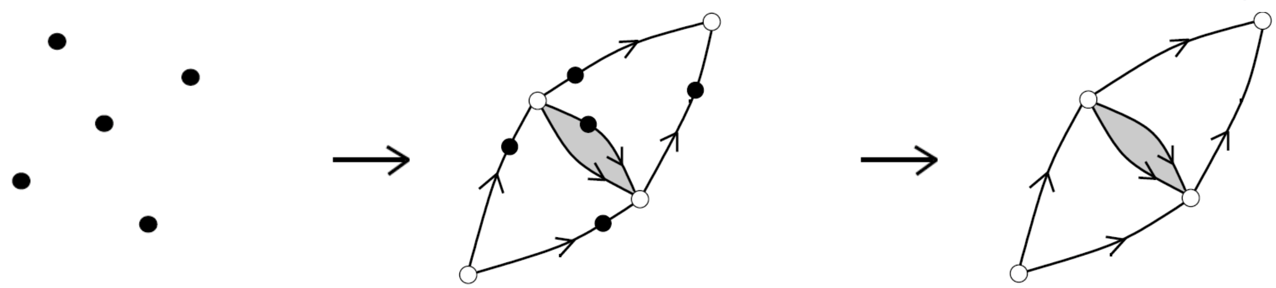


- These correspond to Baxter permutations avoiding 2 4 1 3



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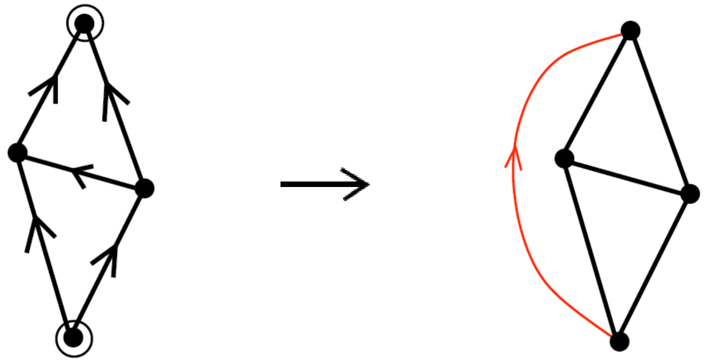
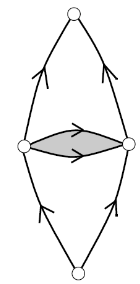
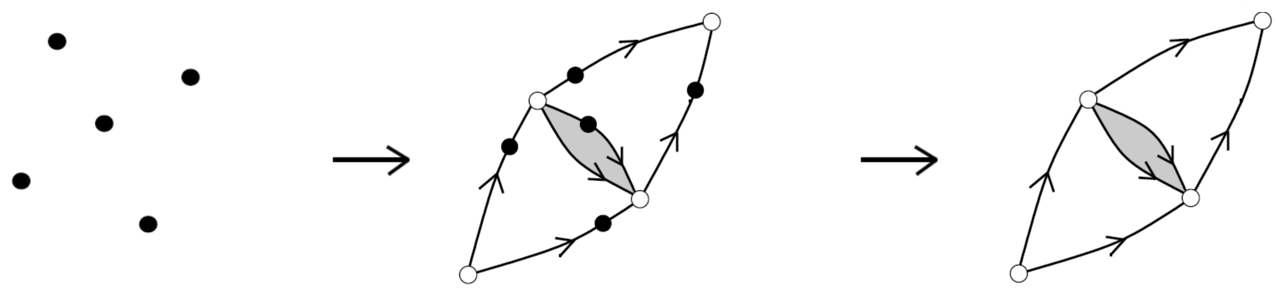


Rooted non-separable map

- Bijective if bipolar orientation avoids the pattern



- These correspond to Baxter permutations avoiding 2 4 1 3



Theorem: rooted non-separable maps with *n+1* edges are in bijection with Baxter permutations of size *n* avoiding 2 4 1 3

(Further specialization: series-parallel maps are in bijection with permutations avoiding 2 4 1 3 and 3 1 4 2)

Two approaches

Baxter permutations

[Bonichon, Bousquet-Mélou, F'08]

Generating tree
2 parameters

Plane bipolar orientations

Baxter permutations avoiding 2 4 1 3

[Dulucq, Gire, West'96]

Generating tree
∞ parameters

Rooted non-separable maps

Two approaches

Baxter permutations ⟷ Plane bipolar orientations

[Bonichon, Bousquet-Mélou, F'08]

Generating tree
2 parameters

Baxter permutations avoiding 2 4 1 3 ⟷ Rooted non-separable maps

[Dulucq, Gire, West'96]

Generating tree
∞ parameters

Open problem: top approach to show connected 1342-avoiding permutations to be in bijection with bicubic maps [Bona'97]