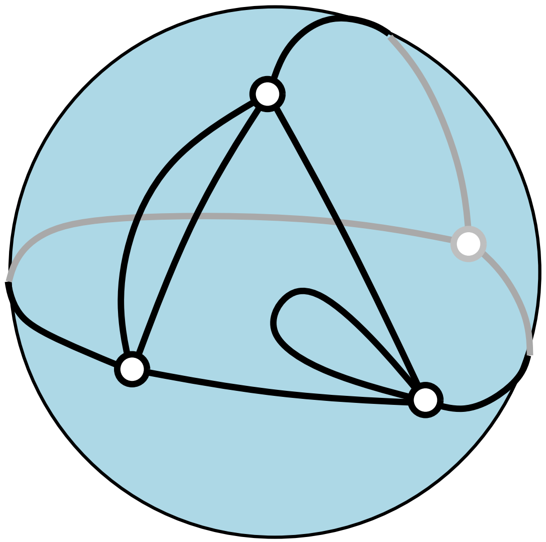
La fonction à deux points et à trois points des quadrangulations et cartes

Éric Fusy (CNRS/LIX)
Travaux avec Jérémie Bouttier et Emmanuel Guitter

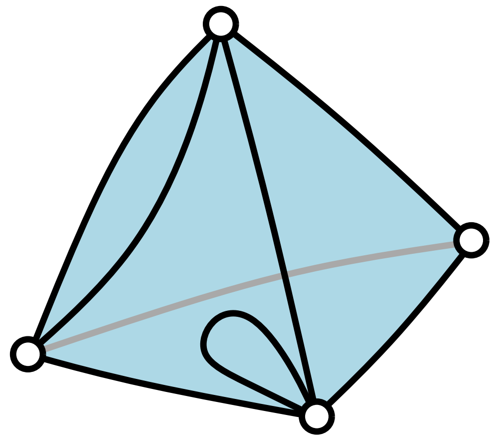
Séminaire Calin, LIPN, Mai 2014

Maps

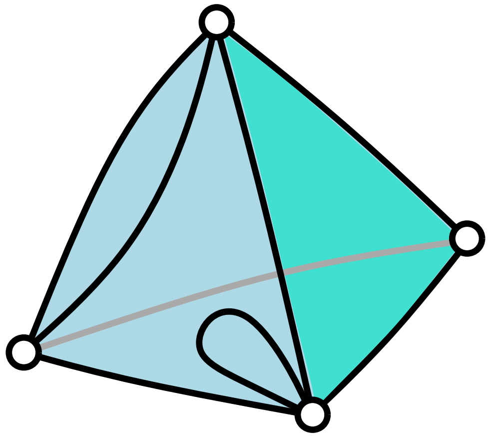
Def. Planar map = connected graph embedded on the sphere



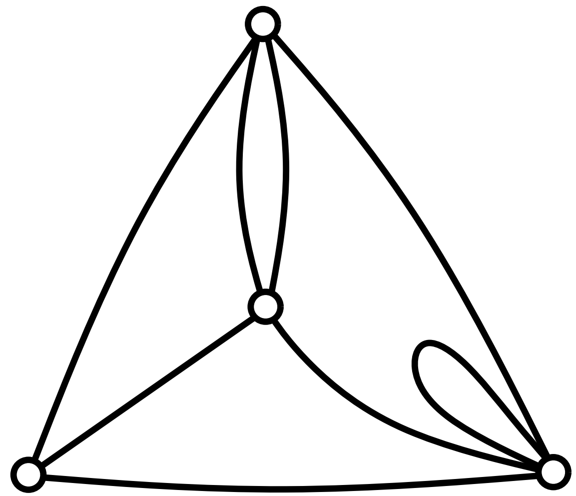
$=$



Easier to draw in the plane (by choosing a face to be the outer face)



$\Rightarrow$



Maps as random discrete surfaces

Natural questions:

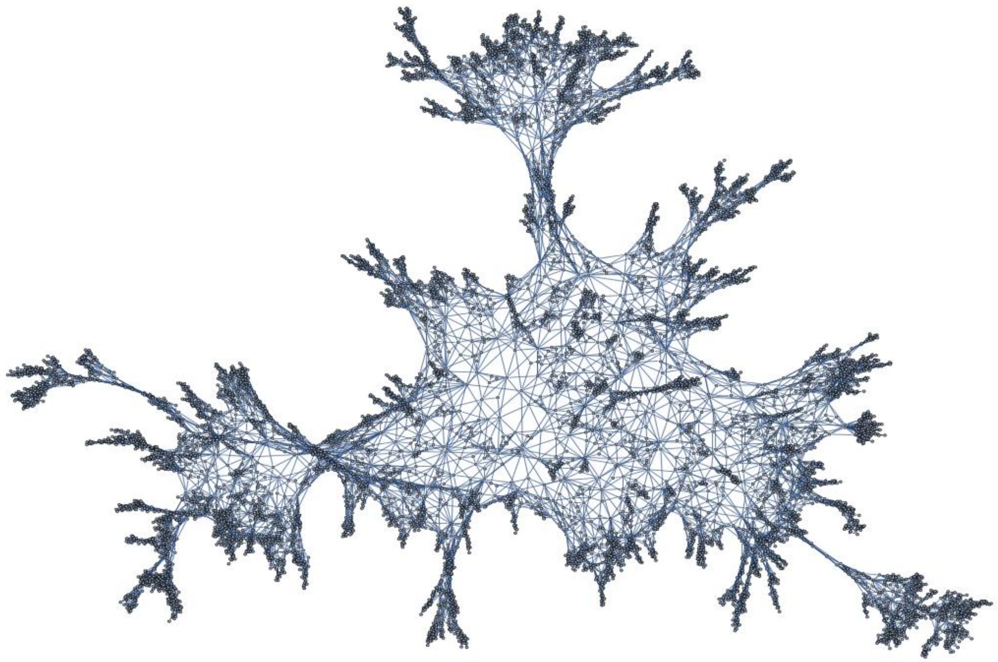
- Typical distance between (random) vertices in random maps
  the order of magnitude is $n^{1/4}$ ($\neq n^{1/2}$ in random trees)

random quadrang. $\begin{cases} \text{- [Chassaing-Schaeffer'04] probabilistic} \\ \text{- [Bouttier Di Francesco Guitter'03] exact GF expressions} \end{cases}$

- How does a random map (rescaled by $n^{1/4}$) "look like" ?

convergence to the "Brownian map"

[Le Gall'13, Miermont'13]



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Counting (rooted) maps

with a marked corner

- Very simple counting formulas ([Tutte'60s]), for instance
  Let $q_n = \#\{\text{rooted quadrangulations with } n \text{ faces}\}$
  $m_n = \#\{\text{rooted maps with } n \text{ edges}\}$

  Then $m_n = q_n = \frac{2}{n+2} 3^n \frac{(2n)!}{n!(n+1)!}$

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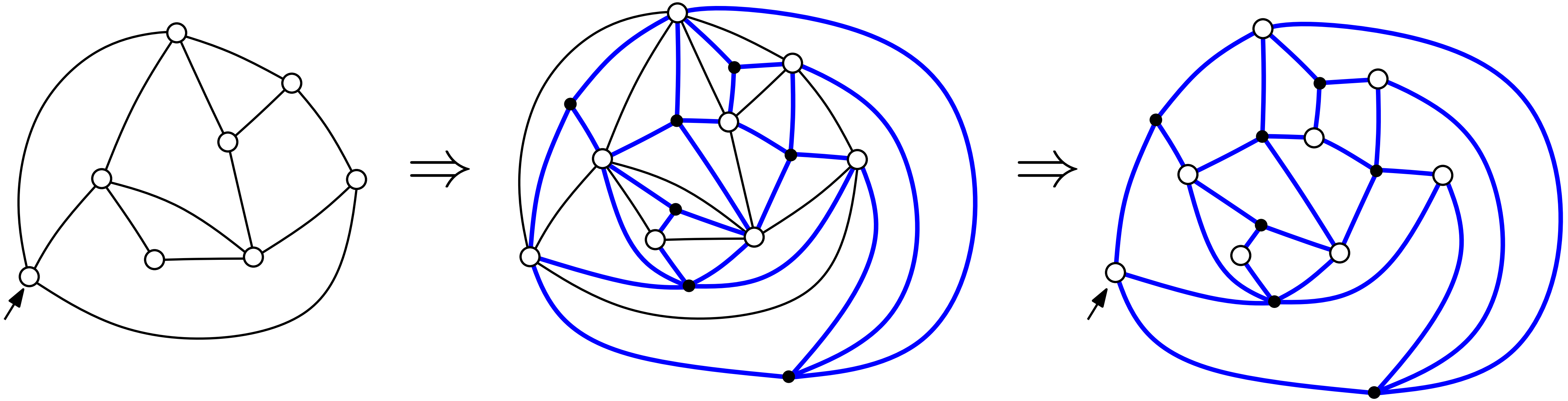
- Proof of $m_n = q_n$ by easy local bijection:



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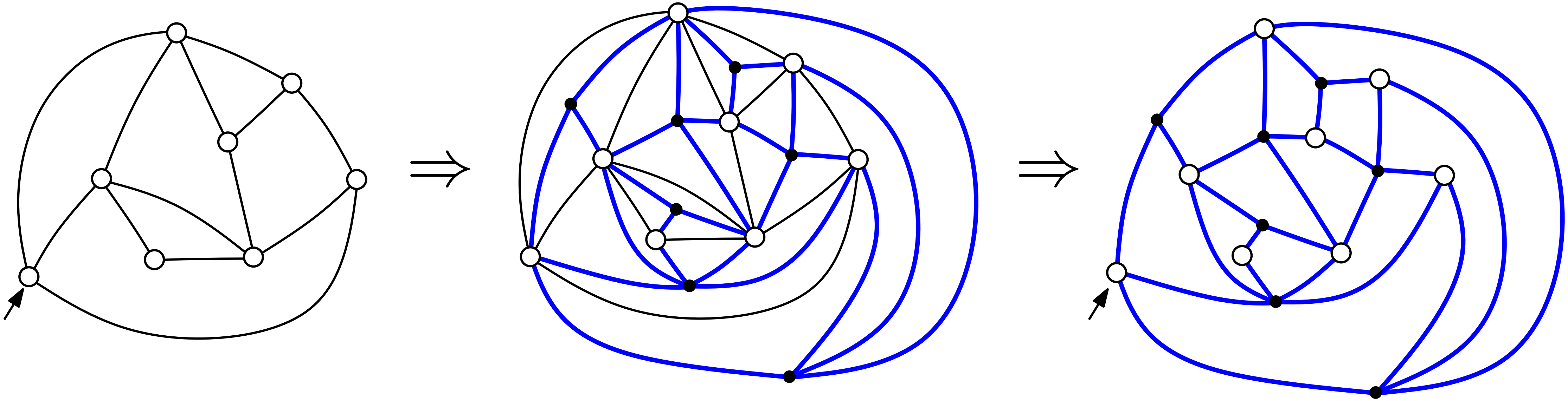

But this bijection does not preserve distance-parameters (only bounds)

The k-point function

- Let $\mathcal{M} = \cup_n \mathcal{M}[n]$ be a family of maps (quadrangulations, general, ...) where n is a size-parameter ($\#$ faces for quad., $\#$ edges for gen. maps)
- Let $\mathcal{M}^{(k)}$ = family of maps from $\mathcal{M}$ with k marked vertices $v_1, \ldots, v_k$

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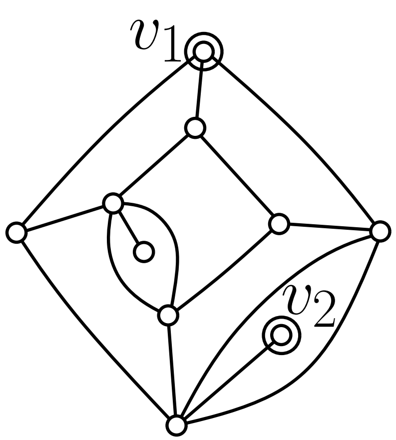
Refinement by distances :

For $D = (d_{i,j})_{1 \leq i < j \leq k}$ any $\binom{k}{2}$-tuple of positive integers

let $\mathcal{M}_D^{(k)} :=$ subfamily of $\mathcal{M}^{(k)}$ where $\mathrm{dist}(v_i, v_j) = d_{ij}$ for $1 \leq i < j \leq k$

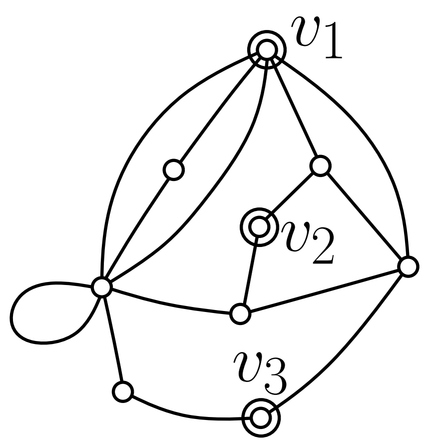
> The counting series $G_D \equiv G_D(g)$ of $\mathcal{M}_D^{(k)}$ with respect to the size is called the **k-point function** of $\mathcal{M}$



$k = 2$

$d_{12} = 3$

quadrangulation



$k = 3$

$d_{12} = 2$

$d_{13} = 2$

$d_{23} = 3$

general map

Exact expressions for the k-point function

- For the two-point functions:

  - quadrangulations **[Bouttier Di Francesco Guitter'03]**

  - maps with prescribed (bounded) face-degrees **[Bouttier Guitter'10]**

  - general maps **[Ambjørn Budd'13]**

  - general hypermaps, general constellations **[Bouttier F Guitter'13]**

- For the three-point functions

  - quadrangulations **[Bouttier Guitter'08]**

  - general maps & bipartite maps **[F Guitter'14]**

Exact expressions for the k-point function

Outline of the talk

- For the two-point functions:

1 - quadrangulations **[Bouttier Di Francesco Guitter'03]**

uses Schaeffer's bijection

- maps with prescribed (bounded) face-degrees **[Bouttier Guitter'10]**

3 - general maps **[Ambjørn Budd'13]**

based on clever observation on Miermont's bijection

- general hypermaps, general constellations **[Bouttier F Guitter'13]**

- For the three-point functions

2 - quadrangulations **[Bouttier Guitter'08]**

uses Miermont's bijection

4 - general maps & bipartite maps **[F Guitter'14]**

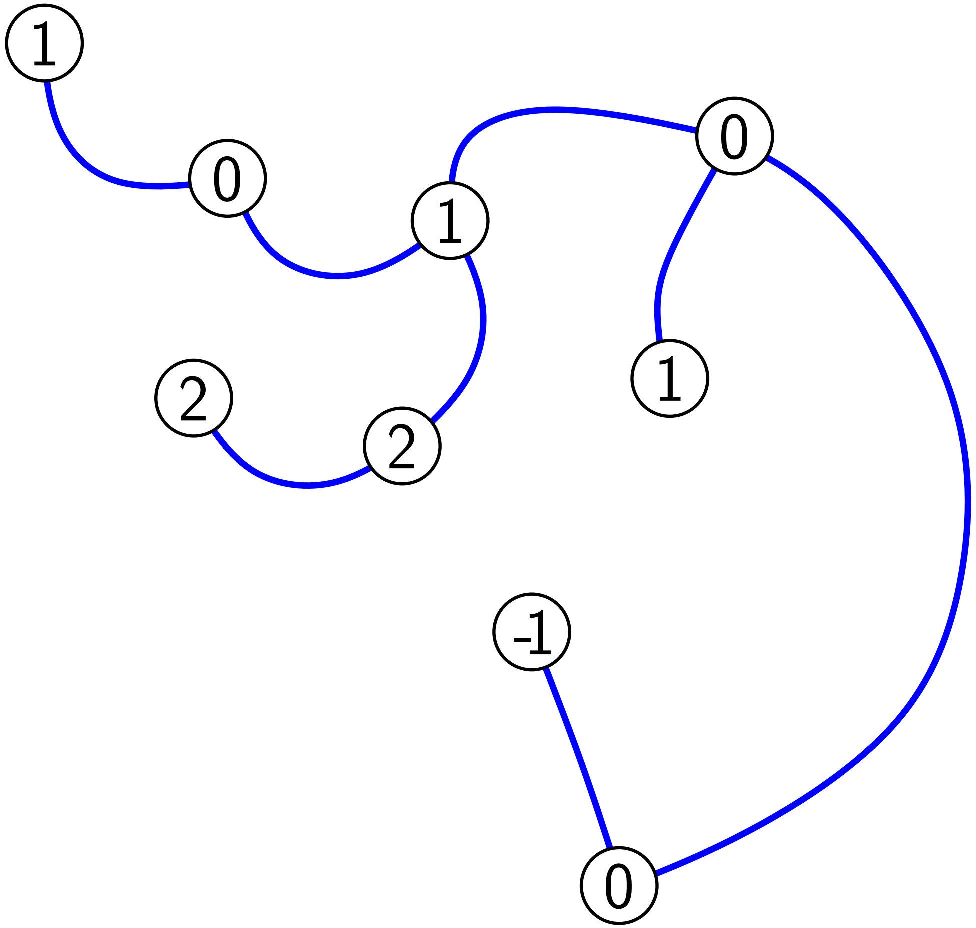
uses AB bijection

Computing the two-point function of quadrangulations using the Schaeffer bijection

Well-labelled trees

Well-labelled tree = plane tree where

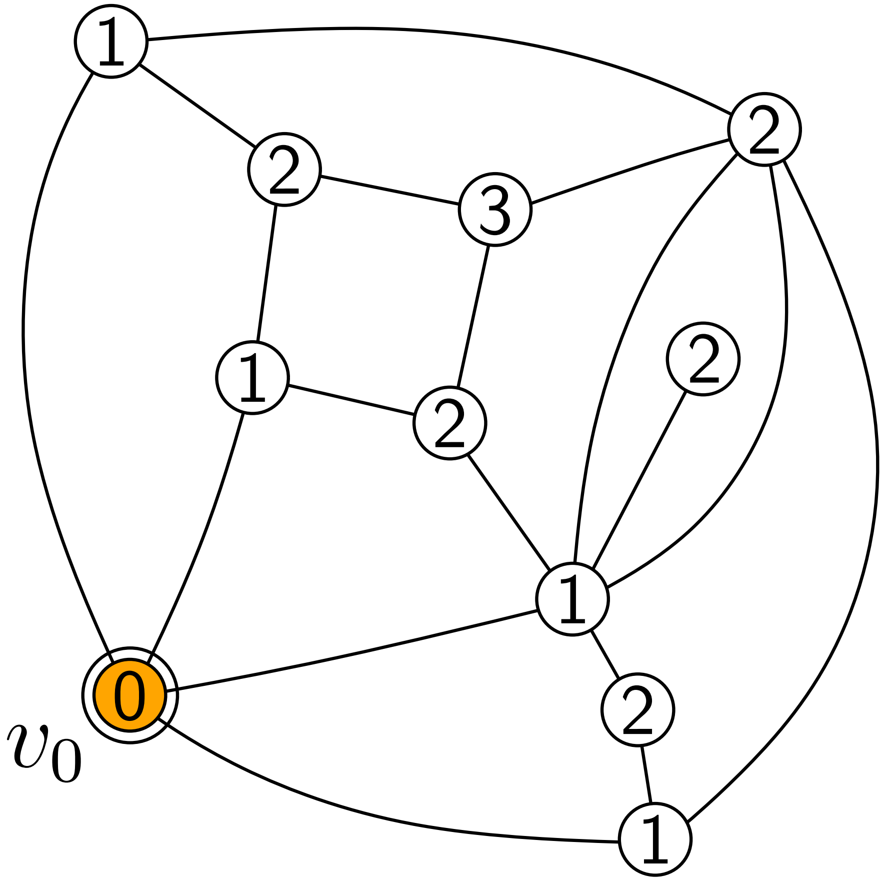
- each vertex v has a label $\ell(v) \in \mathbb{Z}$
- each edge $e = \{u, v\}$ satisfies $|\ell(u) - \ell(v)| \leq 1$



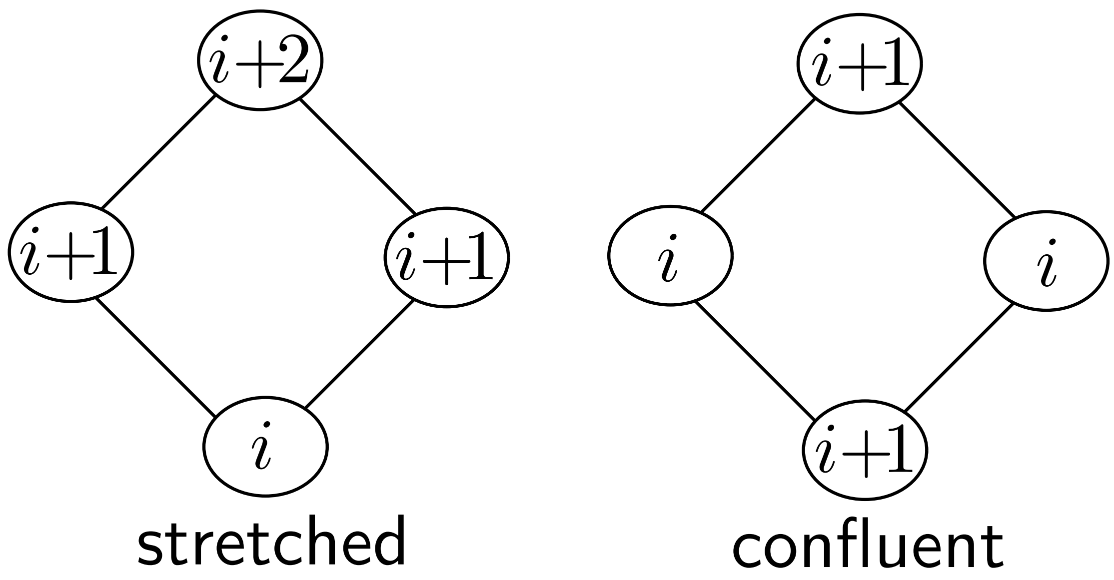
Pointed quadrangulations, geodesic labelling

Pointed quadrangulation = quadrangulation with a marked vertex v_0

Geodesic labelling with respect to v_0: $\ell(v) = \mathrm{dist}(v_0, v)$



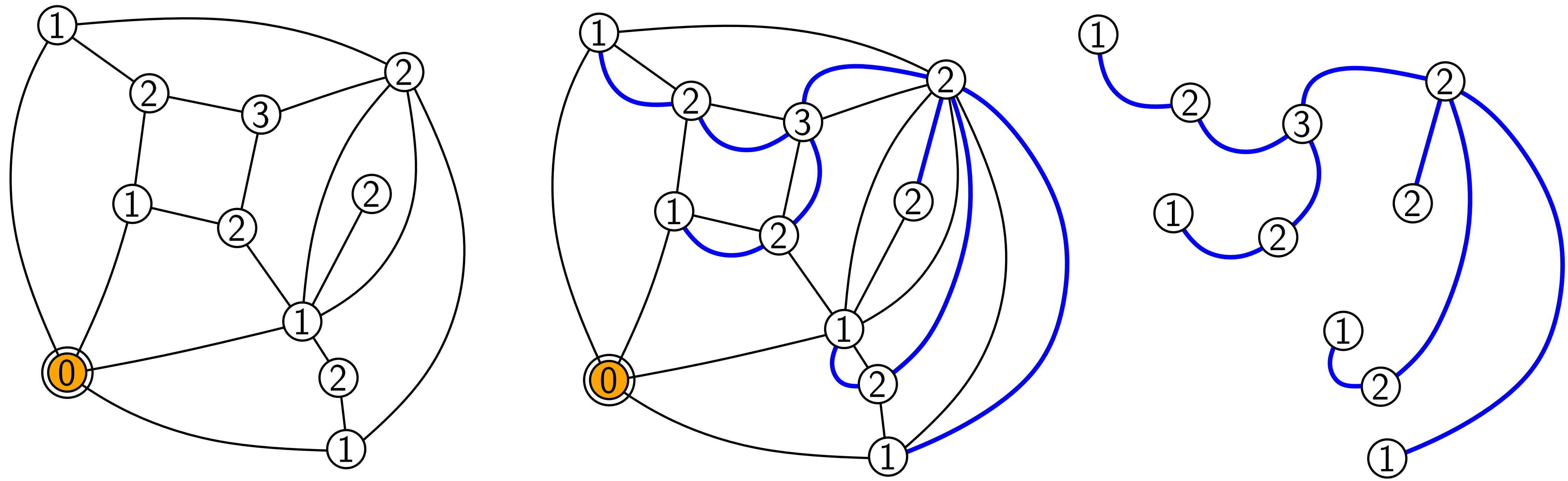
Rk: two types of faces



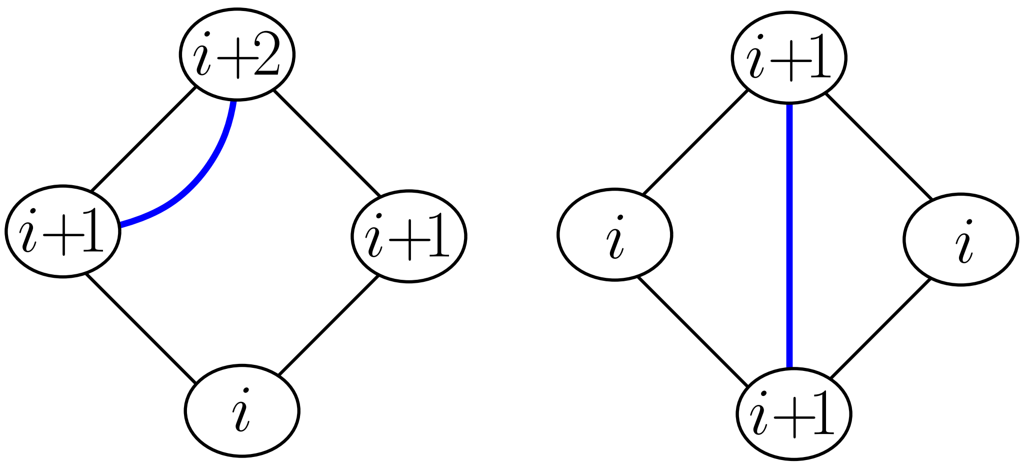
The Schaeffer bijection **[Schaeffer'99]**, also **[Cori-Vauquelin'81]**

Pointed quadrangulation $\Rightarrow$ well-labelled tree with min-label=1

n faces $\qquad\qquad$ n edges



Local rule in each face:

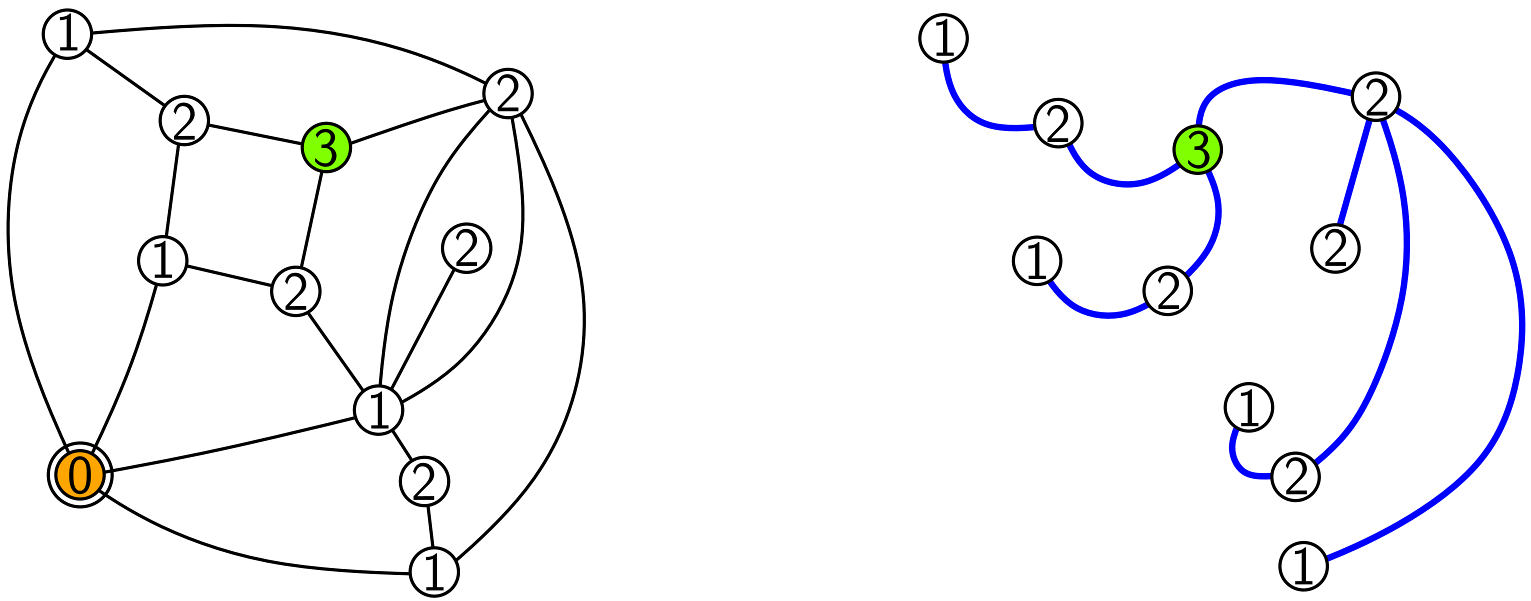


The 2-point function of quadrangulations (1)

Denote by $G_d \equiv G_d(g)$ the two-point function of quadrangulations

bijection $\Rightarrow G_d(g) =$ GF of well-labelled trees with min-label=1
and with a marked vertex of label d

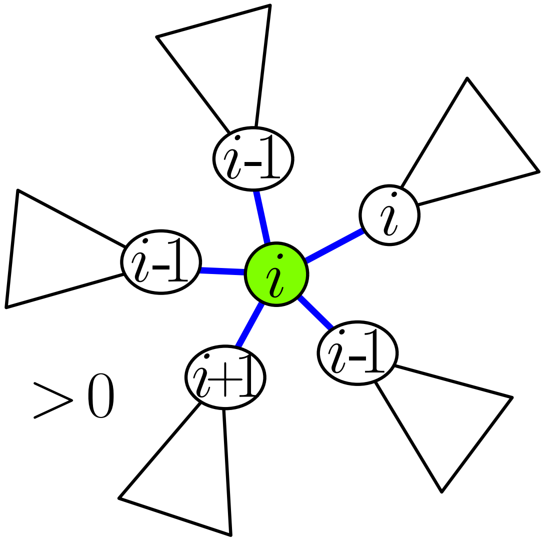
$d = 3$



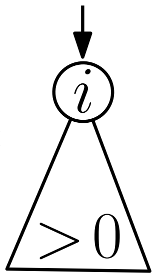
Rk: $G_d = F_d - F_{d-1} = \Delta_d F_d$

where $F_d \equiv F_d(g) =$ GF of well-labelled trees with positive labels
and with a marked vertex of label d

The 2-point function of quadrangulations (2)



$$\Rightarrow \qquad F_i = \log \frac{1}{1-g(R_{i-1}+R_i+R_{i+1})}$$

with $R_i =$ 

GF rooted well-labelled trees with positive labels and label i at the root

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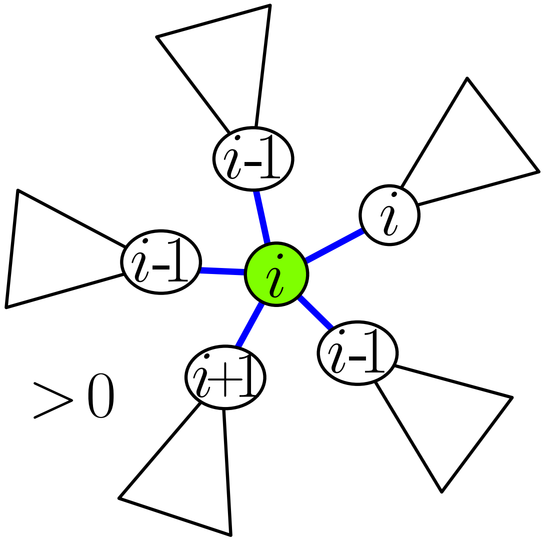

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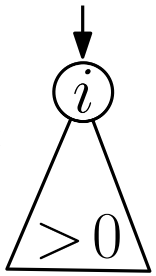
with $R_i =$ 

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Equ. for R_i: $\boxed{R_i = \frac{1}{1-g(R_{i-1}+R_i+R_{i+1})}}$ (so $F_i = \log(R_i)$, $G_d = \log(\frac{R_d}{R_{d-1}})$)

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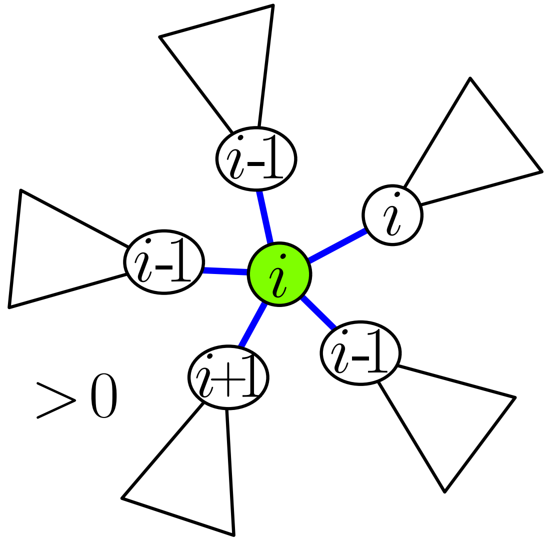

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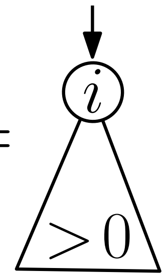
with $R_i =$ 

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- Exact expression for R_i **[BDG'03]**

$$\boxed{R_i = R\frac{[i]_x[i+3]_x}{[i+1]_x[i+2]_x}}$$

with the notation $[i]_x = \frac{1-x^i}{1-x}$

with $R \equiv R(g)$ and $x \equiv x(g)$ given by $\begin{cases} R = 1+3gR^2 \\ x = gR^2(1+x+x^2) \end{cases}$

$$R(g) = \frac{1-S}{6g} \qquad x(g) = \frac{\sqrt{6}}{2}\frac{S^{1/2}\sqrt{1-(1+6g)S}-S-24g+1}{-1+S+6g}$$

with $S = \sqrt{1-12g}$

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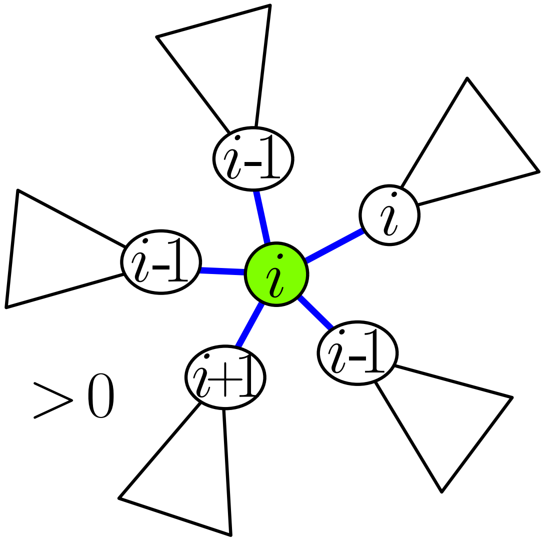

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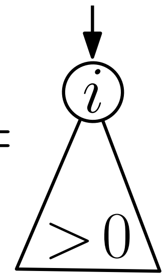
with $R_i =$ 

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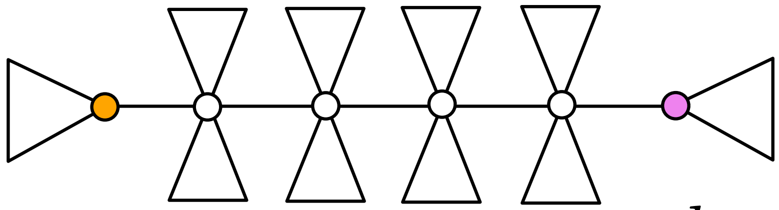
Final 2-point function expression: $\boxed{G_d = \log\left(\frac{[d]_x^2[d+3]_x}{[d-1]_x[d+2]_x^2}\right)}$

Asymptotic considerations

- Two-point function of (plane) trees:

$$\boxed{G_d(g) = (gR^2)^d}$$

with $R = 1 + gR^2 = \frac{1-\sqrt{1-4g}}{2g}$



$d = 5$

G_d is the d th power of a series having a square-root singularity

$\Rightarrow d/n^{1/2}$ converges in law (Rayleigh law, density $\alpha \exp(-\alpha^2)$)

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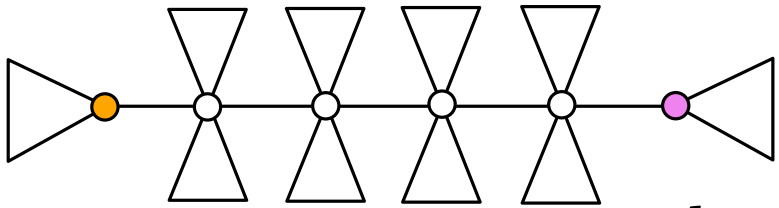

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- Two-point function of quadrangulations:

$$\boxed{G_d(g) \sim_{d\to\infty} a_1 x^d + a_2 x^{2d} + \cdots}$$

where $x = x(g)$ has a quartic singularity

$\Rightarrow d/n^{1/4}$ converges to an explicit law **[BDG'03]**

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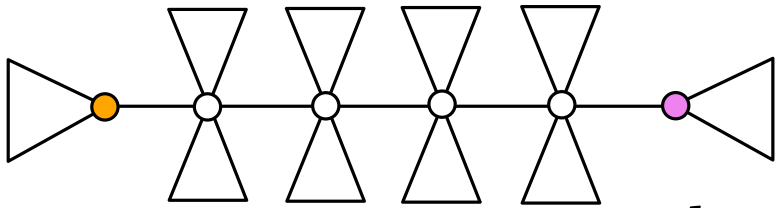

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where $x = x(g)$ has a quartic singularity

$\Rightarrow d/n^{1/4}$ converges to an explicit law **[BDG'03]**

Convergence in the two cases "follows" from (proof by Hankel contour) **[Banderier, Flajolet, Louchard, Schaeffer'03]**: for $0 < s < 1$,

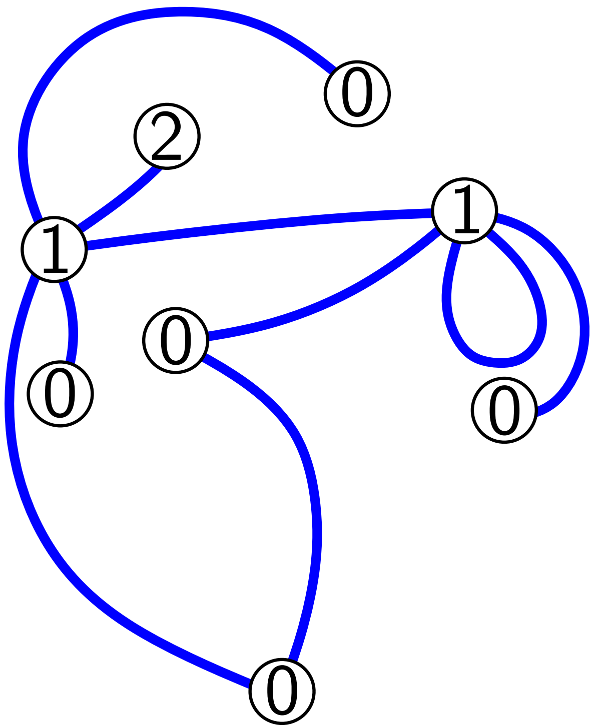
$$x(g) \underset{g\to 1}{\sim} 1 - (1-g)^s \Rightarrow [g^n]x^{\alpha n^s} \sim \frac{1}{2\pi n}\int_0^\infty e^{-t}\mathrm{Im}(\exp(-\alpha t^s e^{i\pi s}))\mathrm{d}t$$

Computing the two-point and three-point function of quadrangulations using Miermont's bijection

Well-labelled maps

Well-labelled map = map where

- each vertex v has a label $\ell(v) \in \mathbb{Z}$
- each edge $e = \{u, v\}$ satisfies $|\ell(u) - \ell(v)| \leq 1$



a well-labelled map M with 3 faces

Rk: Well-labelled tree = well-labelled map with one face

Very-well-labelled quadrangulations

Very-well-labelled quadrangulation $=$ quadrangulation where

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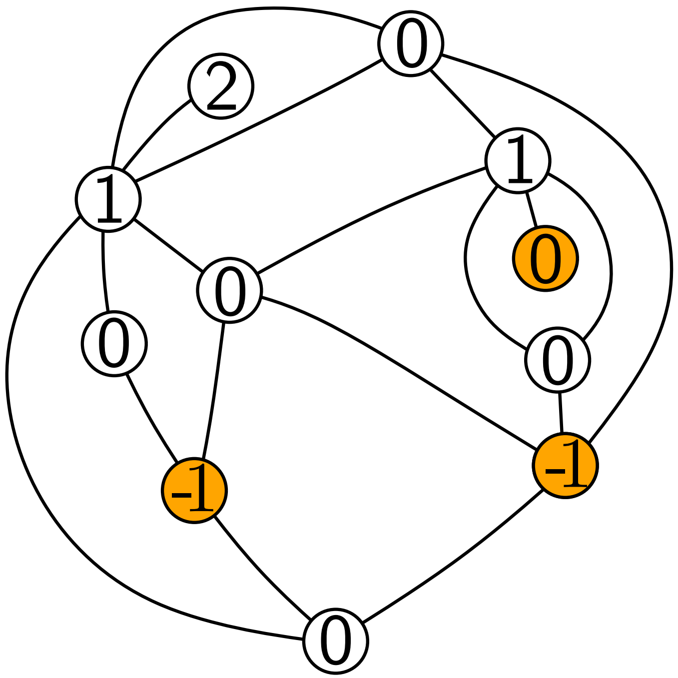
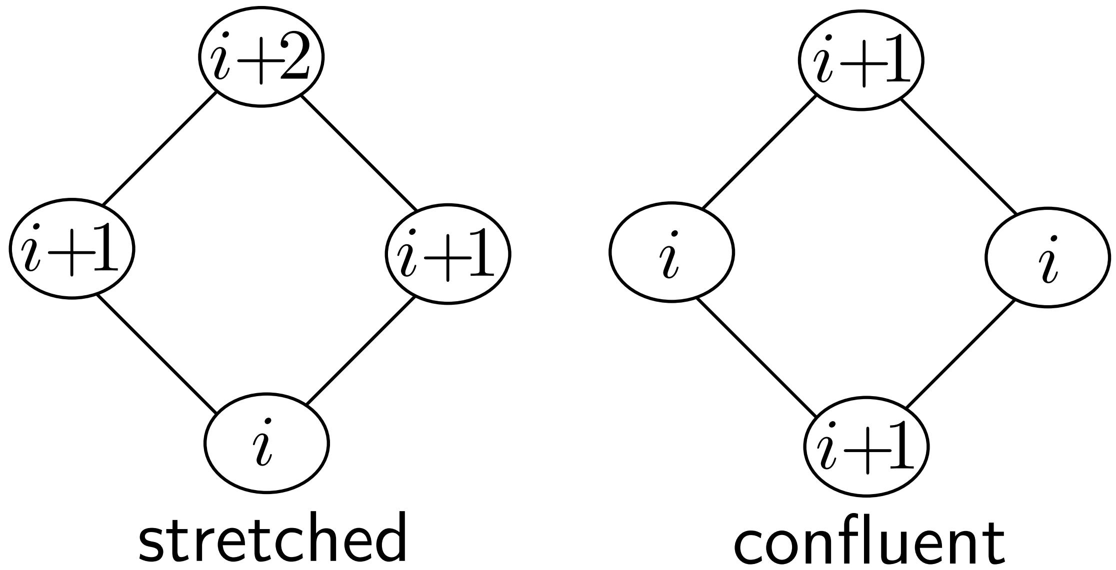


a very-well-labelled quadrangulation Q with 3 local min

Rk: two types of faces



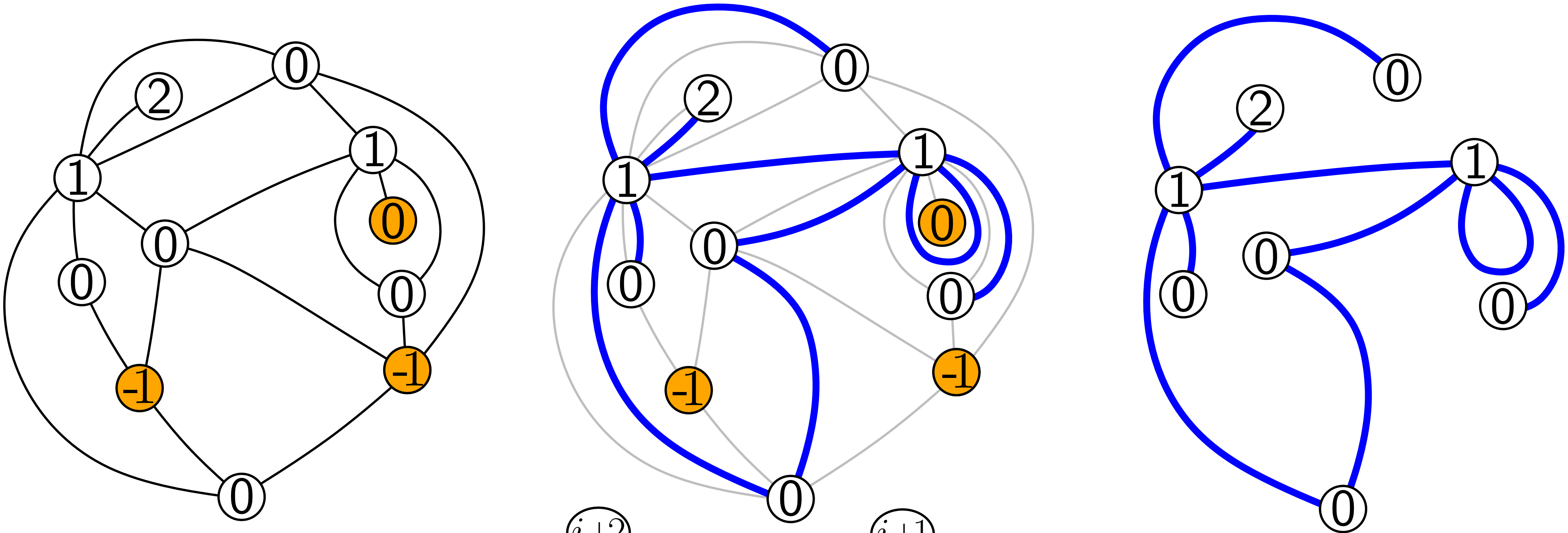
Def: local min$=$ vertex with all neighbours of larger label

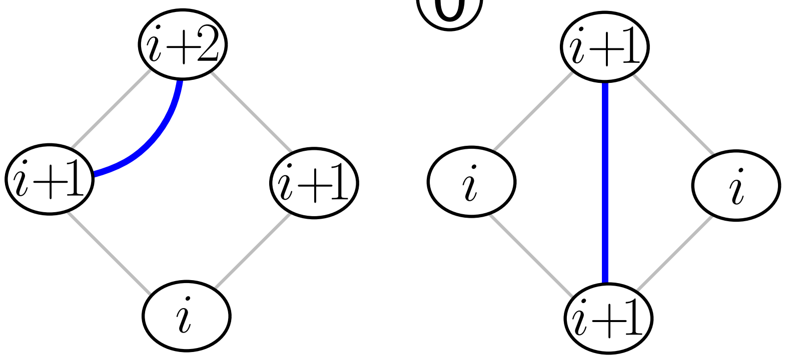
Rk: Geodesic labelling $\Leftrightarrow$ there is just one local min, of label 0

The Miermont bijection [Miermont'07], [Ambjørn, Budd'13]

Very-well labelled quadrangulation Q $\Rightarrow$ well-labelled map M

n faces $\qquad\qquad$ n edges





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local min v $\xleftrightarrow[\ell(v)=\min(f)-1]{}$ face f

non-local min $\xleftrightarrow[\text{same label}]{}$ vertex

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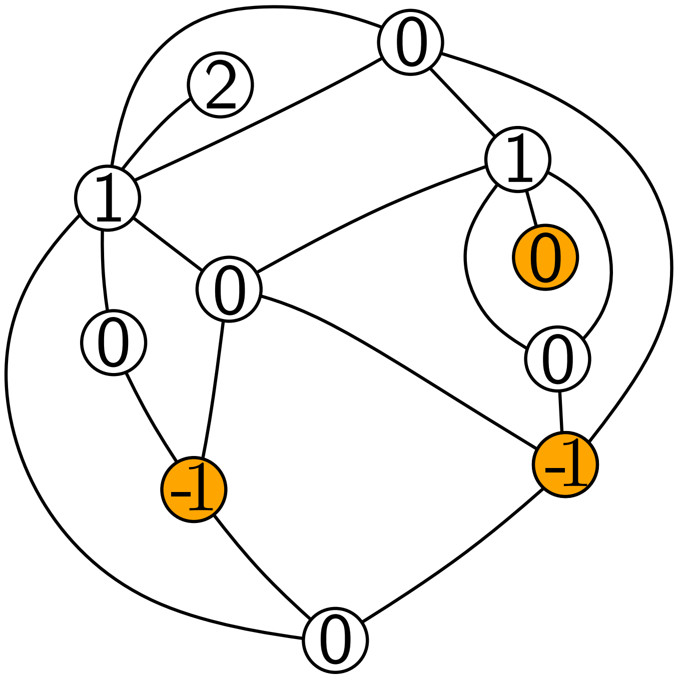
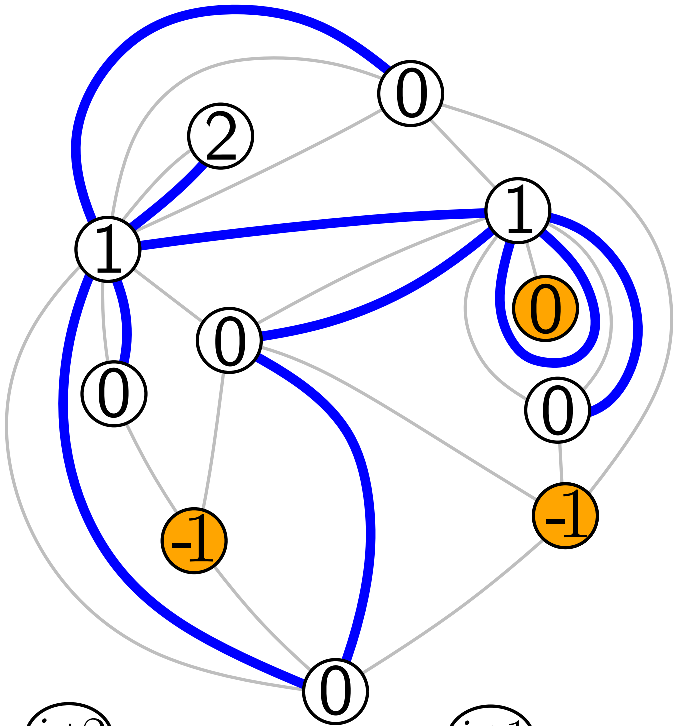
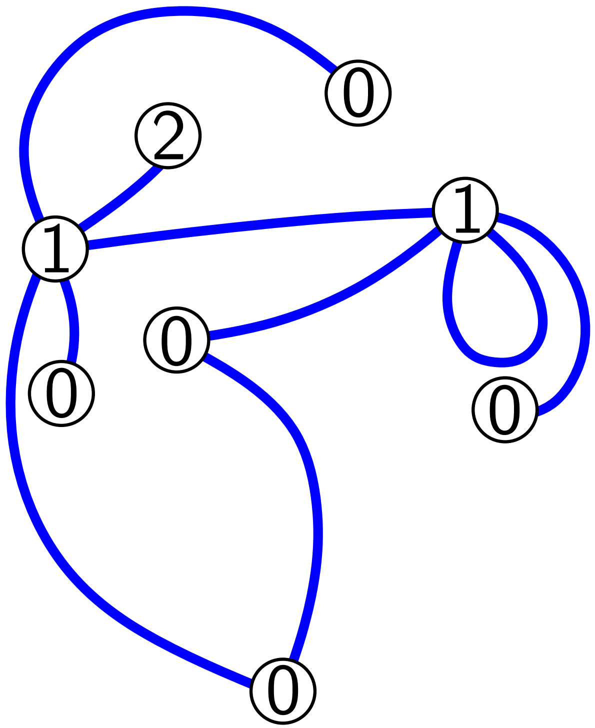
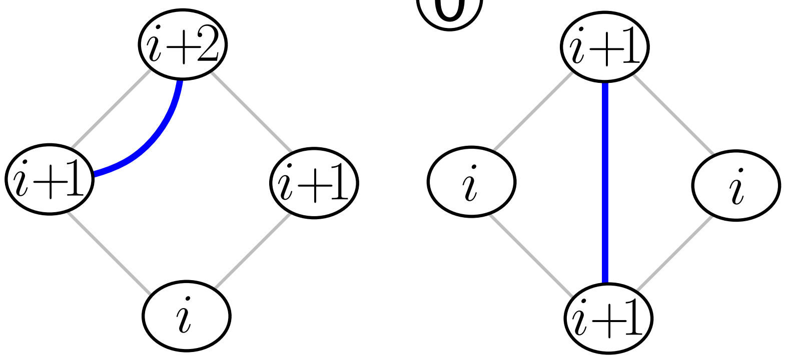






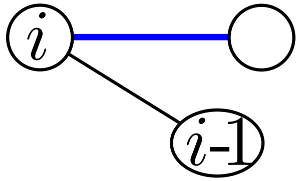


recover the Schaeffer bijection (case of one local min, of label 0)

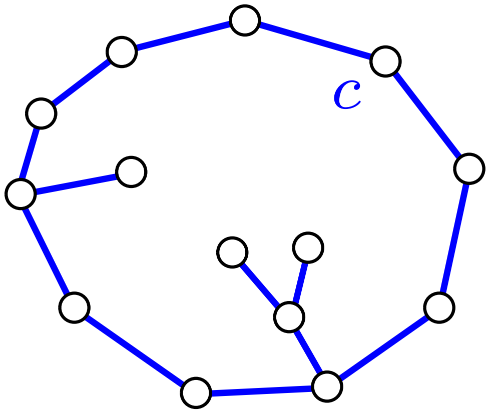
local min v $\overset{\longleftrightarrow}{\ell(v) = \min(f) - 1}$ face f

non-local min $\underset{\text{same label}}{\longleftrightarrow}$ vertex

Proof of the stated properties

implies


(follows from the local rules)

$\Rightarrow$


From each corner c in a "face" of M starts a label-decreasing path of Q that stays in the face and ends at a local min of Q

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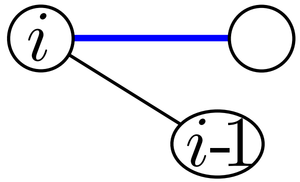

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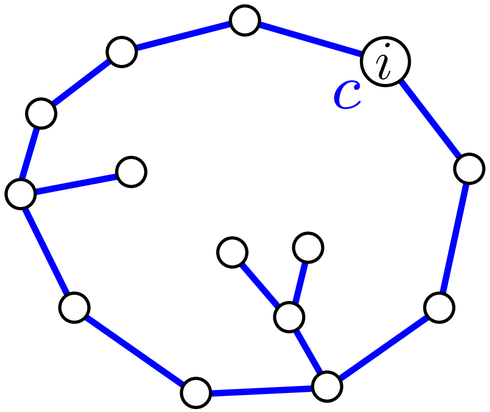

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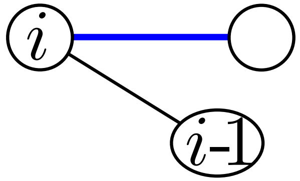
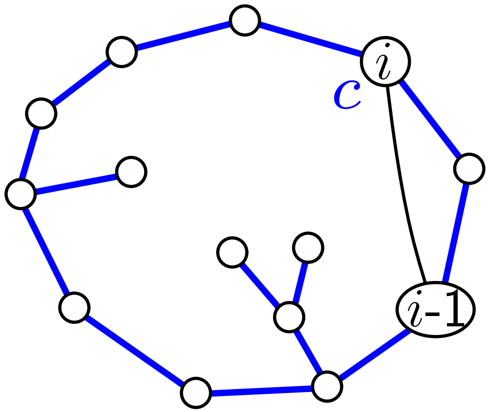


(follows from the local rules)

$\Rightarrow$



From each corner c in a “face” of M starts a label-decreasing path of Q that stays in the face and ends at a local min of Q

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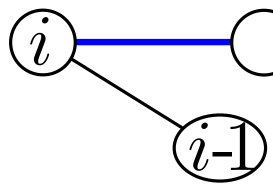

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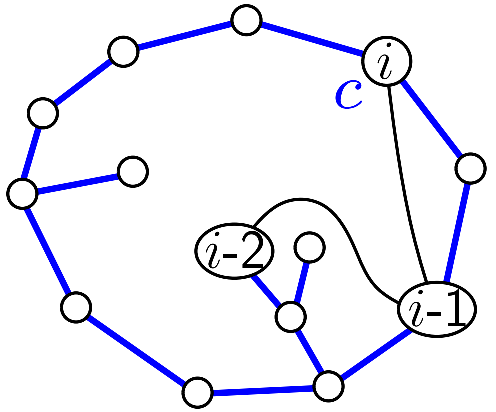

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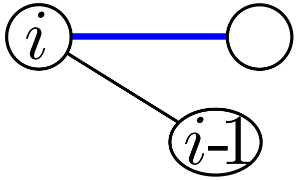

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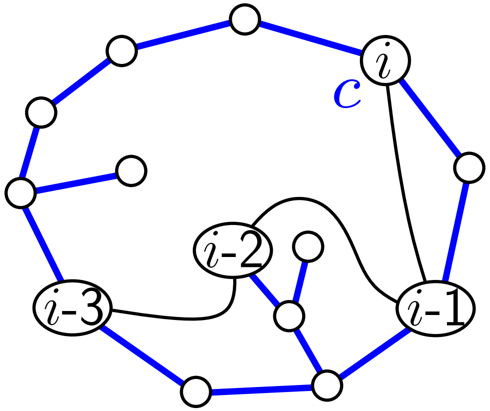

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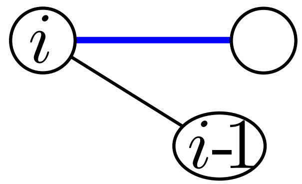

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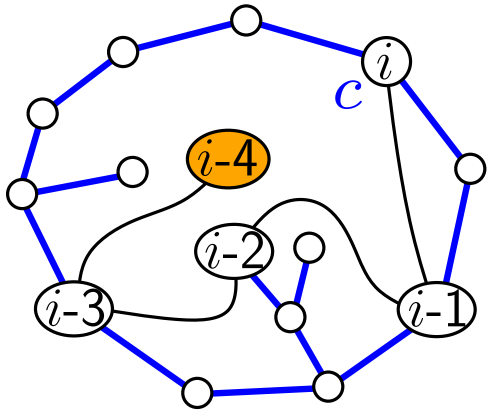

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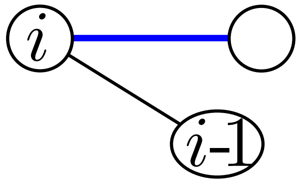

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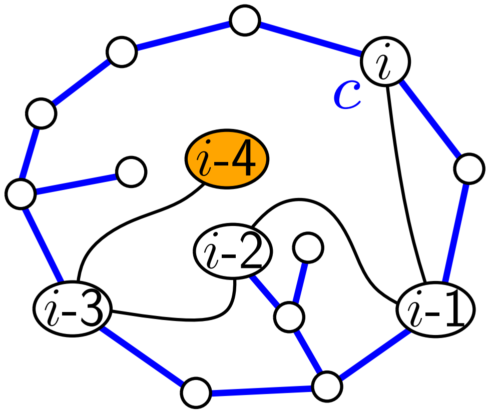

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Let $n = \#$ faces of Q, $p = \#$ local min of Q, $f = \#$ "faces" of M

	$\#V$	$\#E$	$\#F$
Q	$n+2$	$2n$	n
M	$n+2-p$	n	$f = k-1+p$

Euler's relation, with
$k = \#$ connected comp. of M

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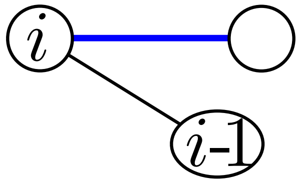

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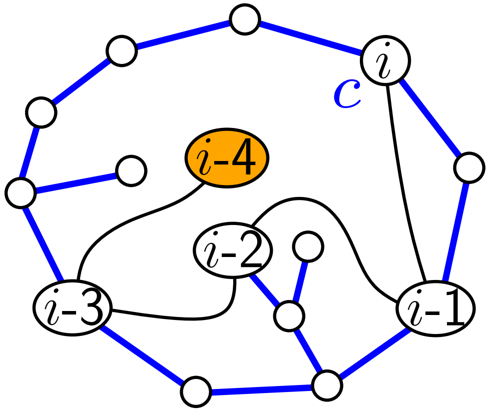

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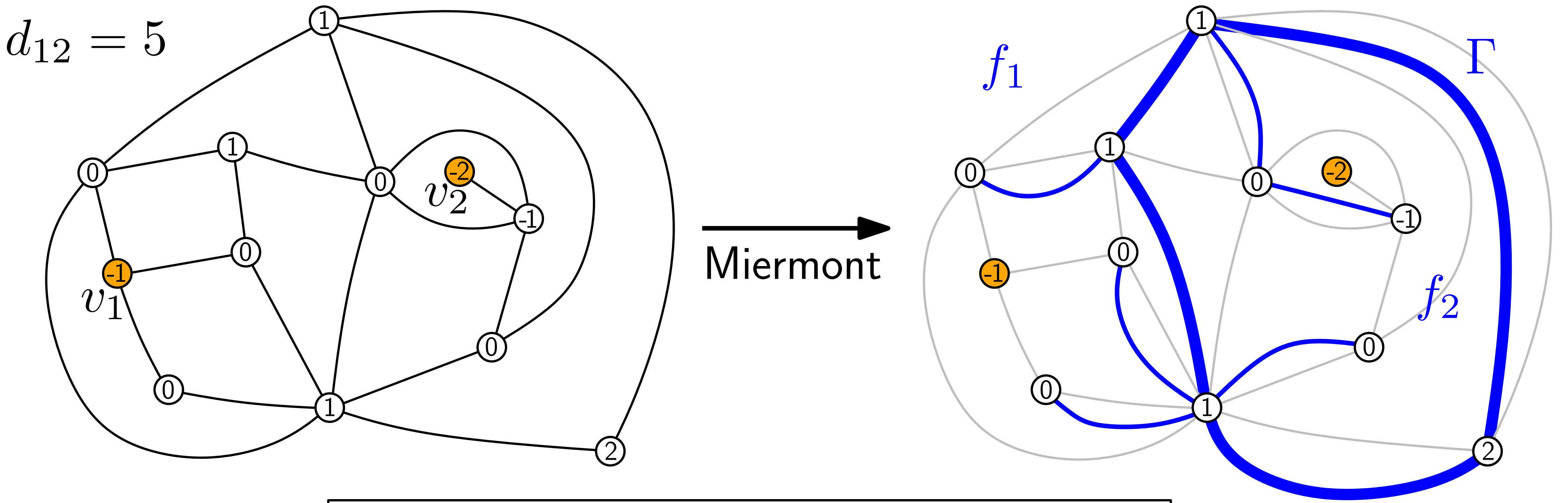
Drawing above $\Rightarrow f \leq p$

Hence $k = 1$ (M connected) $f = p$, and there is exactly one local min of Q in each face of M

The case of two local min

Γ the boundary, here $\min_\Gamma = 1$



$$\boxed{\mathrm{dist}(v_1, v_2) = 2 \cdot \min_\Gamma - \ell(v_1) - \ell(v_2)}$$

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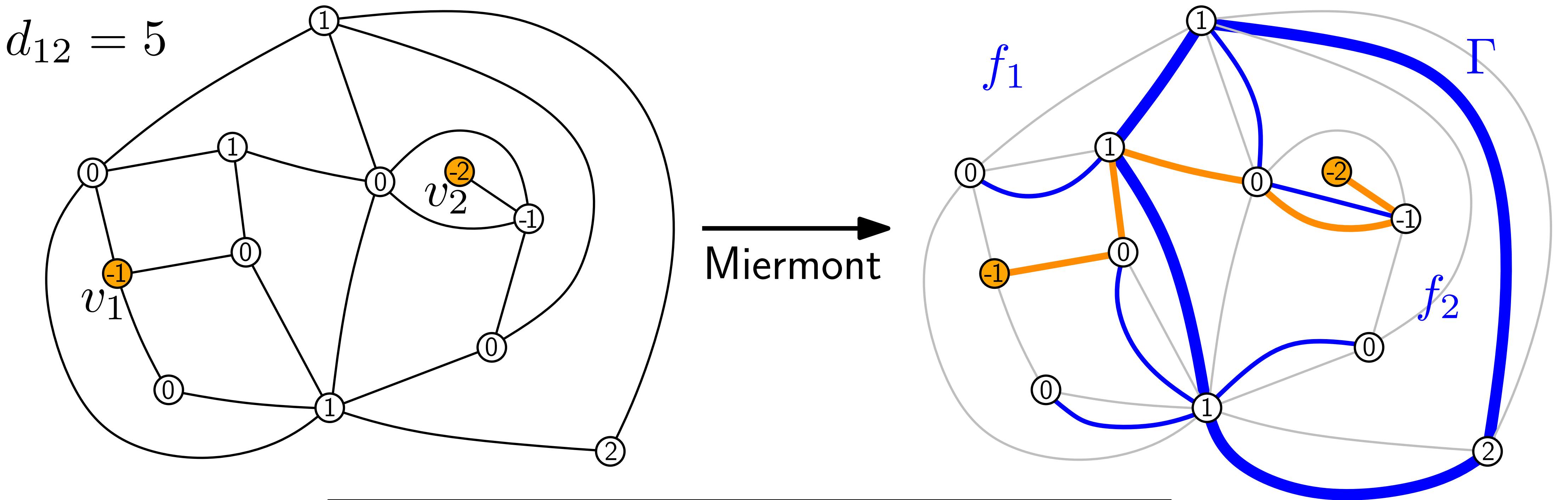

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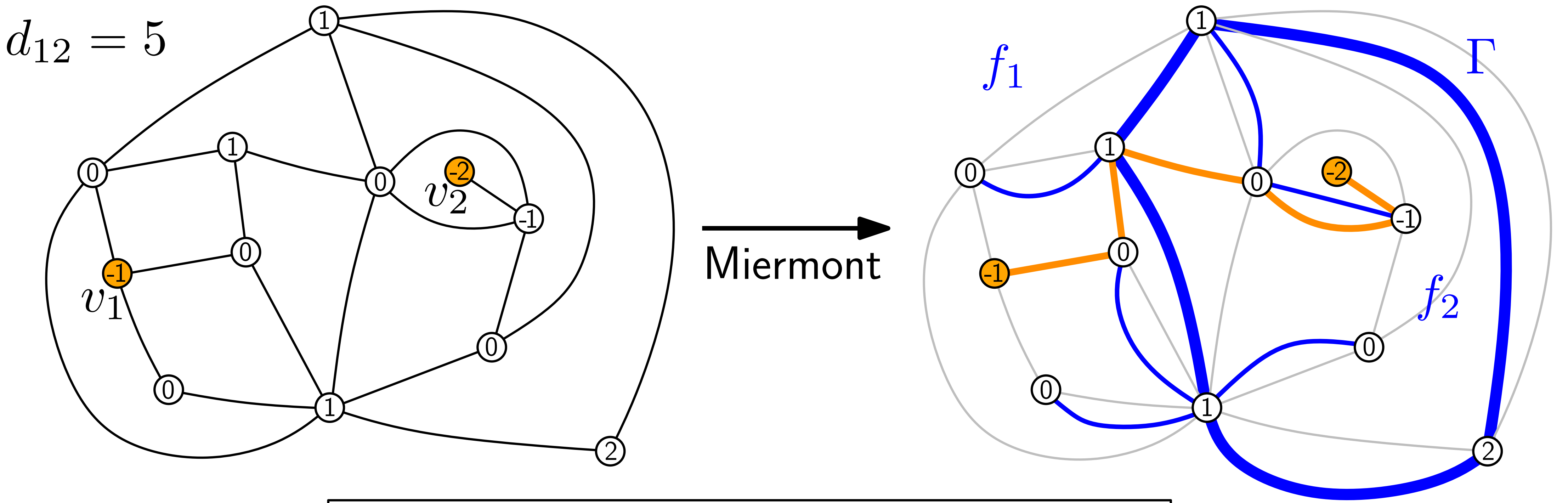
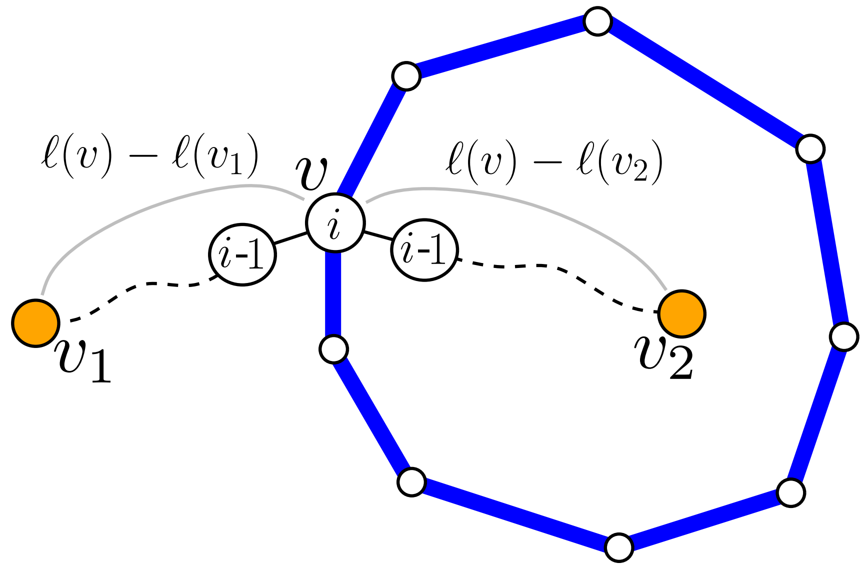


$$\boxed{\mathrm{dist}(v_1, v_2) = 2 \cdot \min_\Gamma - \ell(v_1) - \ell(v_2)}$$

Proof: $\forall v \in \Gamma$, a shortest path $v_1 \to v \to v_2$ has length $2\ell(v) - \ell(v_1) - \ell(v_2)$ (because of the existence of a label-decreasing path on each side)



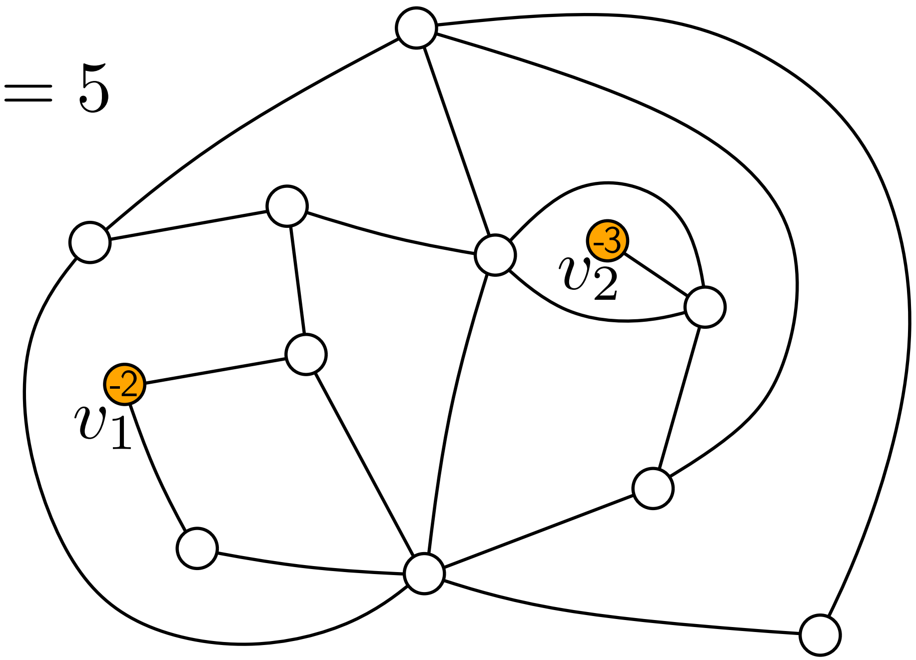
Another way of computing the 2-point function

[Bouttier, Guitter'08] Let $d \geq 2$ and let $s, t \geq 1$ such that $s + t = d$

A bi-pointed quadrangulation Q where $d_{12} = d$ has a unique very-well labelling $\ell(.)$ with two local min, at v_1, v_2, and $\ell(v_1) = -s$, $\ell(v_2) = -t$.

$\ell(.)$ is given by $\boxed{\ell(v) = \min(\mathrm{dist}(v_1, v) - s,\ \mathrm{dist}(v_2, v) - t)}$

$d = 5$



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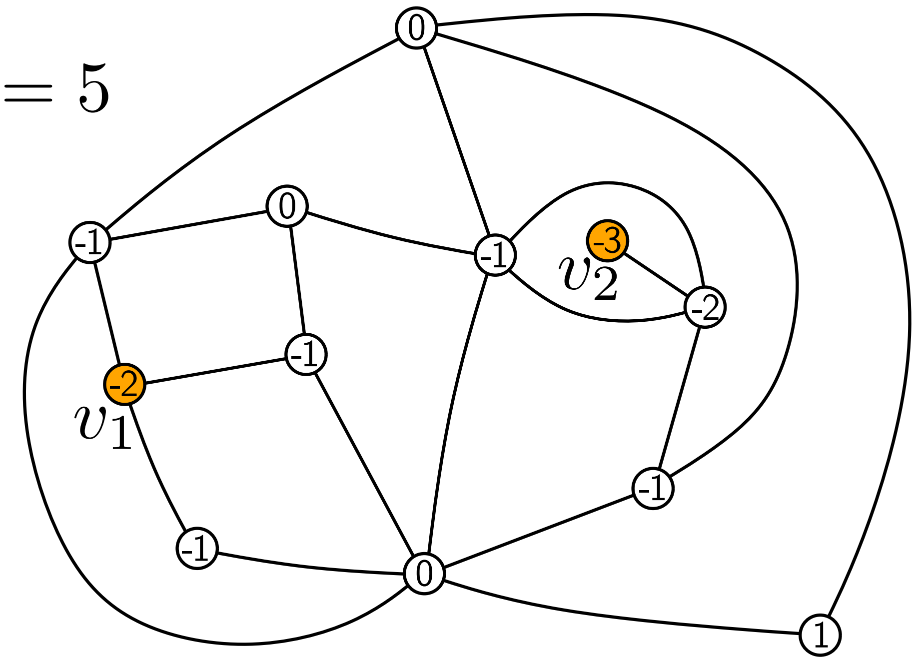

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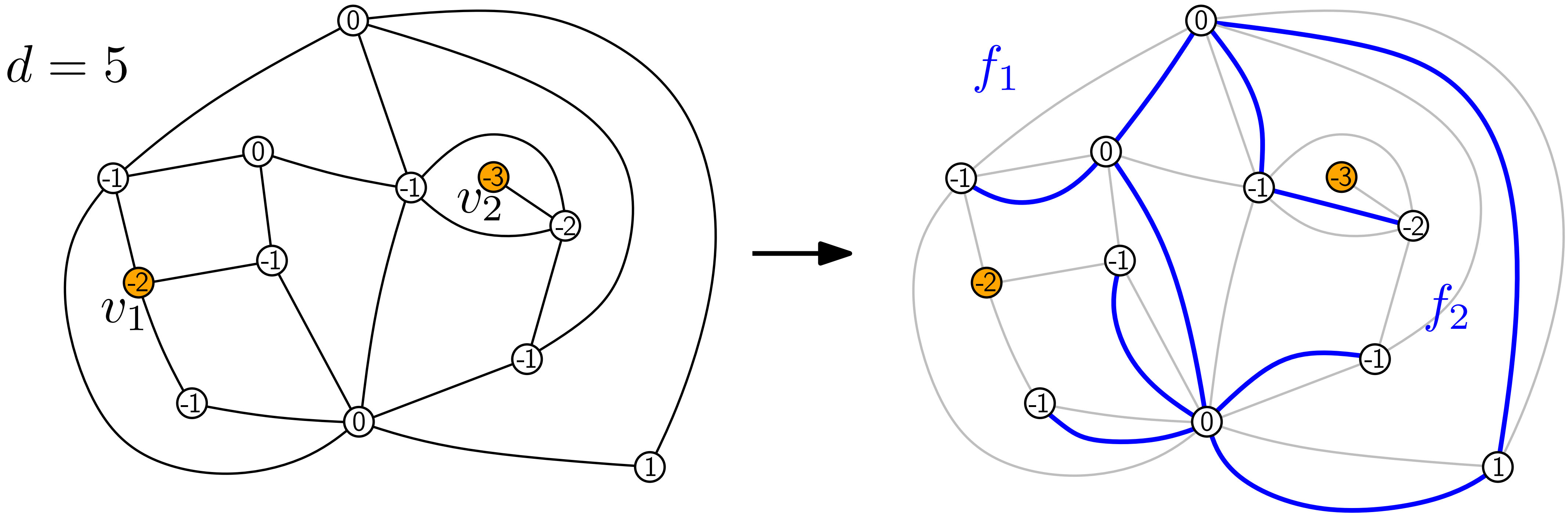

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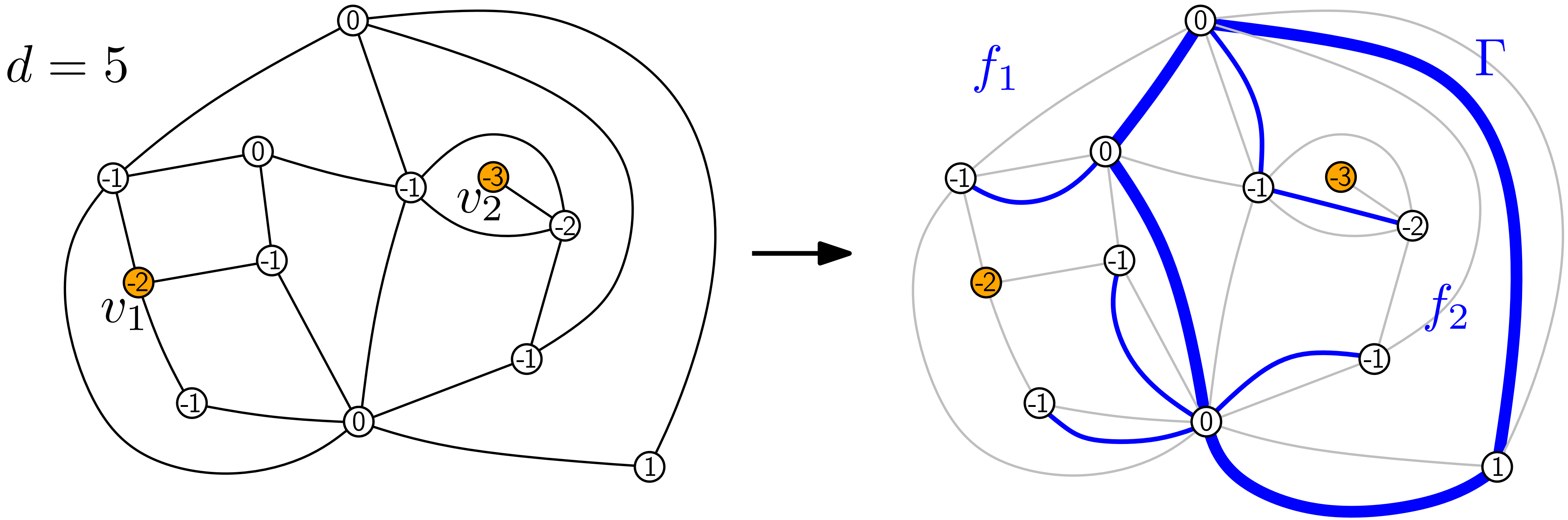

The associated well-labelled map with two faces f_1, f_2 satisfies:

- $\min(f_1) = -s + 1$, $\min(f_2) = -t + 1$
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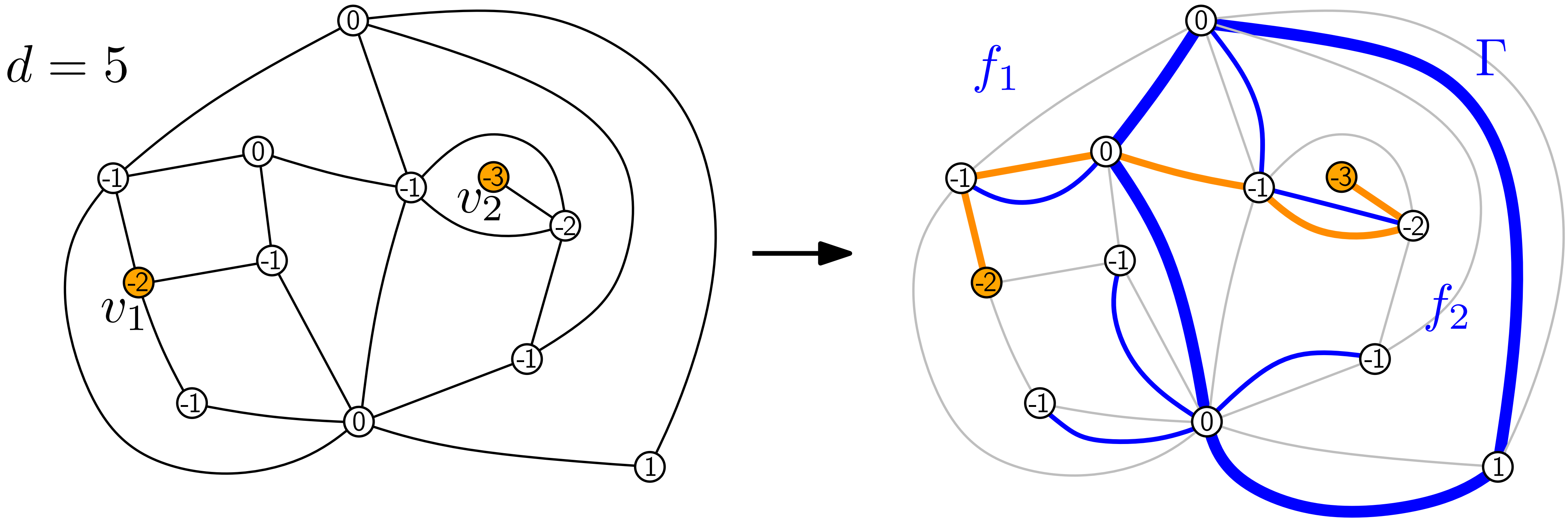

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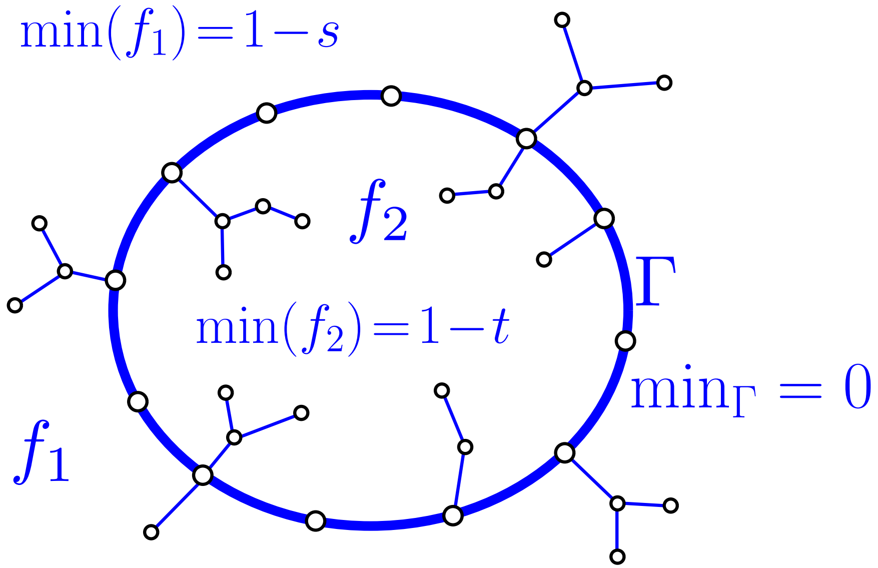
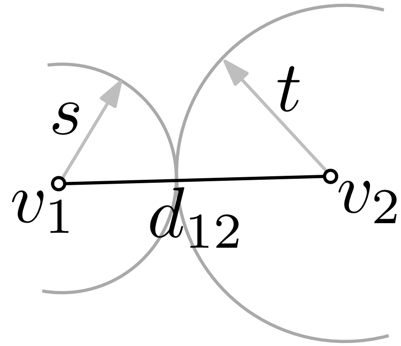
Another way of computing the 2-point function

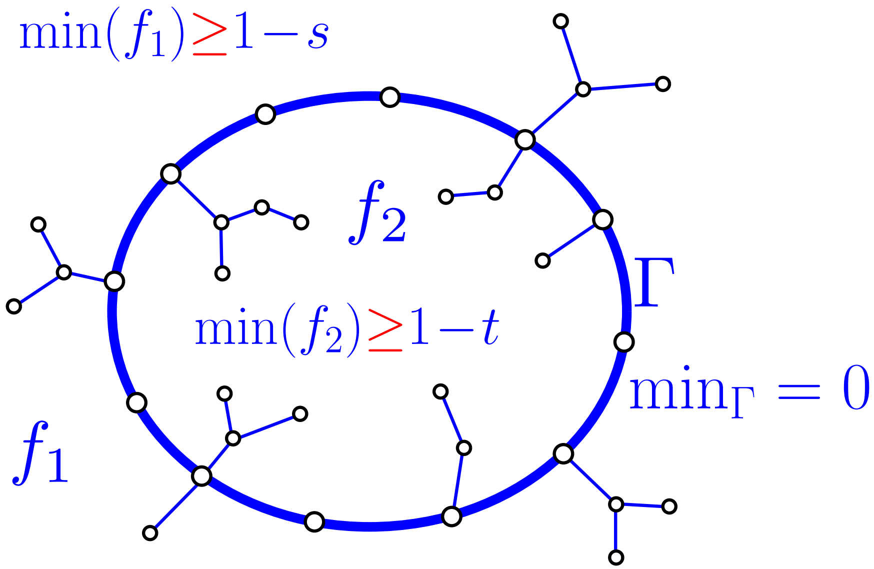
We conclude that, for $d = s + t$ $(s, t \geq 1)$ $G_d(g)$ is the series of





1st method corresponds to $t = 0$

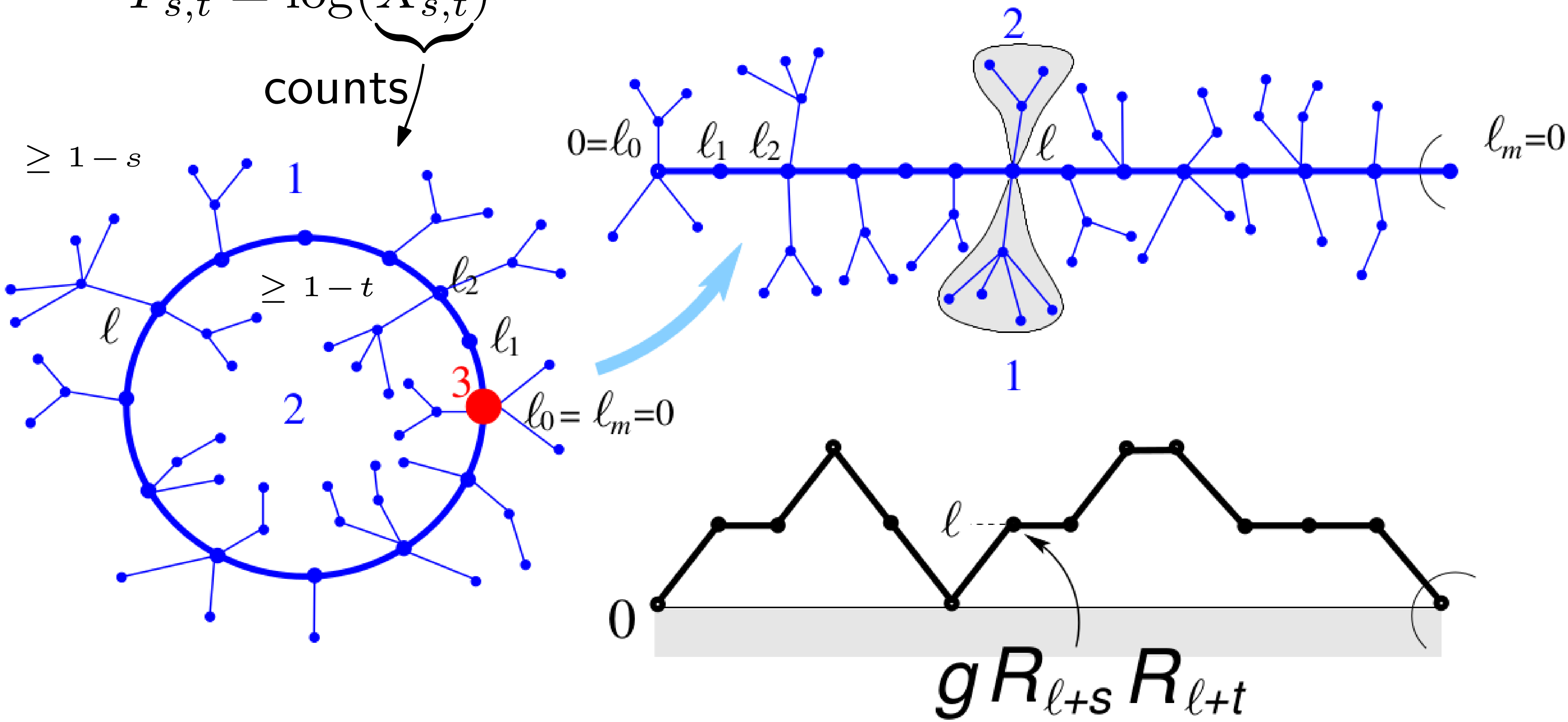
Or ($\Delta :=$ discrete differentiation) $G_d = \Delta_s \Delta_t F_{s,t}$, where $F_{s,t}$ counts



Another way of computing the 2-point function

Then by the link between cyclic and sequential excursions:

$$F_{s,t} = \log(\underbrace{X_{s,t}}_{\text{counts}})$$



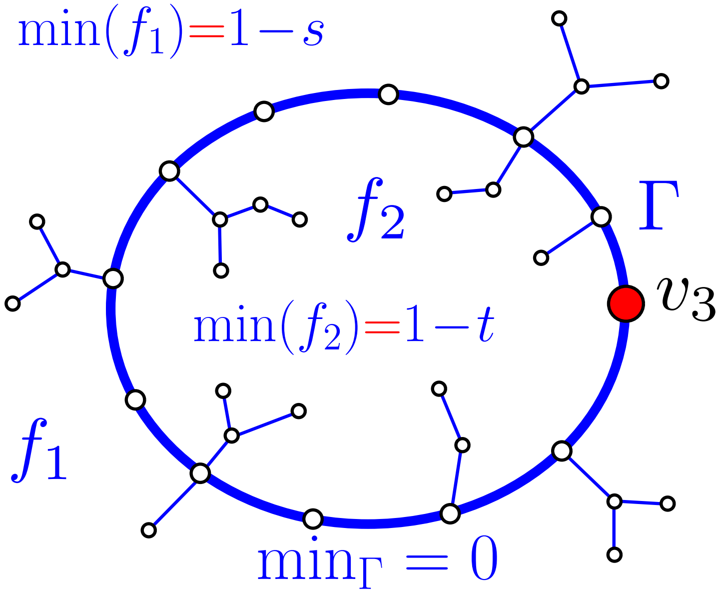
Equation for $X_{s,t}$: $X_{s,t} = 1 + gR_sR_tX_{s,t}(1 + gR_{s+1}R_{t+1}X_{s+1,t+1})$

solution (guessing/checking): $X_{s,t} = \frac{[3]_x[s+1]_x[t+1]_x[s+t+3]_x}{[1]_x[s+3]_x[t+3]_x[s+t+1]_x}$

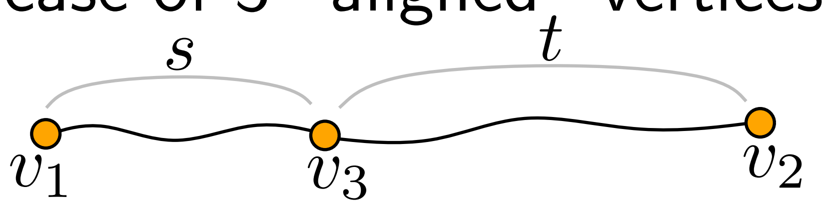
$\Rightarrow$ recover $\boxed{G_d = \log\left(\frac{[s+t]_x^2[s+t+3]_x}{[s+t-1]_x[s+t+2]_x^2}\right)}$

A first covered case for the 3-point function

[Bouttier, Guitter'08] This solves the case of 3 "aligned" vertices



$\Longleftrightarrow$



tri-pointed quadrangulations with

$$d_{12} = s + t,\ d_{13} = s,\ d_{23} = t$$

i.e., v_3 is on a geodesic path from v_1 to v_2 at respective distances s, t from v_1, v_2

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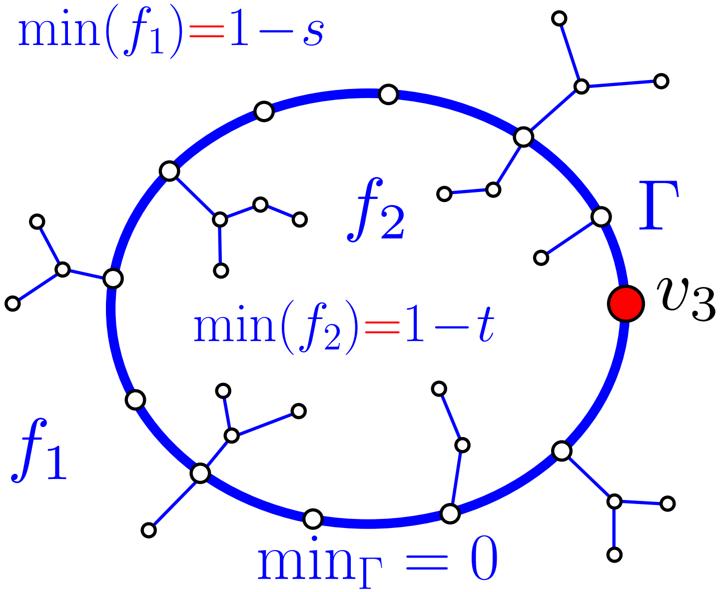

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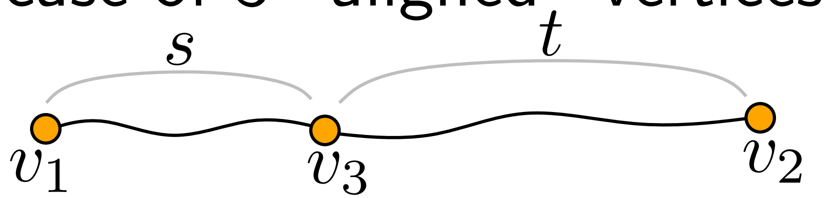

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Hence $G_{s+t,s,t}(g) = \Delta_s \Delta_t X_{s,t}$ where $X_{s,t} = \frac{[3]_x [s+1]_x [t+1]_x [s+t+3]_x}{[1]_x [s+3]_x [t+3]_x [s+t+1]_x}$

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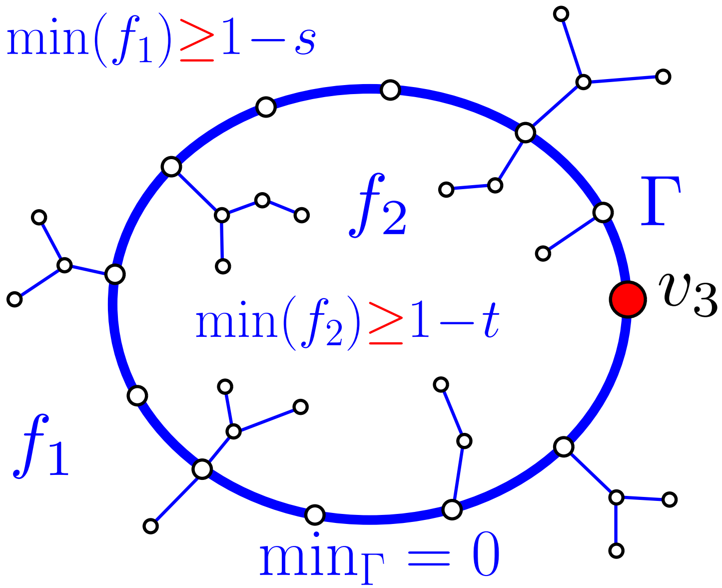

The different cases for the 3-point function

[Bouttier, Guitter'08] $D = (d_{12}, d_{13}, d_{23})$ can be achieved only if

$$\begin{cases} d_{12} \leq d_{13} + d_{23} \\ d_{13} \leq d_{12} + d_{23} \\ d_{23} \leq d_{12} + d_{13} \end{cases}$$

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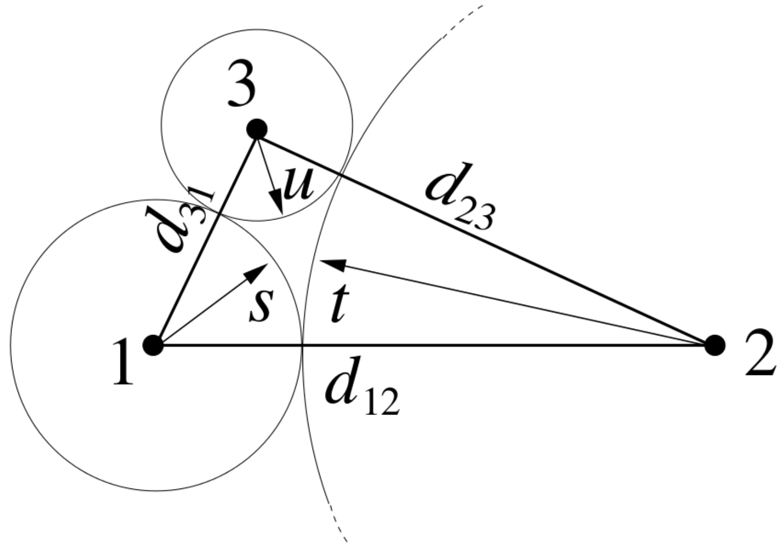

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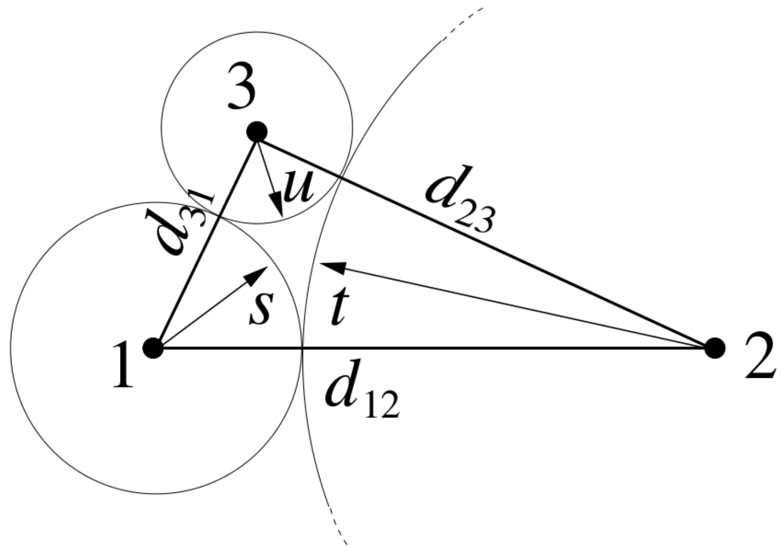

- 3 points are distinct $\Rightarrow$ at most one of s, t, u is zero
- One of s, t, u (say u) is zero $\Leftrightarrow$ aligned points (preceding slide)
- Generic case: $s, t, u > 0$ (non-aligned points)

The generic case

[Bouttier, Guitter'08] write D as

$$d_{12} = s + t$$
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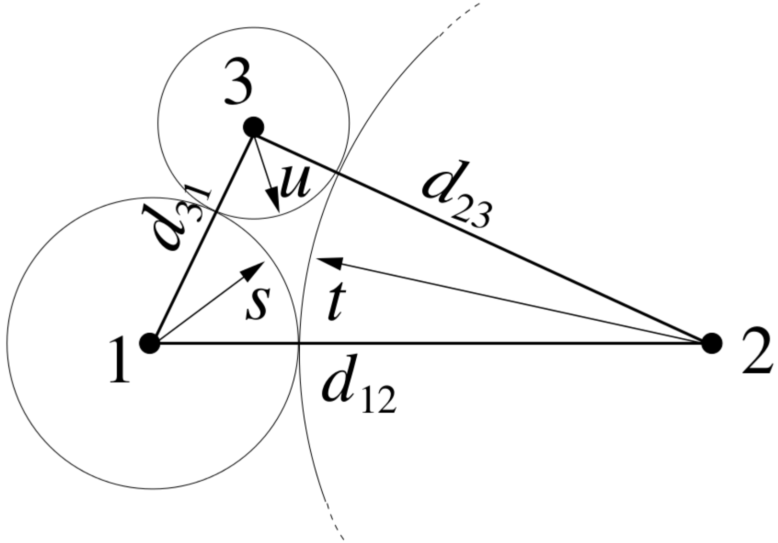

Endow Q with unique very-well labelling with 3 local min at v_1, v_2, v_3 and where $\ell(v_1) = -s,\ \ell(v_2) = -t,\ \ell(v_3) = -u$

Apply the Miermont bijection $\Rightarrow$

obtain a 3-face well-labelled map where

$$\min(f_1) = 1 - s \qquad \min_{\Gamma_{12}} = 0$$
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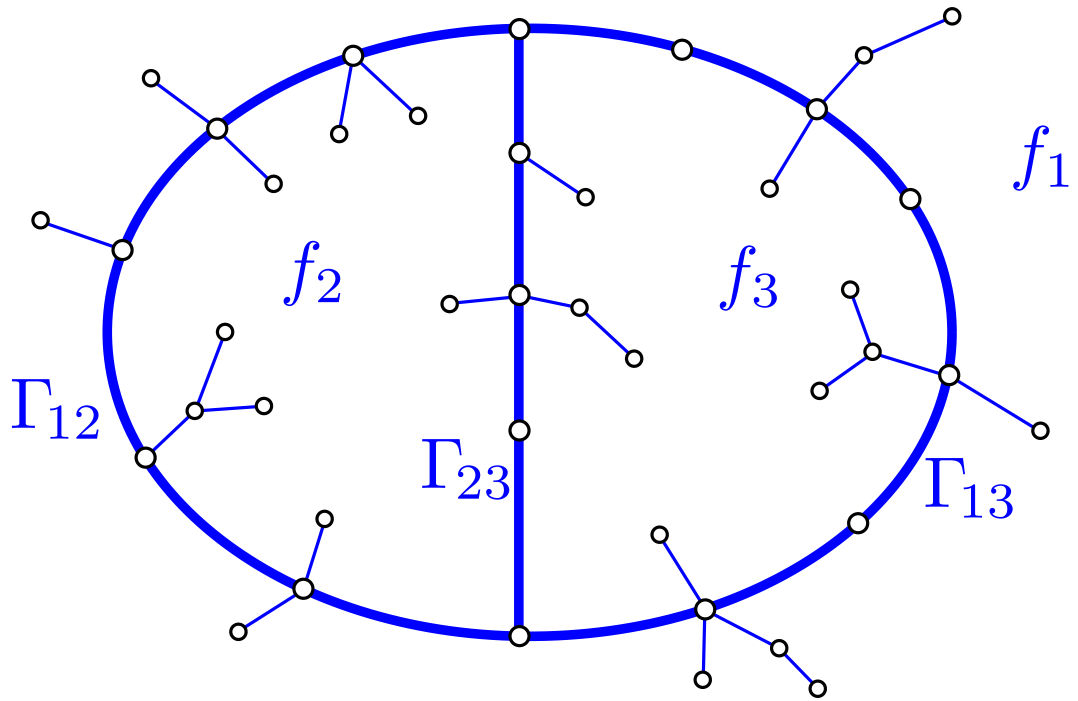

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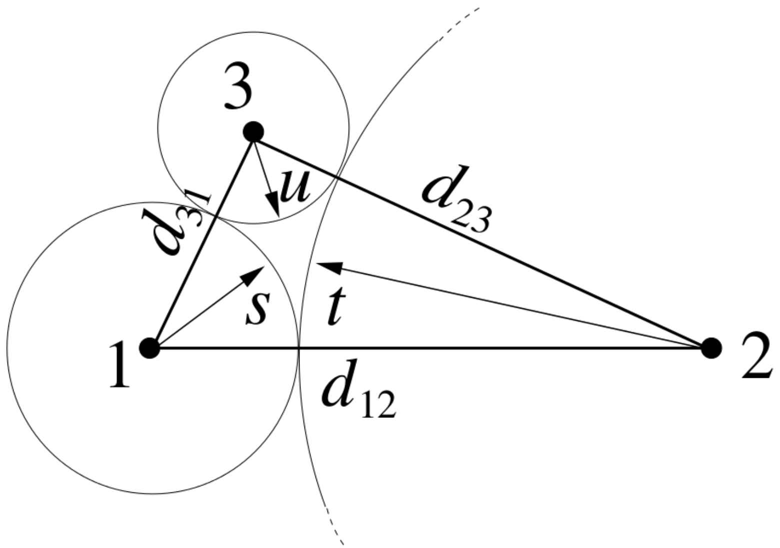

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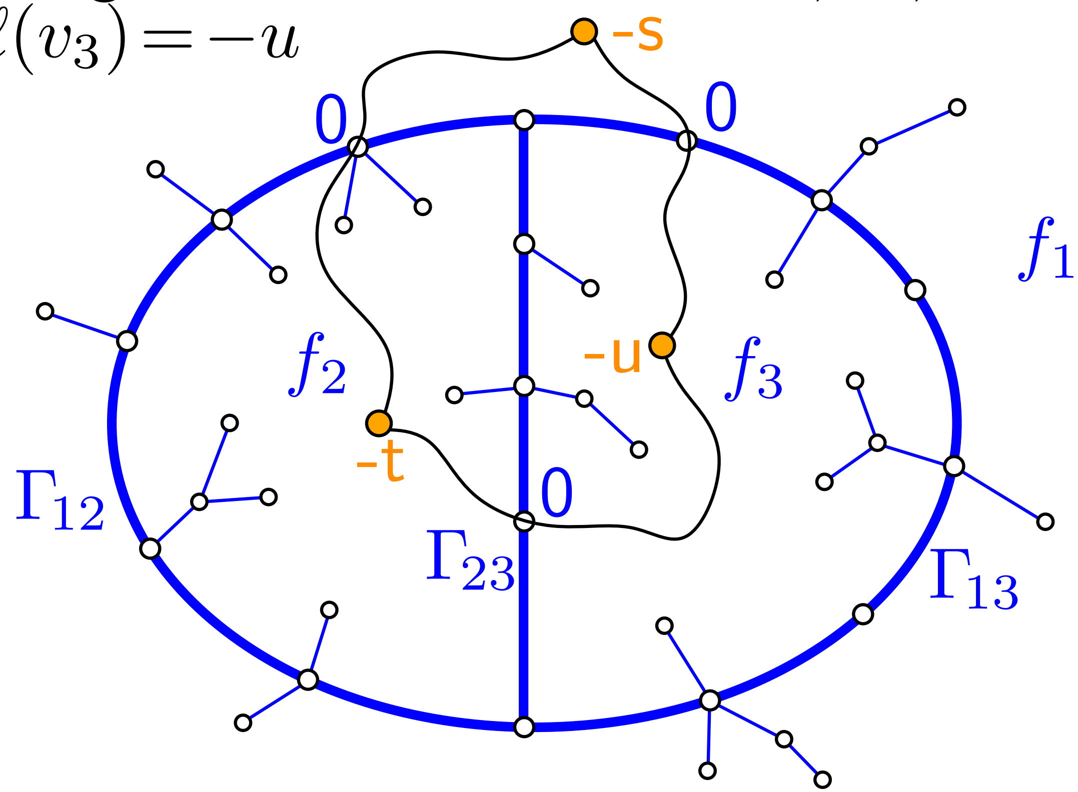

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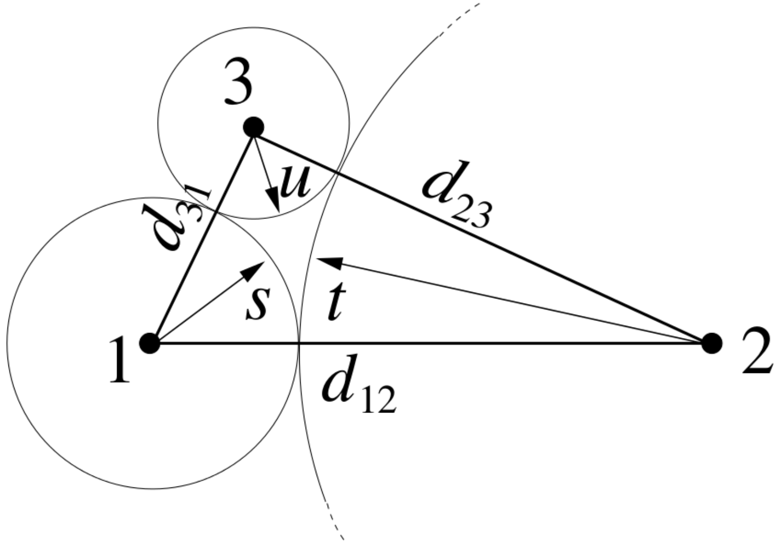

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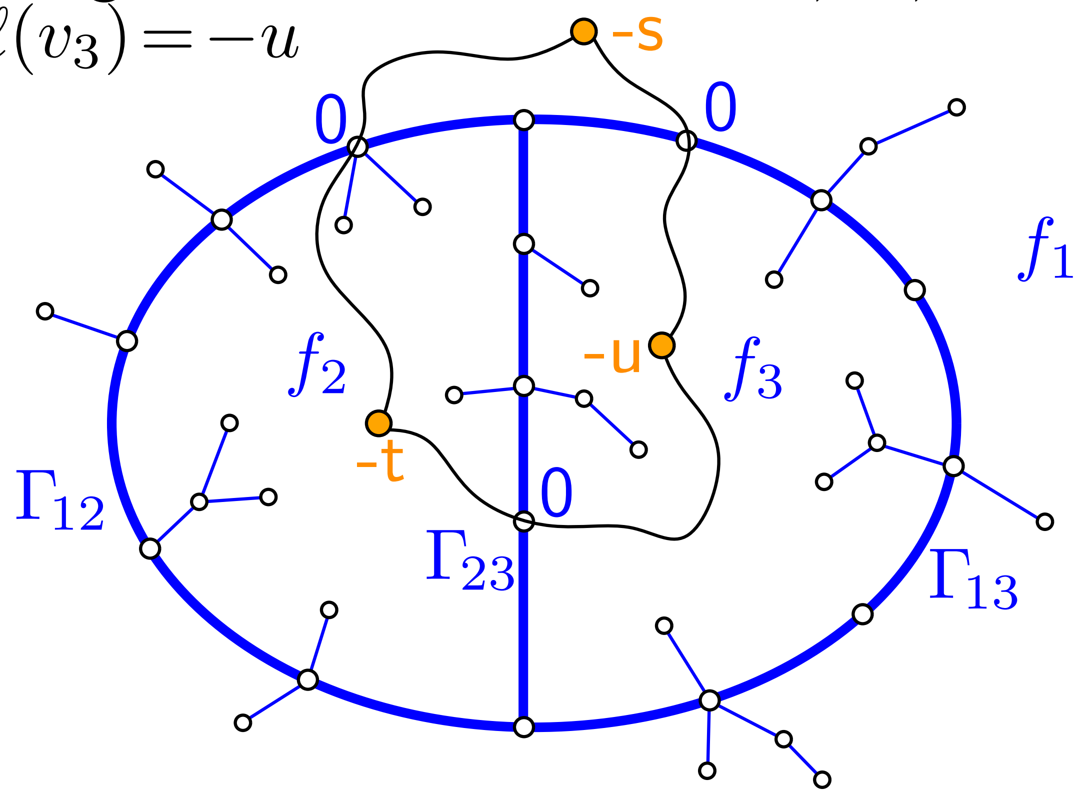
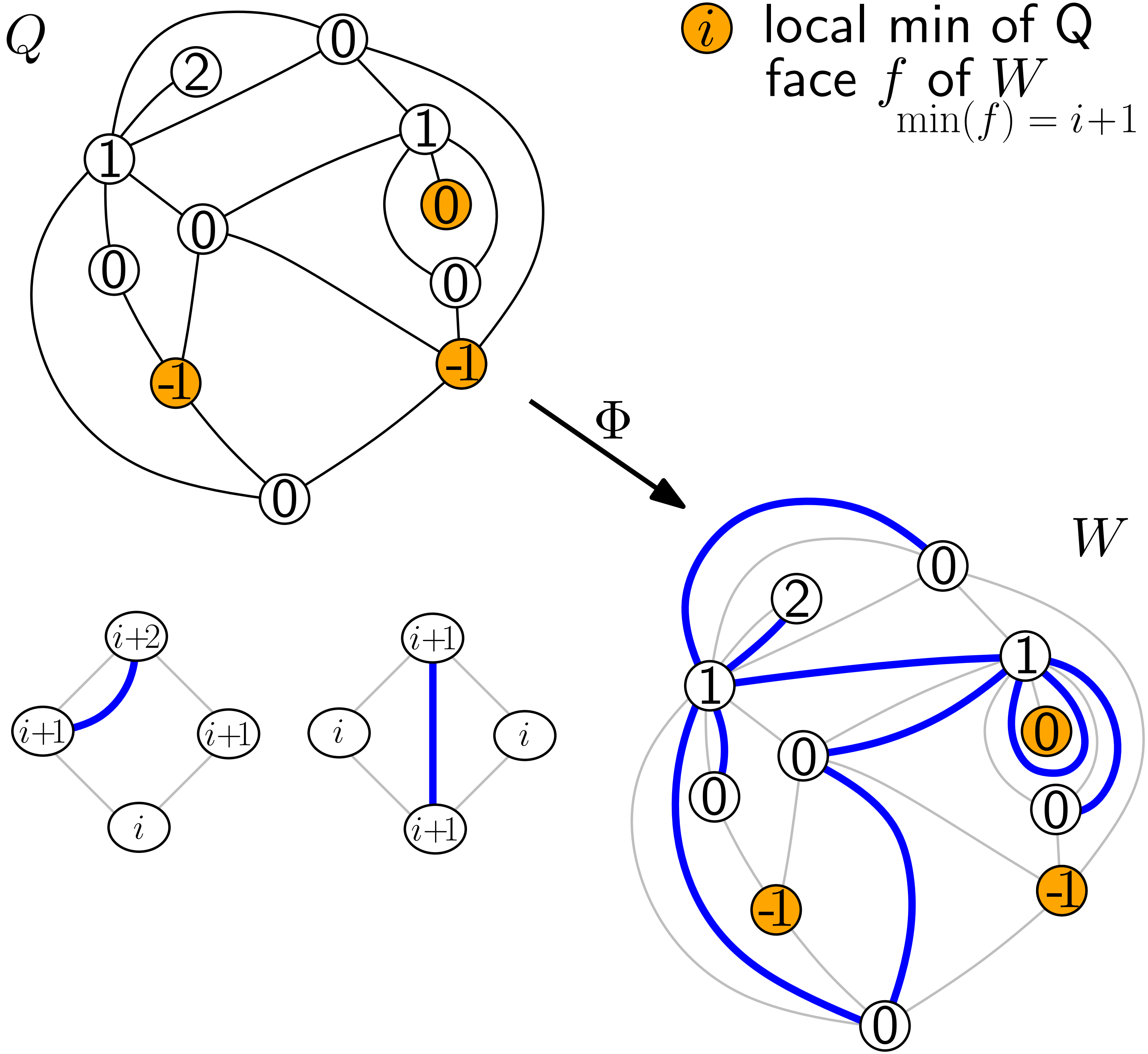


$\Rightarrow$ expression of $G_{d_{12},d_{13},d_{23}}(g)$ as $\Delta_s \Delta_t \Delta_u F_{s,t,u}$, with $F_{s,t,u}(g)$ explicit

Computing the two-point function of general maps using the Ambjørn-Budd bijection

The Ambjørn-Budd bijection Λ [Ambjørn-Budd'13]

Recall the Miermont bijection Φ (reformulated by Ambjørn-Budd)



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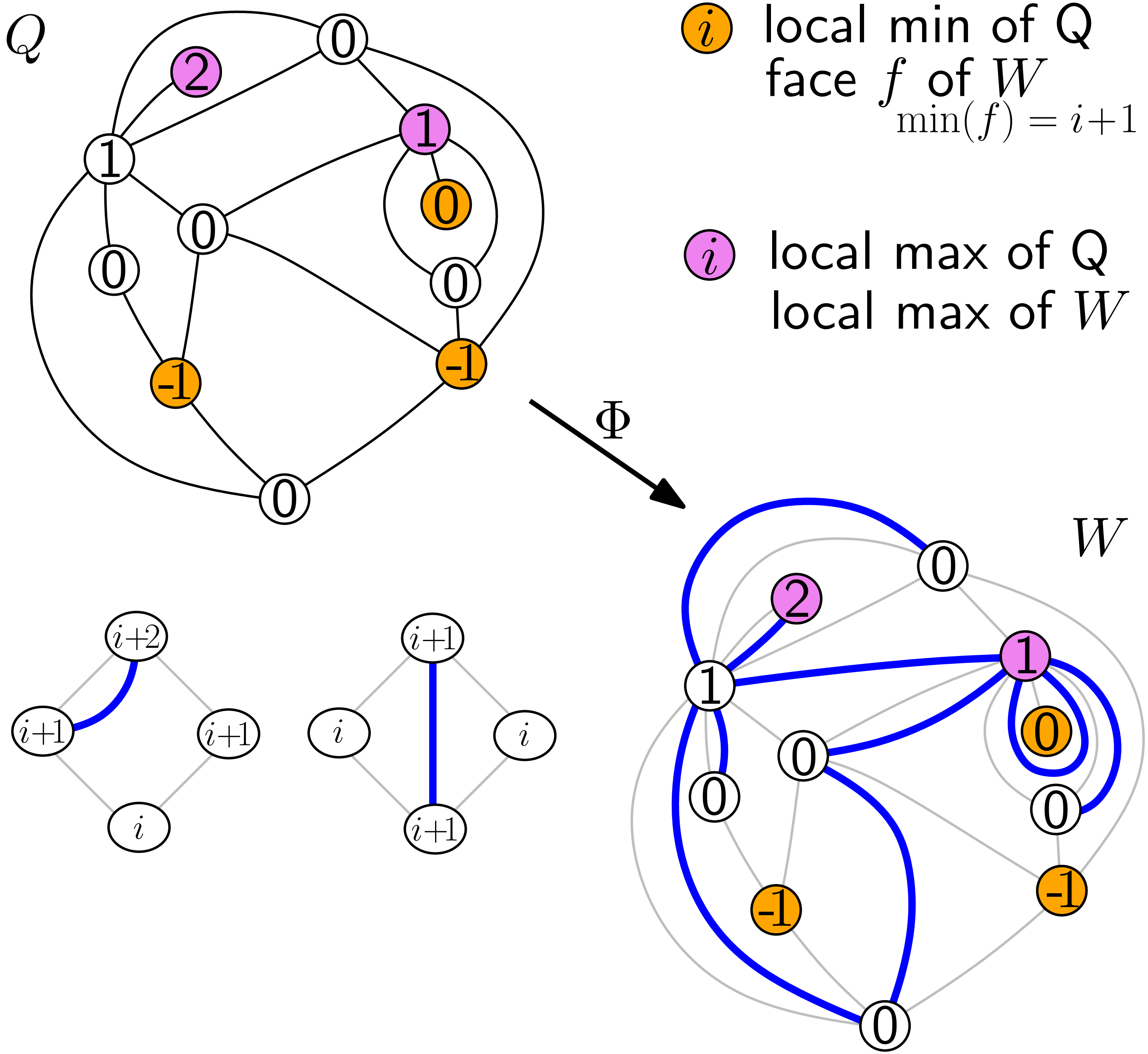

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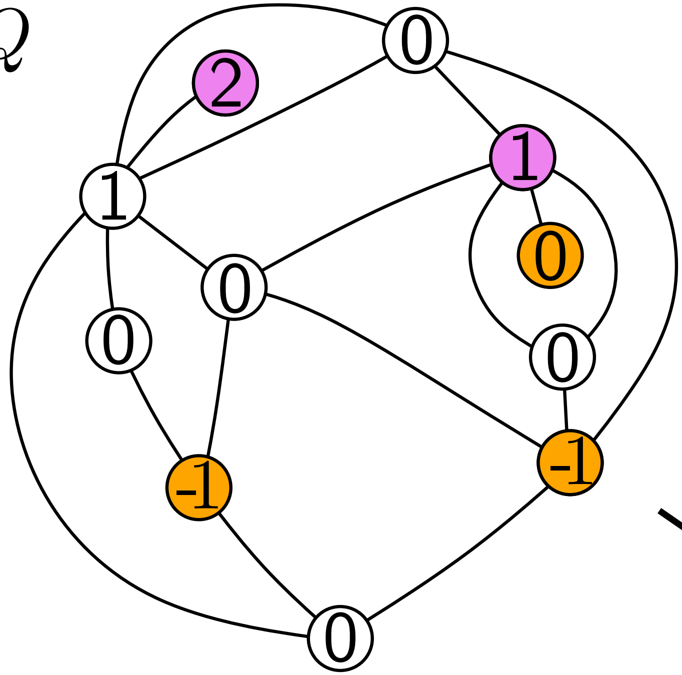
Q



i local min of Q, face f of W, $\min(f) = i{+}1$

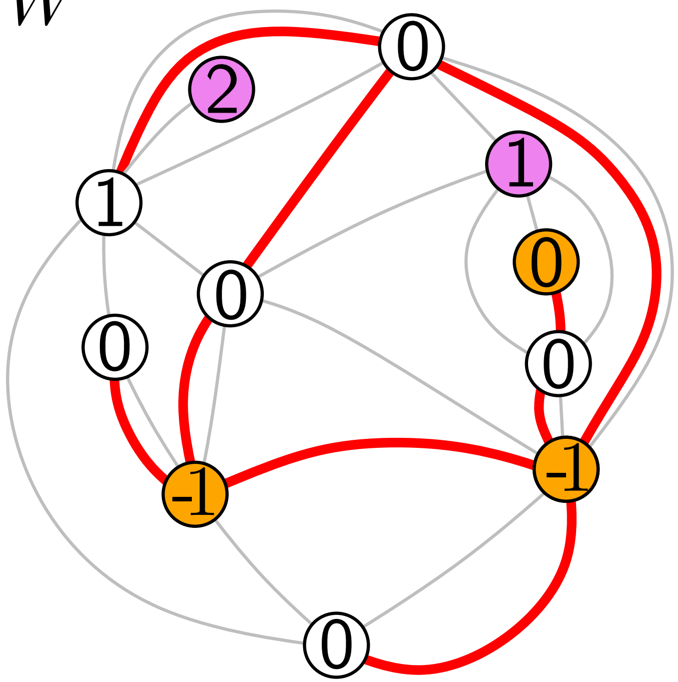
i local max of Q, local max of W

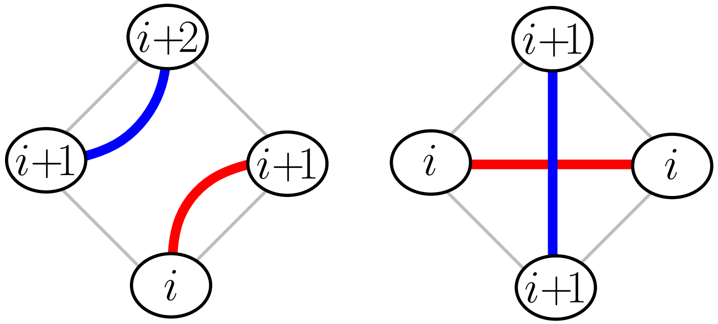
Let $op : \mathbf{Z} \to \mathbf{Z}$, $i \to$ -i

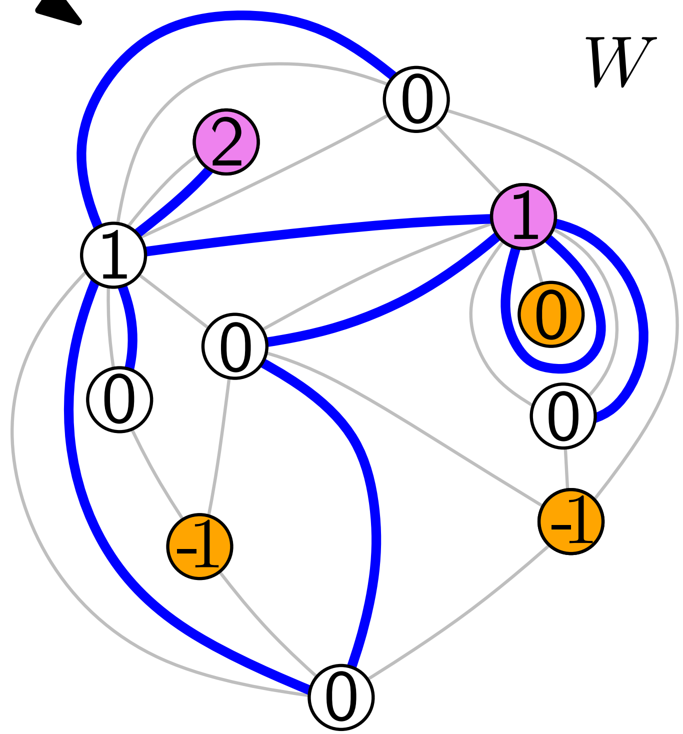
$\Phi^{-} = op \circ \Phi \circ op$

Φ

W^{-}



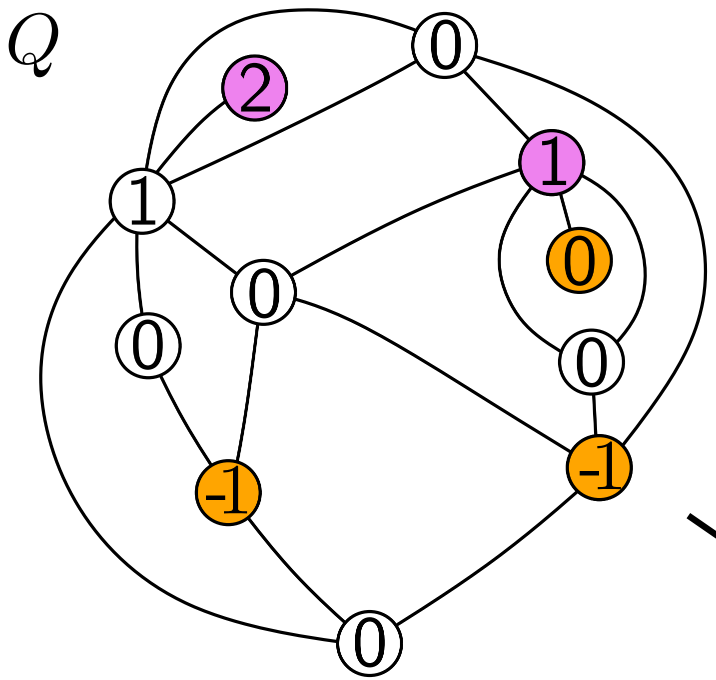




The Ambjørn-Budd bijection Λ [Ambjørn-Budd'13]

Recall the Miermont bijection Φ (reformulated by Ambjørn-Budd)

Q



Let $op : \mathbf{Z} \to \mathbf{Z}$
$\qquad\qquad i \to \text{-}i$

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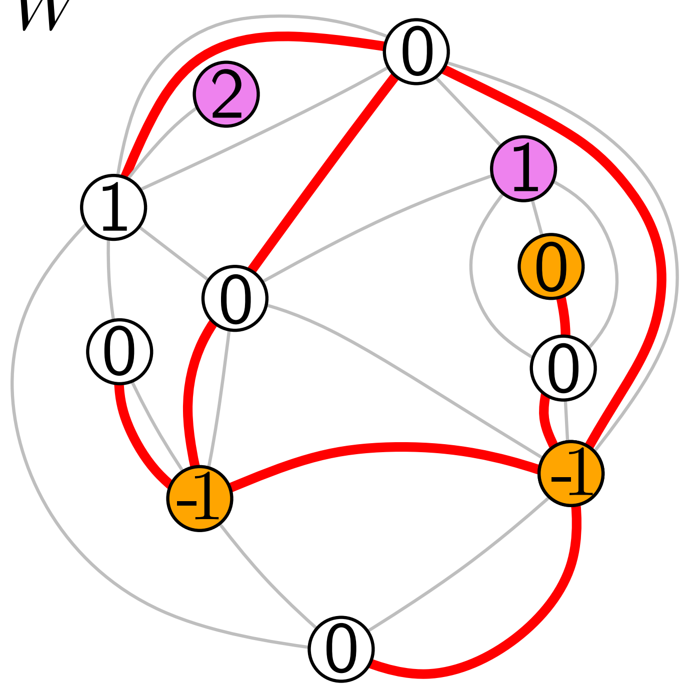
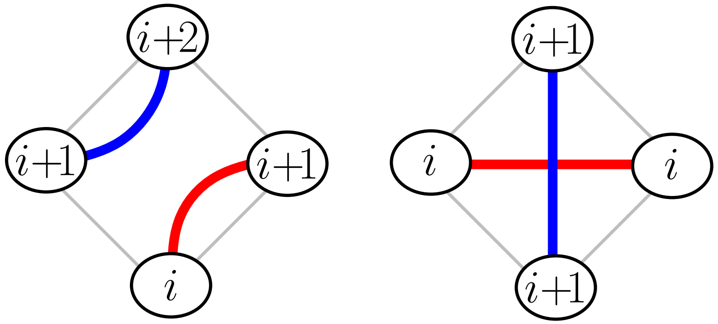
Φ

(orange i) local min of Q
face f of W
$\min(f) = i+1$
local min of W^-

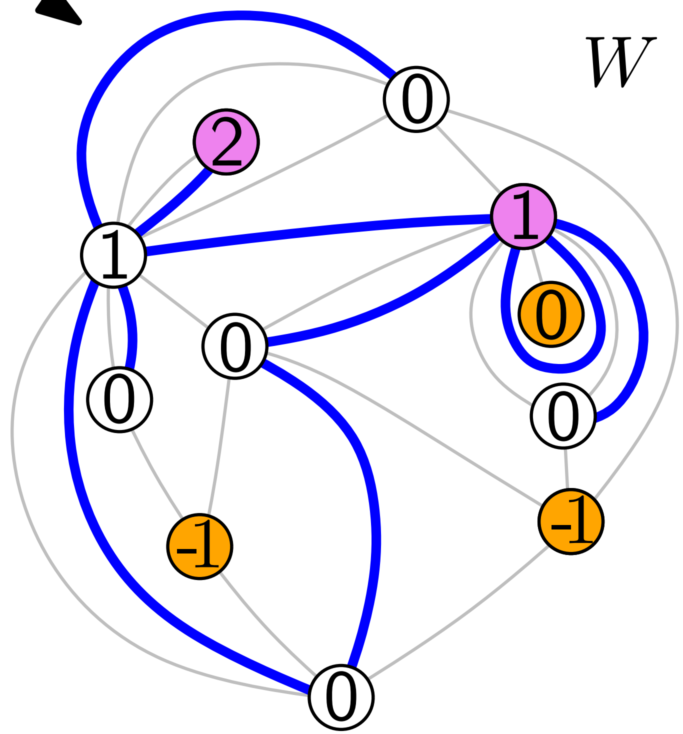
(pink i) local max of Q
local max of W
face f of W^-
$\max(f) = i-1$

W^-





W



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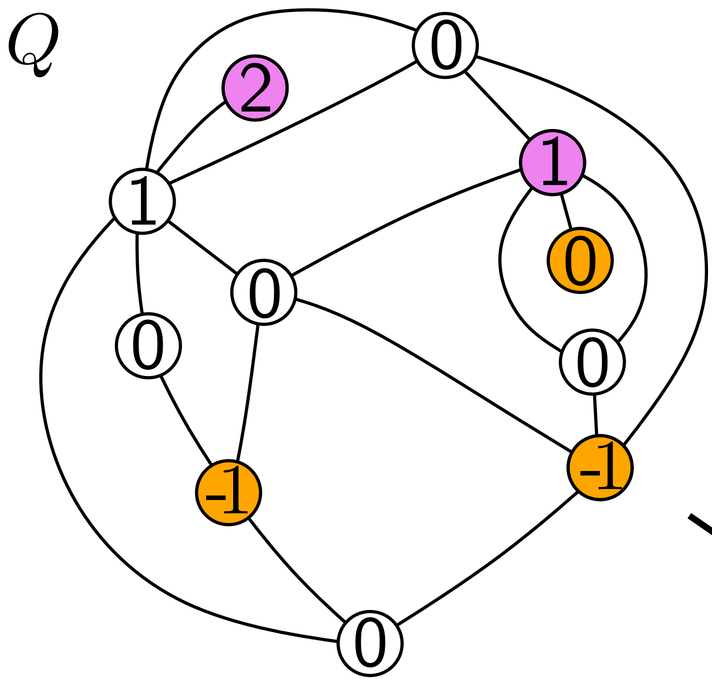

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local min of W^-

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local max of W
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Let $op : \mathbf{Z} \to \mathbf{Z}$
$\quad i \to -i$

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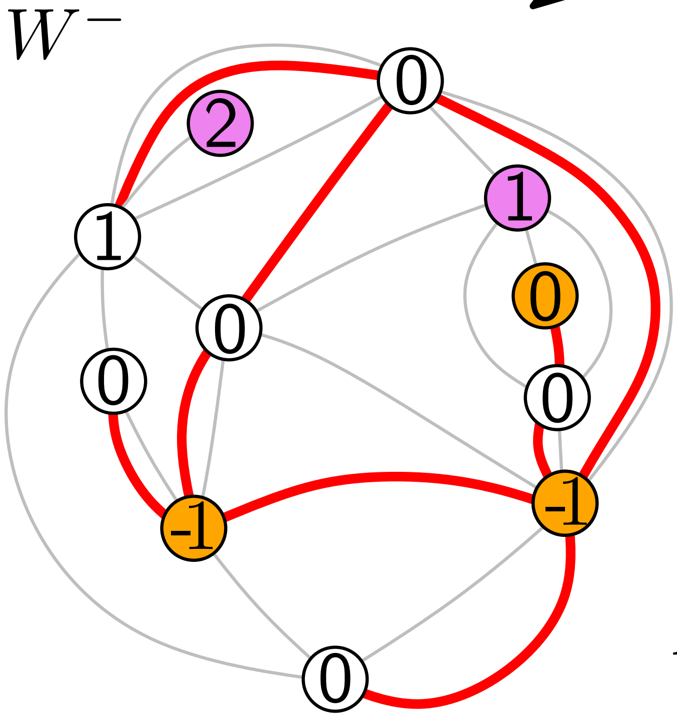
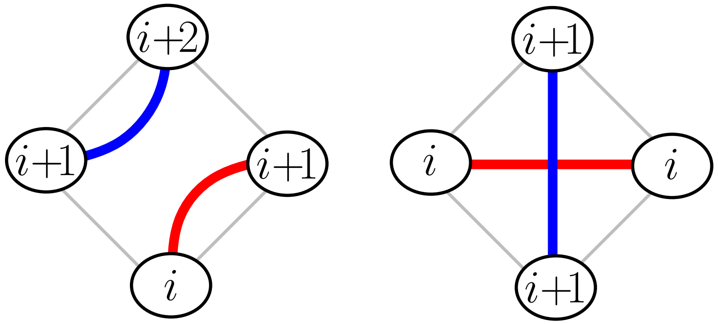
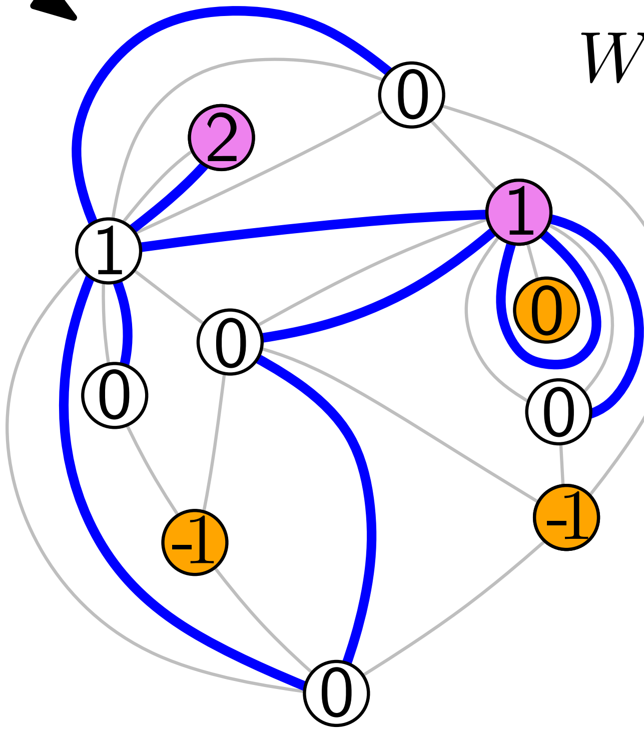
Φ





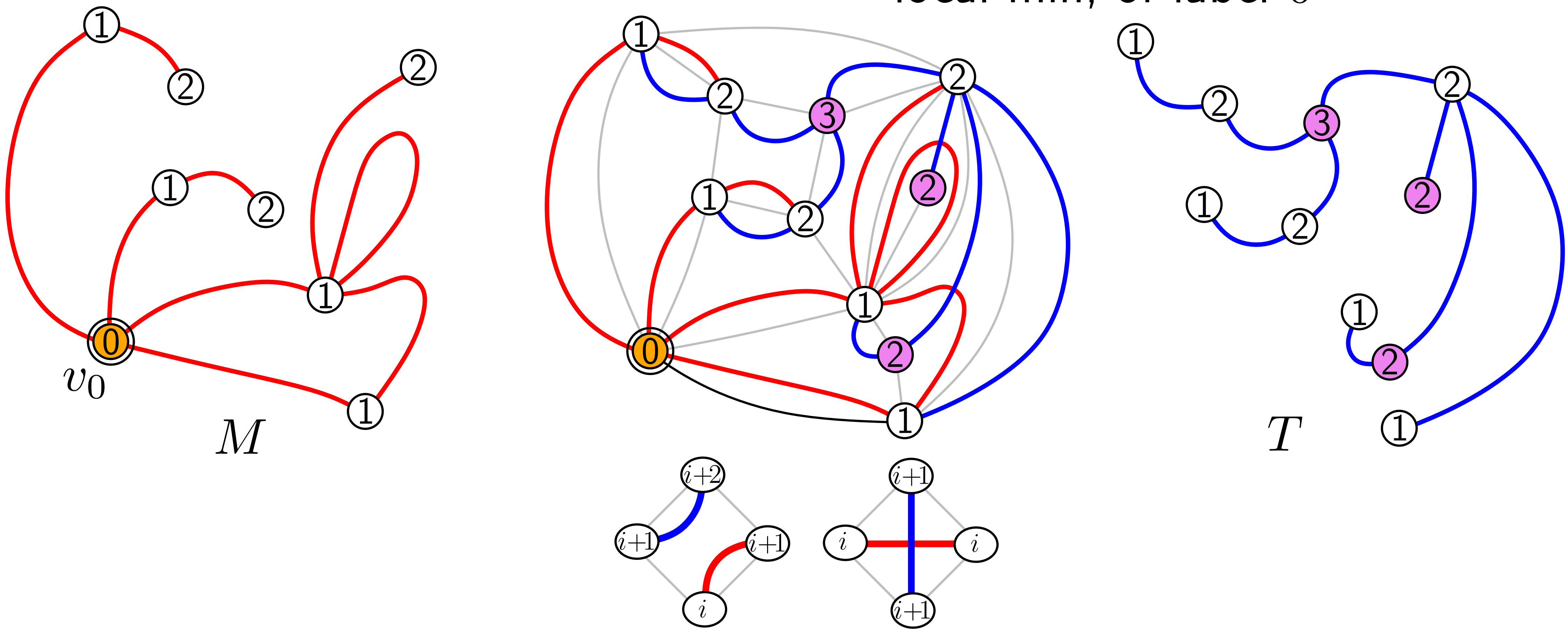


Λ

Λ is a new "duality" relation
for well-labelled map

The bijection Λ applied to pointed maps

Rk: pointed maps+geodesic labelling $\leftrightarrow$ well-labelled maps with one local min, of label 0



$\Rightarrow$ pointed maps n edges $\leftrightarrow$ well-labelled trees min-label=1 and n edges
(as for quadrang., but this time vertex of $M \neq v_0 \leftrightarrow$ non-local max of T)

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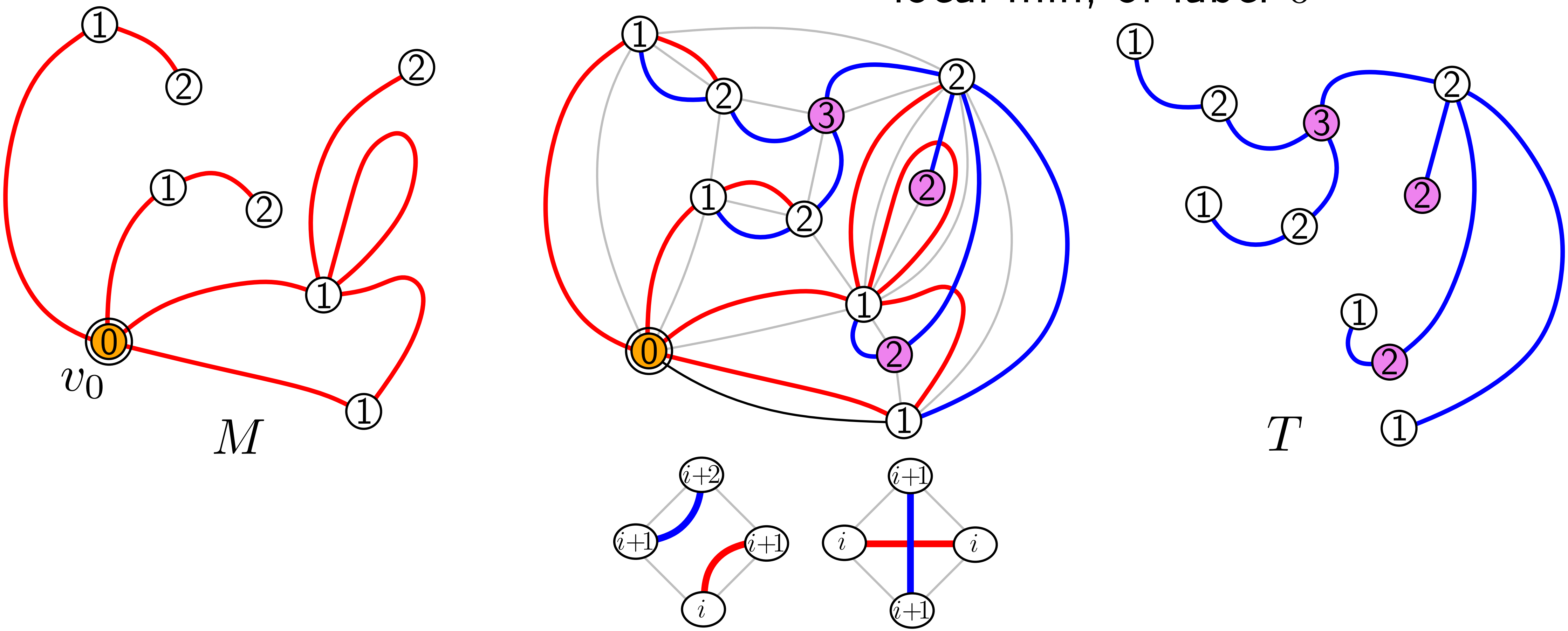

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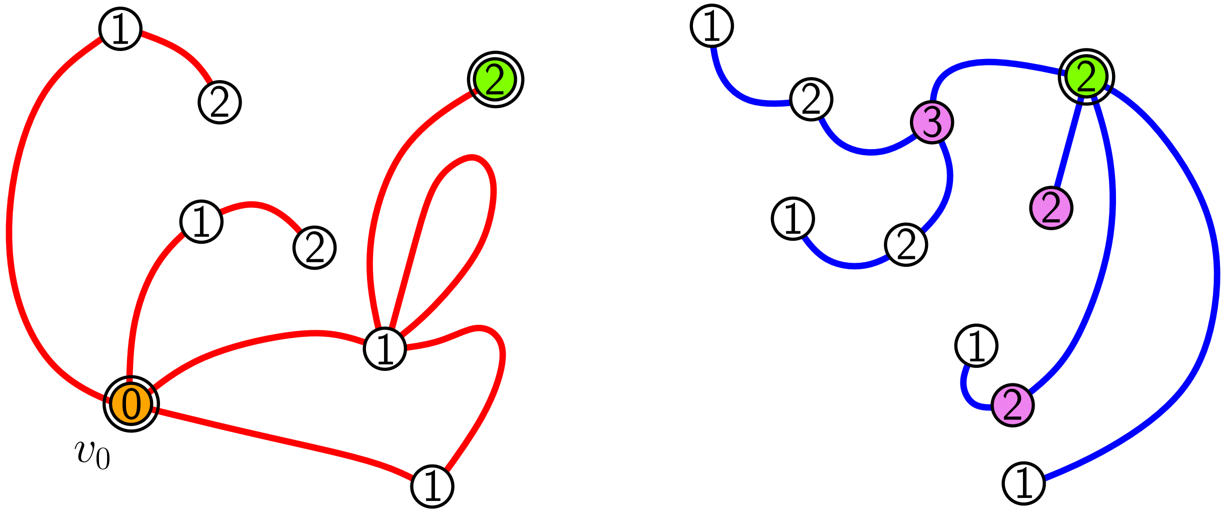
Rk: In that case, Φ^- gives a new bijection from pointed quadrangulations with n faces to pointed maps with n edges that preserves the distances to the pointed vertex (not the case with the easy local bijection)

The two-point function of general maps

Let $G_d(g)$ the 2-point function of general maps

$d = 2$



AB bijection $\Rightarrow$ $G_d(g)$ is the series of well-labelled trees with min-label 1 with a marked non local max of label d

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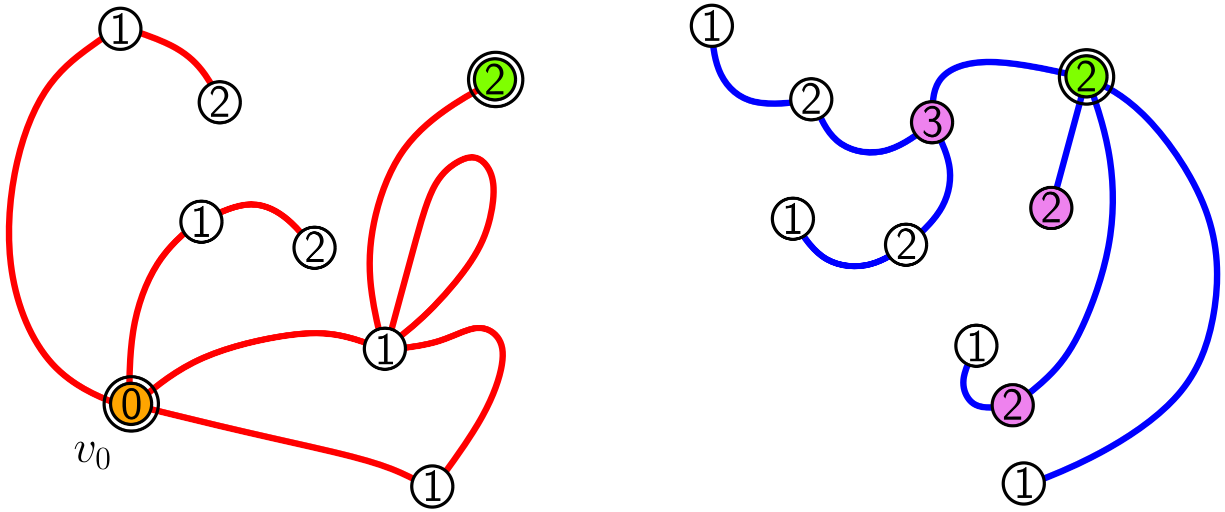

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$G_d = F_d - F_{d-1}$, with $F_d(g) :=$ the series of well-labelled trees with positive labels and a marked non local max of label d

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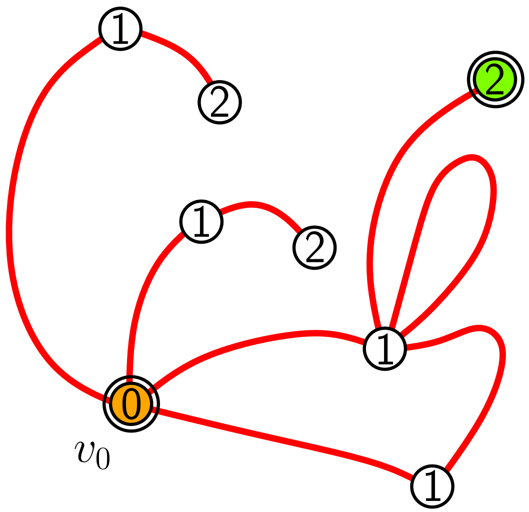

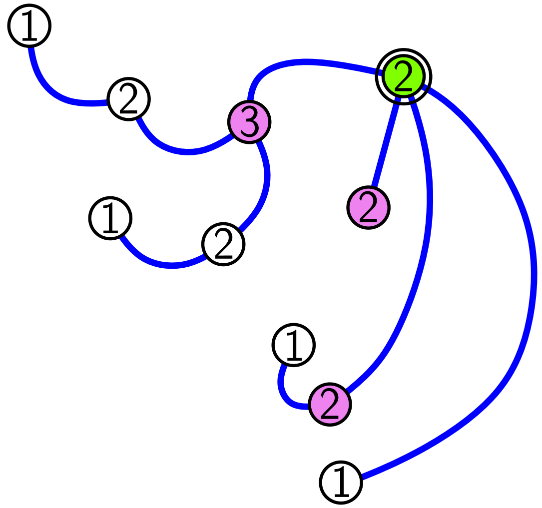

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$G_d = F_d - F_{d-1}$, with $F_d(g) :=$ the series of well-labelled trees with positive labels and a marked non local max of label d

$$F_i = \log \frac{1}{1-g(R_{i-1}+R_i+R_{i+1})} - \log \frac{1}{1-g(R_{i-1}+R_i)}$$
$$= \log(1 + gR_iR_{i+1})$$

$\Rightarrow$ $\boxed{G_d = \log\left(\frac{[d+1]_x^3[d+3]}{[d]_x[d+2]_x^3}\right)}$ for general maps

The two-point function of general maps

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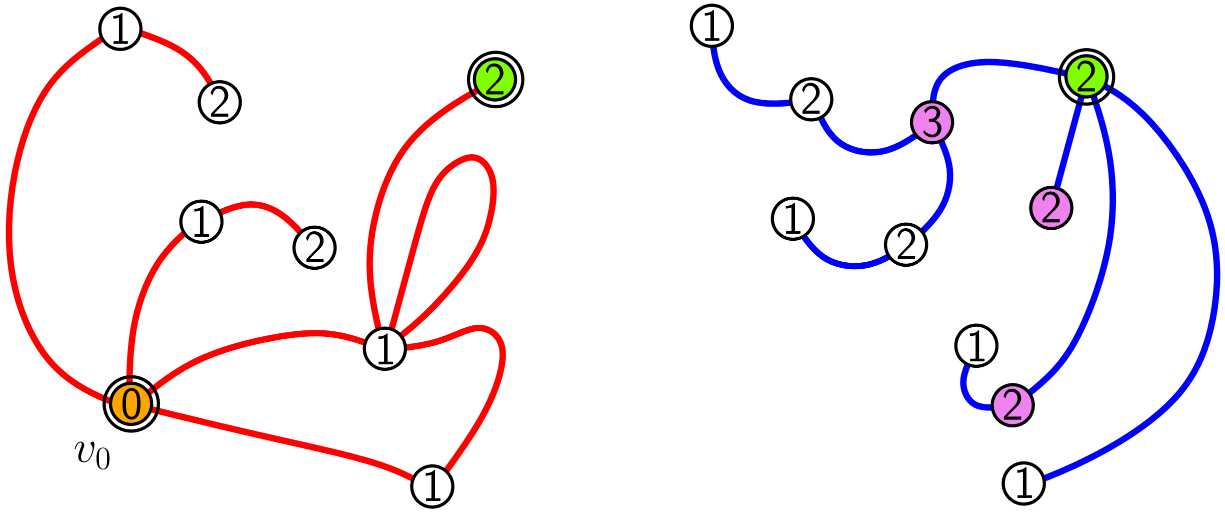

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$$\Rightarrow \boxed{G_d = \log\left(\frac{[d+1]_x^3[d+3]}{[d]_x[d+2]_x^3}\right)} \text{ for general maps}$$

recall $G_d = \log\left(\frac{[d]_x^2[d+3]_x}{[d-1]_x[d+2]_x^2}\right)$ for quadrang. (same asymptotic laws)

The case of two local min

Let M a well-labelled map with two local min v_1, v_2

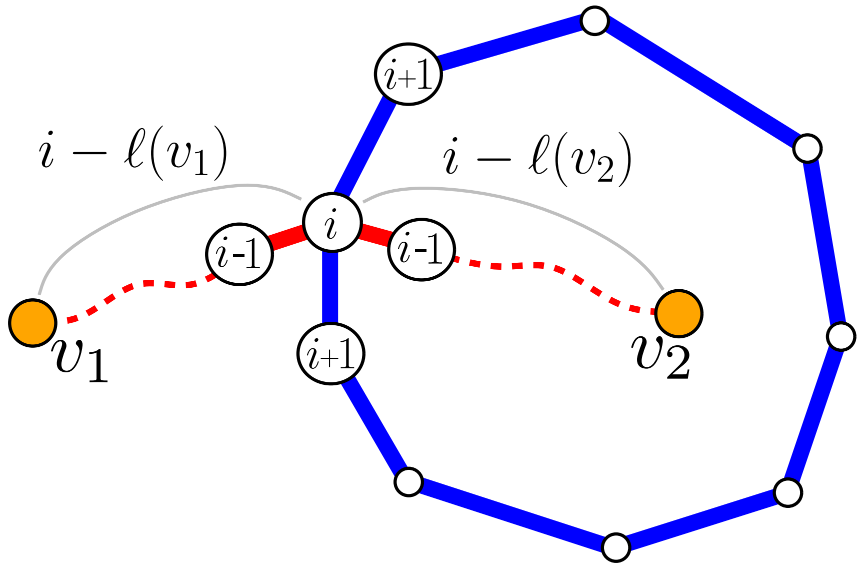
Let $M' = \Lambda(M)$, let f_1, f_2 the two faces of M'

Let Γ the (cycle) boundary of M', $i := \min_\Gamma$

Two cases:

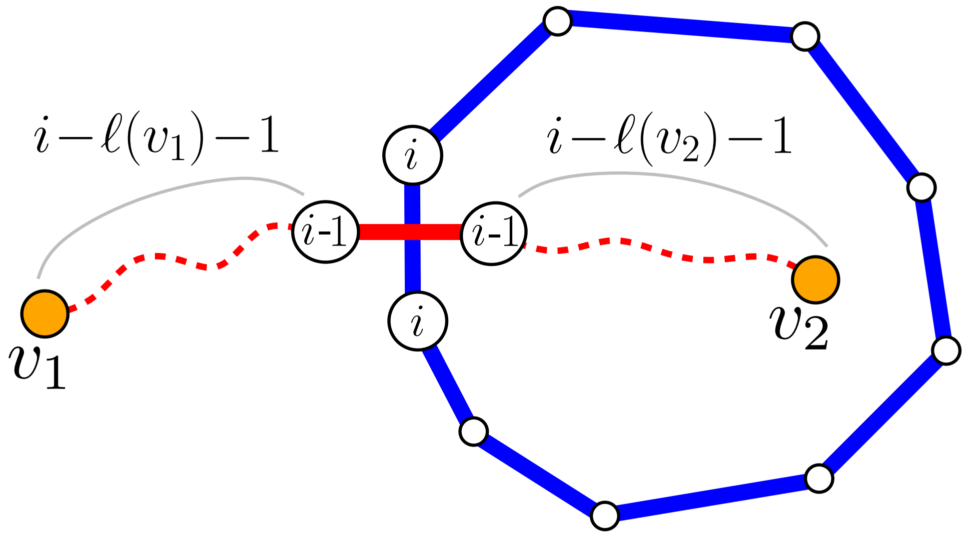
A): no edge of labels $i - i$ on Γ

$$\boxed{\mathrm{dist}_M(v_1, v_2) = 2i - \ell(v_1) - \ell(v_2)}$$



B): $\exists$ an edge of labels $i - i$ on Γ

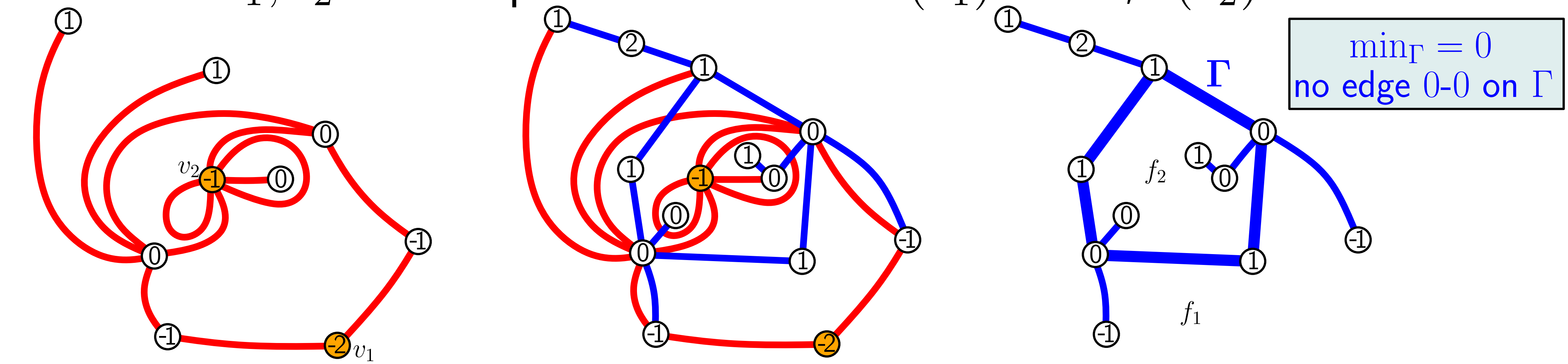
$$\boxed{\mathrm{dist}_M(v_1, v_2) = 2i - \ell(v_1) - \ell(v_2) - 1}$$



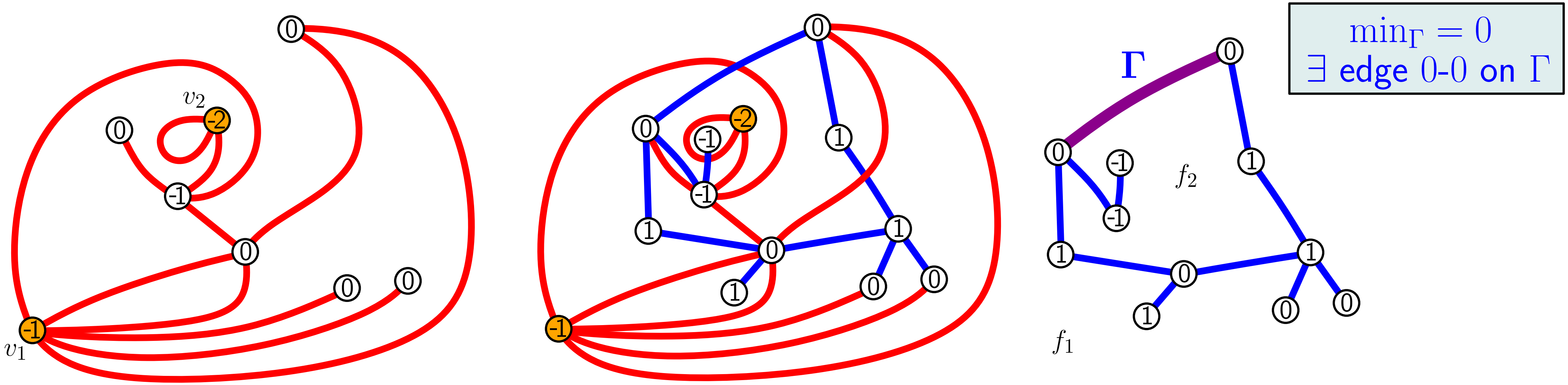
2 other ways to compute the 2-point function

[F, Guitter'14] For $d \geq 1$, let M a bi-pointed map with $d_{12} = d$

A) Write d as $s + t$ with $s, t \geq 1$. Endow M with unique well-labelling where v_1, v_2 are unique local min and $\ell(v_1) = -s$, $\ell(v_2) = -t$



B) Write d as $s + t - 1$ with $s, t \geq 1$. Endow M with unique well-labelling where v_1, v_2 are unique local min and $\ell(v_1) = -s$, $\ell(v_2) = -t$



2 other ways to compute the 2-point function

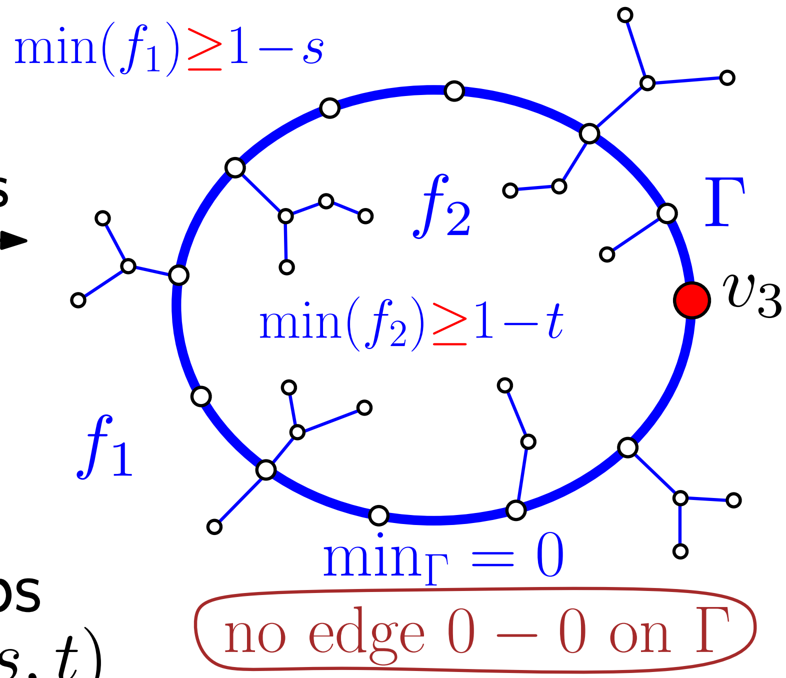
Case (A): $G_{s+t}(g) = \Delta_s \Delta_t \log(\underbrace{N_{s,t}}_{\text{counts}})$

$$X_{s,t} = \frac{N_{s,t}}{1-gR_sR_tN_{s,t}}$$

$\Rightarrow$ exact expression for $N_{s,t}$

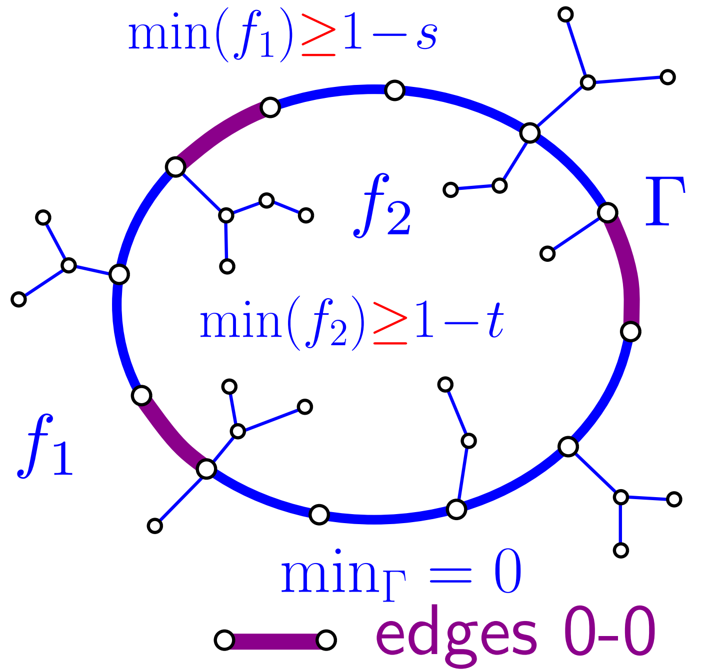
recover $G_{s+t} = \log\left(\frac{[s+t]_x^2[s+t+3]_x}{[s+t-1]_x[s+t+2]_x^2}\right)$

Rk: $\Delta_s\Delta_t N_{s,t}$ gives GF of tri-pointed maps with aligned points: $d_{12}, d_{13}, d_{23} = (s+t, s, t)$



Case (B): $G_{s+t-1}(g) = \Delta_s \Delta_t \underbrace{\log\left(\frac{1}{1-gR_sR_tN_{s,t}}\right)}_{\text{counts}}$

recover $G_{s+t-1} = \log\left(\frac{[s+t-1]_x^2[s+t+2]_x}{[s+t-2]_x[s+t+1]_x^2}\right)$



3-point function: generic (non-aligned) case

Case A: $d_{12} + d_{13} + d_{23}$ even

parametrize as: $$d_{12} = s + t \qquad \text{with } s, t, u > 0$$
$$d_{13} = s + u$$
$$d_{23} = t + u$$

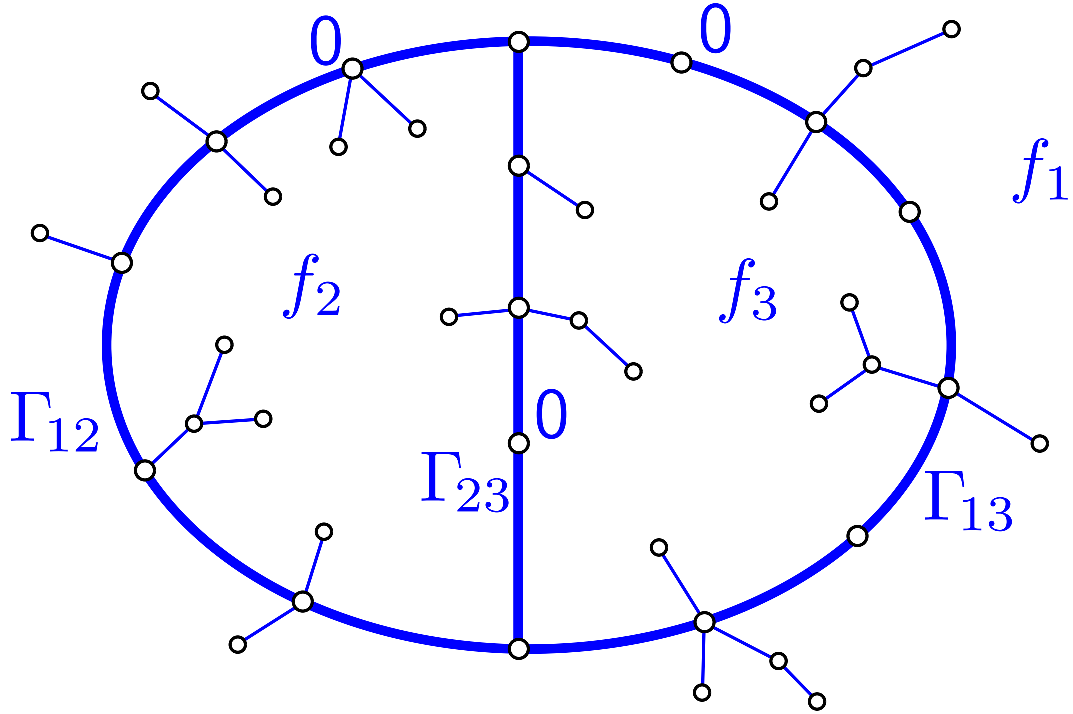
endow tri-pointed map with unique "$(-s, -t, -u)$-well-labelling"
and apply the AB bijection Λ

$\min(f_1) = 1 - s \qquad \min_{\Gamma_{12}} = 0$

$\min(f_2) = 1 - t \qquad \min_{\Gamma_{13}} = 0$

$\min(f_3) = 1 - u \qquad \min_{\Gamma_{23}} = 0$

and no edge 0-0 on Γ



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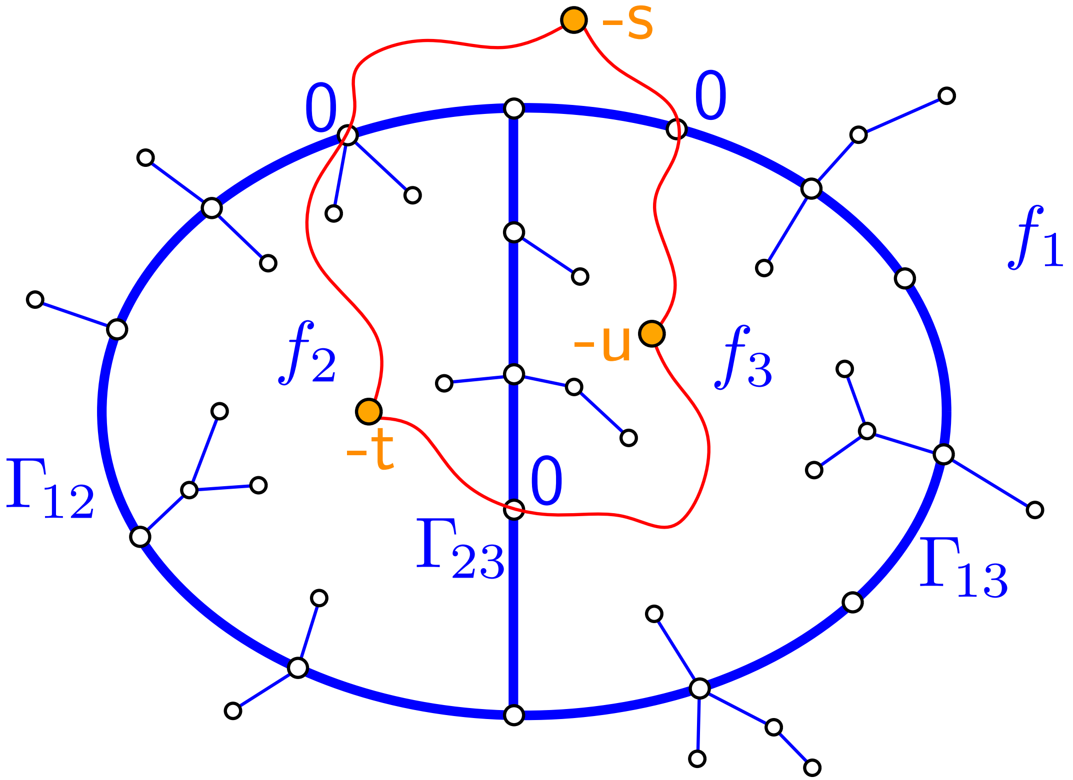

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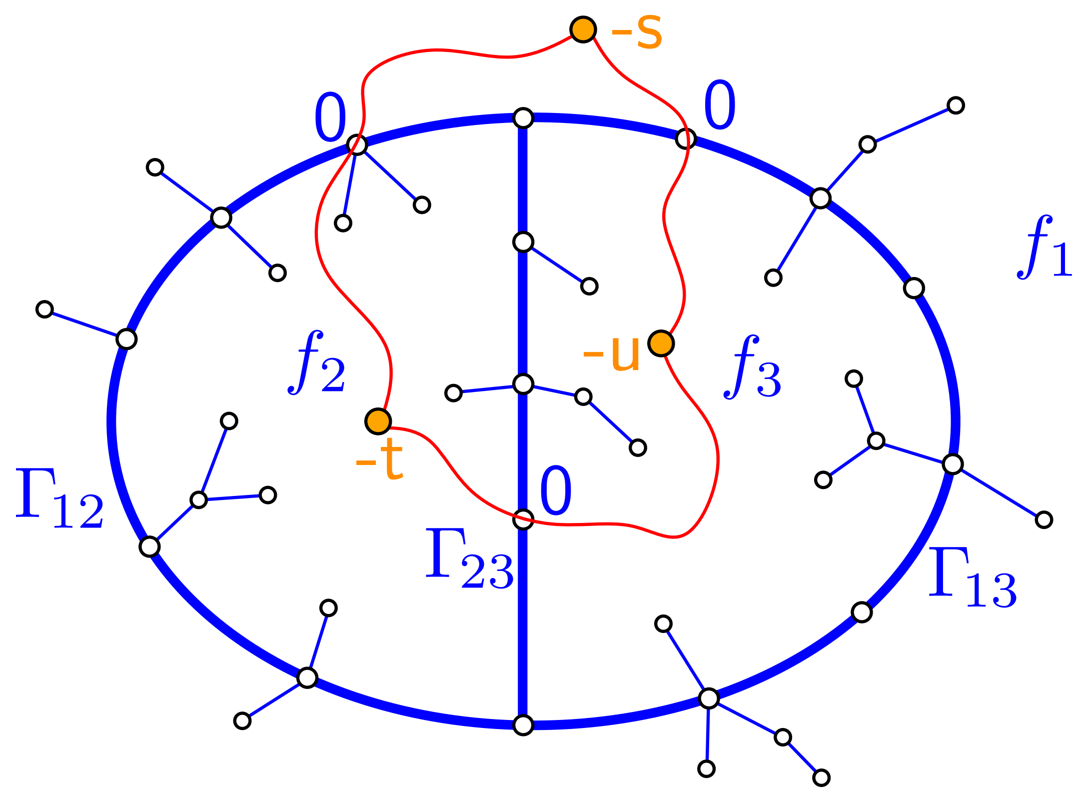


$\Rightarrow$ expression of $G_{d_{12},d_{13},d_{23}}(g)$ as $\Delta_s \Delta_t \Delta_u F^{\text{even}}_{s,t,u}$, with $F^{\text{even}}_{s,t,u}(g)$ explicit

3-point function: generic (non-aligned) case

Case B: $d_{12} + d_{13} + d_{23}$ odd (did not exist for quadrang.)

parametrize as:

$$d_{12} = s + t - 1$$
$$d_{13} = s + u - 1$$
$$d_{23} = t + u - 1$$

with $s, t, u > 0$

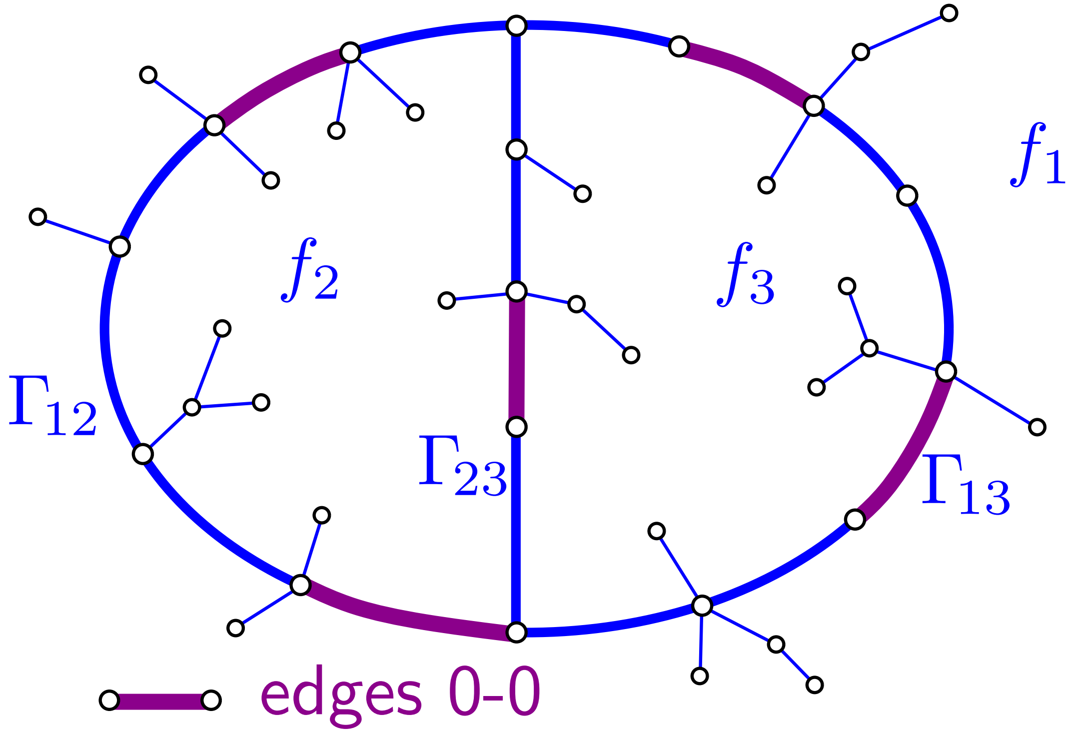
endow tri-pointed map with unique "$(-s, -t, -u)$-well-labelling"
and apply the AB bijection Λ

$\min(f_1) = 1 - s$ $\quad \min_{\Gamma_{12}} = 0$

$\min(f_2) = 1 - t$ $\quad \min_{\Gamma_{13}} = 0$

$\min(f_3) = 1 - u$ $\quad \min_{\Gamma_{23}} = 0$

and there is an edge 0-0
on each of $\Gamma_{12}, \Gamma_{13}, \Gamma_{23}$



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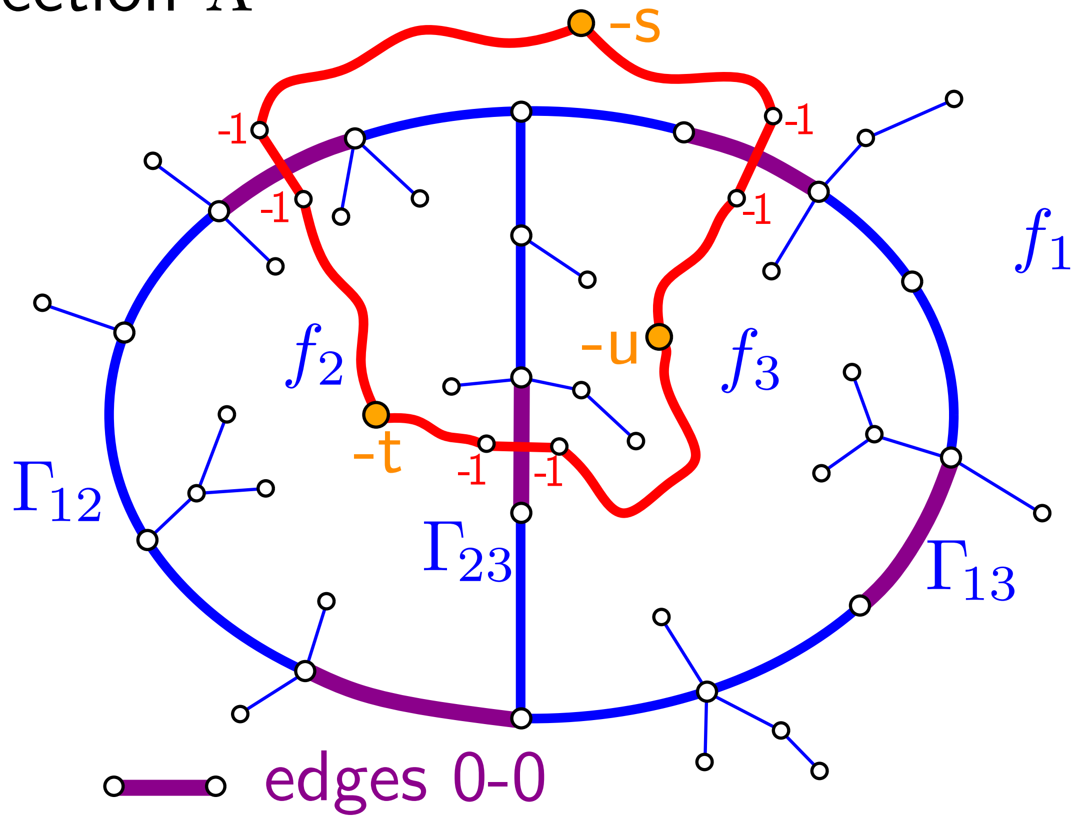

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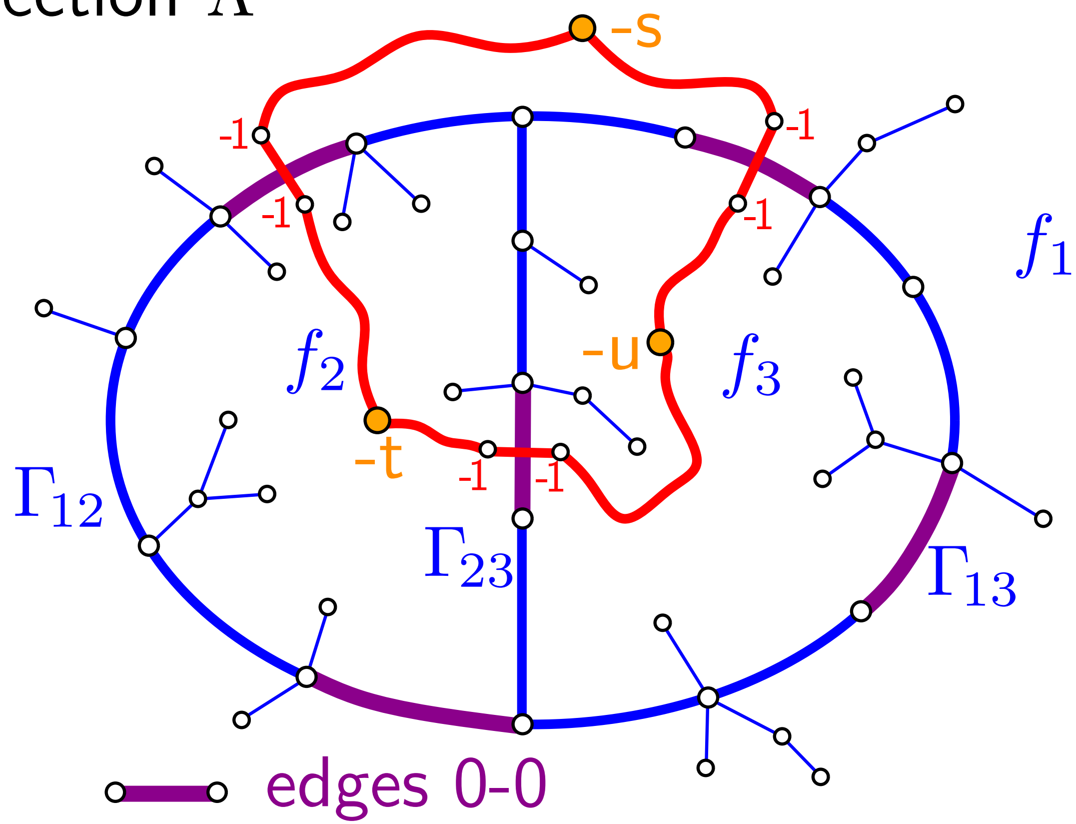

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and there is an edge 0-0
on each of $\Gamma_{12}, \Gamma_{13}, \Gamma_{23}$

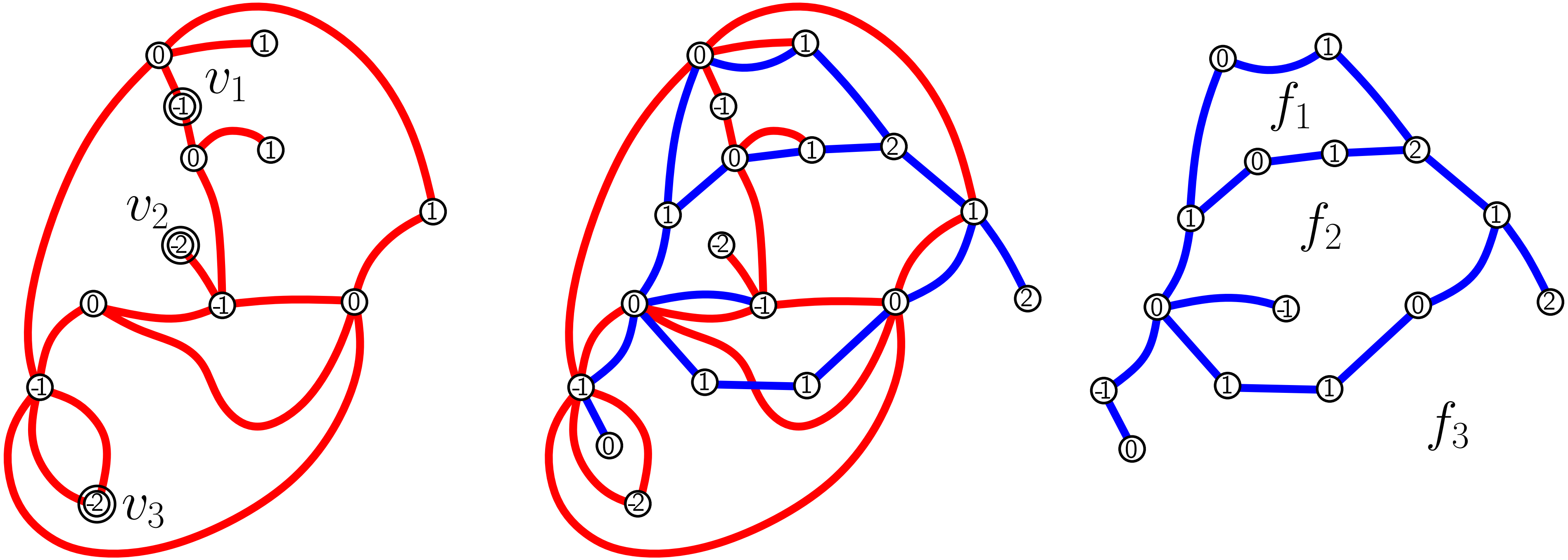
$\Rightarrow$ expression of $G_{d_{12},d_{13},d_{23}}(g)$ as $\Delta_s \Delta_t \Delta_u F^{\mathrm{odd}}_{s,t,u}$, with $F^{\mathrm{odd}}_{s,t,u}(g)$ explicit

Examples

Case A:

$d_{12} = 3$
$d_{13} = 3$
$d_{23} = 4$

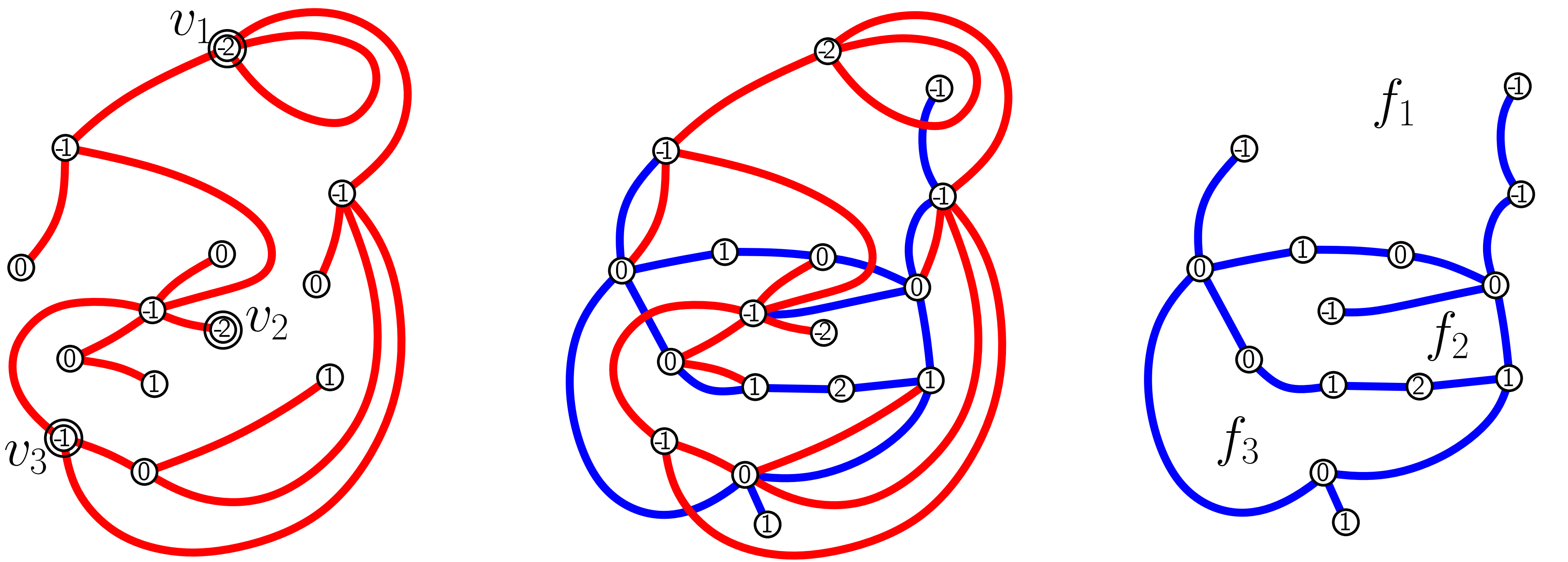
$s = 1$
$t = 2$
$u = 2$



Case B:

$d_{12} = 3$
$d_{13} = 2$
$d_{23} = 2$

$s = 2$
$t = 2$
$u = 1$



Conclusion and remarks

- There are exact expressions for the 2-point and 3-point functions of quadrangulations and general maps (bijections + GF calculations)

- Asymptotically the limit laws (rescaling by $n^{1/4}$) are the same for the random quad. Q_n of size n as for the random map M_n of size n

  Rk: also follows from **[Bettinelli, Jacob, Miermont'13]**
  $(Q_n, \mathrm{dist}/n^{1/4})$ and $(M_n, \mathrm{dist}/n^{1/4})$ are close as metric spaces, when coupling (M_n, Q_n) by the AB bijection

- We can also obtain similar expressions for bipartite maps (associated well-labelled maps are restricted to have no edge $i - i$)

- The GF expressions $G_D(g)$ for maps/bipartite maps can be extended to expressions $G_D(g, z)$ where z marks the number of faces