

# *A unified bijective method for maps: application to two classes with boundaries*

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**Abstract.** Based on a construction of the first author, we present a general bijection between certain decorated plane trees and certain orientations of planar maps with no counterclockwise circuit. Many natural classes of maps (e.g. Eulerian maps, simple triangulations,...) are in bijection with a subset of these orientations, and our construction restricts in a simple way on the subset. This gives a general bijective strategy for classes of maps. As a non-trivial application of our method we give the first bijective proofs for counting (rooted) simple triangulations and quadrangulations with a boundary of arbitrary size, recovering enumeration results found by Brown using Tutte’s recursive method.

**Résumé.** En nous appuyant sur une construction du premier auteur, nous donnons une bijection générale entre certains arbres décorés et certaines orientations de cartes planaires sans cycle direct. De nombreuses classes de cartes (par exemple les eulériennes, les triangulations) sont en bijection avec un sous-ensemble de ces orientations, et notre construction se spécialise de manière simple sur le sous-ensemble. Cela donne un cadre bijectif général pour traiter les familles de cartes. Comme application non-triviale de notre méthode nous donnons les premières preuves bijectives pour l’énumération des triangulations et quadrangulations simples (enracinées) ayant un bord de taille arbitraire, et retrouvons ainsi des formules de comptage trouvées par Brown en utilisant la méthode récursive de Tutte.

**Keywords:** Triangulation, quadrangulation, maps with boundaries, mobiles, bijection, counting

## 1 Introduction

The enumeration of planar maps (connected graphs embedded on the sphere) has received a lot of attention since the seminal work of Tutte in the 60’s (Tut63). Tutte’s recursive method consists in translating the decomposition of a class of maps (typically obtained by deleting an edge) into a functional equation satisfied by the corresponding generating function. The translation usually requires an additional “catalytic” variable, and the obtained functional equation is solved using the so-called “quadratic method” (GJ83, sec.2.9) or its extensions (BMJ06). The final result is, for many classes of maps, a strikingly simple counting formula. For instance, the number of (rooted) maps with  $n$  edges is  $\frac{2 \cdot 3^n}{(n+1)(n+2)} \binom{2n}{n}$ . Tutte’s method has the advantage of being systematic, but is quite technical in the way of solving the equation and does not give a combinatorial understanding of the simple-looking enumerative formulas.

<sup>†</sup>Supported by the French ANR project A3 and the European project ExploreMaps

As an alternative method, bijective constructions have been developed to obtain more direct combinatorial proofs of the counting formulas, with nice algorithmic applications (random generation and asymptotically optimal encoding in linear time). The first bijections appeared in (CV81) and later in (Sch98) where direct bijections for several classes of maps are described. Typically bijections are from a class of “decorated” plane trees to a class of maps and operate on trees by progressively closing facial cycles. More general constructions have been given subsequently, with application to statistical physics models (Ising, hard particles) on random lattices (BMS02; BFG07) and to the investigation of metric properties of random maps (CS04; BFG04). Even if it has been successfully applied to many classes, e.g. in (PS06; PS03; FPS08; Fus09), the bijective method for maps is up to now not as systematic as Tutte’s recursive method, since for each class of maps one has to “guess” the tree family to match with, and one has to specify the construction from trees to maps.

This article contributes to fill this gap. Based on a construction of the first author (Ber07; BC), we provide in Section 3 a general bijection  $\Phi$  between a set  $\mathcal{D}$  of certain decorated plane trees which we call *mobile*<sup>(i)</sup> and a set  $\mathcal{O}$  of certain orientations on planar maps with no counterclockwise circuit. As it turns out, a map class is often in bijection with a subfamily  $\mathcal{S}$  of  $\mathcal{O}$  on which our construction restricts nicely; typically the orientations in  $\mathcal{S}$  are characterized by degree constraints which can be traced through our construction and yields a degree characterization of the associated mobiles. The mobiles family is then specifiable by a decomposition grammar and amenable to the Lagrange inversion formula for counting. To summarize, our method makes the bijective method more systematic, since it consists in specializing a “master bijection”  $\Phi$  to the class of maps under consideration. The problem of enumerating a class of map  $\mathcal{M}$  therefore reduces to guessing a family of “canonical” orientations (in  $\mathcal{O}$ ) for  $\mathcal{M}$  instead of guessing a family of trees to match with  $\mathcal{M}$  (the first task being often simpler than the second).

We focus here, in Section 4 and Section 5 respectively, on two classes that were not completely covered before, namely simple triangulations and simple quadrangulations with a polygonal boundary and a root-corner incident to the boundary. We show bijectively that the number  $t_{n,k}$  of rooted simple triangulations with  $n+k$  vertices and boundary of length  $k$  and the number  $q_{n,k}$  of rooted simple quadrangulations with  $n+2k$  vertices and boundary of length  $2k$  satisfy

$$t_n^{(k)} = \frac{2(2k-3)!}{(k-1)!(k-3)!} \frac{(4n+2k-5)!}{n!(3n+2k-3)!}, \quad q_n^{(k)} = \frac{3(3k-2)!}{(k-2)!(2k-1)!} \frac{(3n+3k-4)!}{n!(2n+3k-2)!},$$

recovering results found by Brown respectively in (Bro64) and (Bro65) using Tutte’s recursive method. The case without boundaries ( $k=3$  for triangulations,  $k=2$  for quadrangulations) have already received bijective proofs in (PS06; Sch98) (for triangulations) and (FPS08) (for triangulations) and in (Sch98) respectively; our construction actually coincides with (Sch98) for triangulations and with (FPS08) for quadrangulations. The case of triangulations with boundaries has also received a partial bijective interpretation, different from ours, in (PS06) (only one direction is given, from trees to maps, which by injection shows that  $t_{n,k}$  is *at least* the number above, but does not suffice to prove equality). Our approach leads to unified and reasonably short proofs for both triangulations and quadrangulations with boundaries.

## 2 Maps and orientations

**Maps.** A (planar) *map* is a connected planar graph embedded in the oriented sphere and considered up to continuous deformation. A map is *simple* if it has no loop nor multiple edge. The *faces* are the connected

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<sup>(i)</sup> The term *mobile* is borrowed from a famous bijection by Bouttier et al. (BFG04) which can be seen as a specialization of  $\Phi$ .

components of the complementary of the graph. A *plane tree* is a map with a unique face. Cutting an edge  $e$  at its middle point gives two half-edges, each incident to an endpoint of  $e$  (they are both incident to the same vertex if  $e$  is a loop). We shall also consider some maps decorated with dangling half-edges called *stems* (see e.g. Figure 2(a)). A *corner* is the angular section between two consecutive half-edges around a vertex. The degree of a vertex or face is the number of incident corners. A map is a *triangulation* (resp. *quadrangulation*) if every face has degree 3 (resp. 4).

A map is said to be *vertex-rooted* if a vertex is marked, *face-rooted* if a face is marked, and *corner-rooted* if a corner is marked<sup>(ii)</sup>. The marked vertex, face or corner are called the *root-vertex*, *root-face* or *root-corner*. For a corner-rooted map, the marked corner is indicated by a dangling half-edge pointing to that corner; see Figure 1. A corner-rooted map is said to *induce* the vertex-rooted map (resp. face-rooted map) obtained by keeping the root-vertex (resp. root-face) as marked, but otherwise forgetting the root-corner. Given a face-rooted (or corner-rooted) map, vertices and edges are said to be *outer* or *inner* depending on whether they are incident to the root-face or not.

**Orientations.** An *orientation*  $O$  of a map  $M$  is the choice of a direction for each edge of  $M$ . A *circuit* is a directed cycle. A *source* is a vertex incident to no ingoing edge. If  $M$  is face-rooted (resp. vertex-rooted, corner-rooted), then the pair  $(M, O)$  is called a *vertex-rooted orientation* (resp. *face-rooted orientation*, *corner-rooted orientation*). A corner-rooted orientation naturally *induces* a face-rooted orientation and a vertex-rooted orientation. For a vertex  $v$  of  $M$ , the indegree  $\text{in}(v)$  is the number of edges going into  $v$ ; the outdegree  $\text{out}(v)$  is the number of edges going out of  $v$ . For a face  $f \in M$ , the clockwise-degree  $\text{cw}(f)$  is the number of edges incident to  $f$  that have  $f$  on their right; the counterclockwise-degree  $\text{ccw}(f)$  is the number of edges that have  $f$  on their left. For corner-rooted maps, the half-edge indicating the root-corner increases by 1 the indegree of the root-vertex and the clockwise-degree of the root-face.

A vertex-rooted orientation is said to be *accessible* if every vertex is accessible from the root-vertex by a directed path; it is *source-accessible* if in addition the root-vertex is a source. A circuit of a face-rooted (or corner-rooted) orientation is said *clockwise* if the root-face is on its left. The orientation is *minimal* if every circuit is clockwise; it is *clockwise-minimal* if in addition the root-face is a (clockwise) circuit. We extend the definition of accessibility to (face-rooted) clockwise-minimal orientations  $O$  by calling  $O$  *accessible* if it is accessible from one of the vertices incident to the root-face. Observe that  $O$  is in fact accessible from any vertex on the root-face in this case (hence  $O$  is induced only by minimal accessible corner-rooted orientations). Similarly, we call a source-accessible orientation  $O$  *minimal* if  $O$  is minimal for one of the faces incident to the root-vertex. Observe that  $O$  is in fact minimal for every face incident to the root-vertex in this case (hence  $O$  is induced only by minimal accessible corner-rooted orientations).

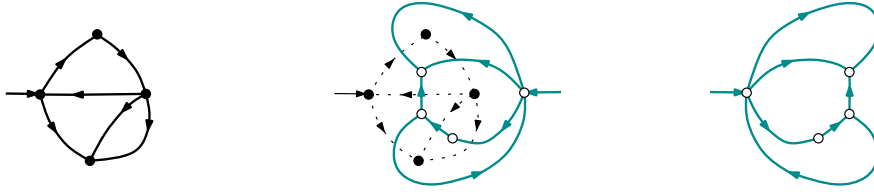
Let  $\delta$  be a positive integer. We denote by  $\mathcal{S}_\delta$  the set of source-accessible minimal orientations such that the root-vertex has degree  $\delta$ . We denote by  $\mathcal{O}_\delta$  the set of clockwise-minimal accessible orientations such that the root-face has degree  $\delta$ . We denote by  $\tilde{\mathcal{S}}_\delta$  the subset of  $\mathcal{S}_\delta$  such that every face incident to the root-vertex has clockwise degree 1. We denote by  $\tilde{\mathcal{O}}_\delta$  the subset of  $\mathcal{O}_\delta$  such that every vertex incident to the root-face has indegree 1.

Given a map  $M$  and a function with vertex-set  $V$  and a function  $\alpha : V \mapsto \mathbb{N}$ , an  $\alpha$ -*orientation* is an orientation of  $M$  such that  $\text{in}(v) = \alpha(v)$  for each  $v \in V$ . The following result is classical (see (Fe104)).

**Lemma 1** *If a face-rooted map  $M$  has an  $\alpha$ -orientation, then  $M$  has a unique minimal  $\alpha$ -orientation.*

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<sup>(ii)</sup> Corner-rooted maps are usually simply called *rooted maps* in the literature. A face-rooted map can be thought as a *plane map* (a connected graph embedded in the plane) by thinking of the root-face as the infinite face.



**Fig. 1:** The dual of an oriented map.

**Duality.** The *dual*  $M^*$  of a map  $M$  is the map obtained by the following two step process; see Figure 1.

1. In each face  $f$  of  $M$ , draw a vertex  $v_f$  of  $M^*$ . For each edge  $e$  of  $M$  separating faces  $f$  and  $f'$  (which can be equal), draw the *dual edge*  $e^*$  of  $M^*$  going from  $v_f$  to  $v_{f'}$  across  $e$ .
2. Flip the drawing of  $M^*$ , that is, inverse the orientation of the sphere.

The dual of a face-rooted map is a vertex-rooted map. Corners of a map and of its dual are in natural correspondence (they face each other); this gives the way of defining the root-corner of the dual of a corner-rooted map; see Figure 1. Duality is involutive on maps and rooted maps.

The *dual* of an orientation of  $M$  is the orientation of the dual map  $M^*$  obtained by applying the following rule at step 1: the dual-edge  $e^*$  of an edge  $e \in M$  is oriented from the left of  $e$  to the right of  $e$ . Observe that the duality is involutive for oriented map (this is the motivation for step 2 in the definition of duality). The clockwise degree (resp. counterclockwise degree) of a face  $f$  of  $M$  is equal to the indegree (resp. outdegree) of the vertex  $v_f$  of  $M^*$  (this is true also with the special convention applying to corner-rooted maps). The following result is classical.

**Lemma 2** *A face-rooted orientation is minimal (resp. clockwise-minimal) if and only if the dual vertex-rooted orientation is accessible (resp. source-accessible).*

Observe that duality maps the set of orientations  $\mathcal{S}_\delta$  (resp.  $\tilde{\mathcal{S}}_\delta$ ) to the set  $\mathcal{O}_\delta$  (resp.  $\tilde{\mathcal{O}}_\delta$ ). Also, minimal accessible orientations (of corner-rooted maps) are self dual. We mention that these orientations, which play an important role below, are in bijection with spanning trees (Ber07).

### 3 Bijections between mobiles and orientations

In this section, we first recall a bijection  $\Phi$  originally due to the first author (Ber07). We then present some extensions of  $\Phi$  which will be convenient for our subsequent goals. Indeed, in the next two sections we will show how to use these extensions in order to count several families of maps.

The bijection  $\Phi$  maps minimal accessible (corner-rooted) orientations with  $n$  edges and pairs of corner-rooted plane trees  $(B, T)$  with  $n+1$  and  $n$  edges respectively. The tree  $B$  is called the (rooted) *mobile* and its vertices are *bicolored* in black and white (in such a way that edges always connect a black and a white vertex). Informally, the bijection  $\Phi$  consists in folding the tree  $T$  (oriented from the root to leaves) around the mobile. More precisely, one glue the vertices of  $T$  on the black corners of the mobile and then erase the edges and white vertices of  $B$  (leaving the edges of  $T$  as edges of a minimal accessible orientation). In what follows we adopt a slightly different presentation, in which the tree  $T$  only appears implicitly in certain *decorations* added to the mobile  $B$ .

We call *decorated mobile* a bicolored (unrooted) plane tree with *outgoing stems* (dangling outgoing half-edges) possibly attached at each black corner; see Figure 2(a). The *excess* of a decorated mobile is the number of edges minus the number of (outgoing) stems. A mobile with excess  $\delta$  is called a  $\delta$ -*mobile*.

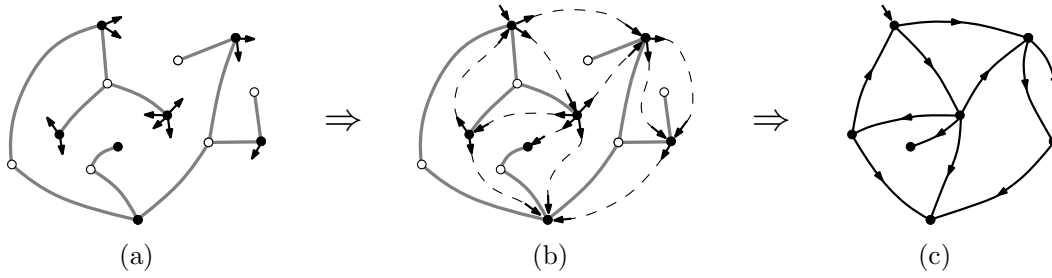


Fig. 2: The rooted closure of a mobile of excess  $\delta = 1$ .

A *fully decorated* mobile is obtained from a decorated mobile by inserting an *incoming stem* (dangling incoming half-edge) in each black corner following an edge of the mobile (and not a stem) in clockwise order around the vertex; the fully decorated mobile is represented in solid lines in Figure 2(b). The *degree*  $\text{deg}(v)$  of a vertex  $v$  of a decorated mobile is the total number of incident half-edges (including the outgoing stems). Moreover, for a black vertex  $b$  the indegree  $\text{in}(b)$  and out-degree  $\text{out}(b)$  are respectively the number of incident ingoing and outgoing stems incident to  $b$  in the fully-decorated mobile (so that  $\text{deg}(b) = \text{in}(b) + \text{out}(b)$ ).

### 3.1 Bijection between 1-mobiles and minimal accessible orientations

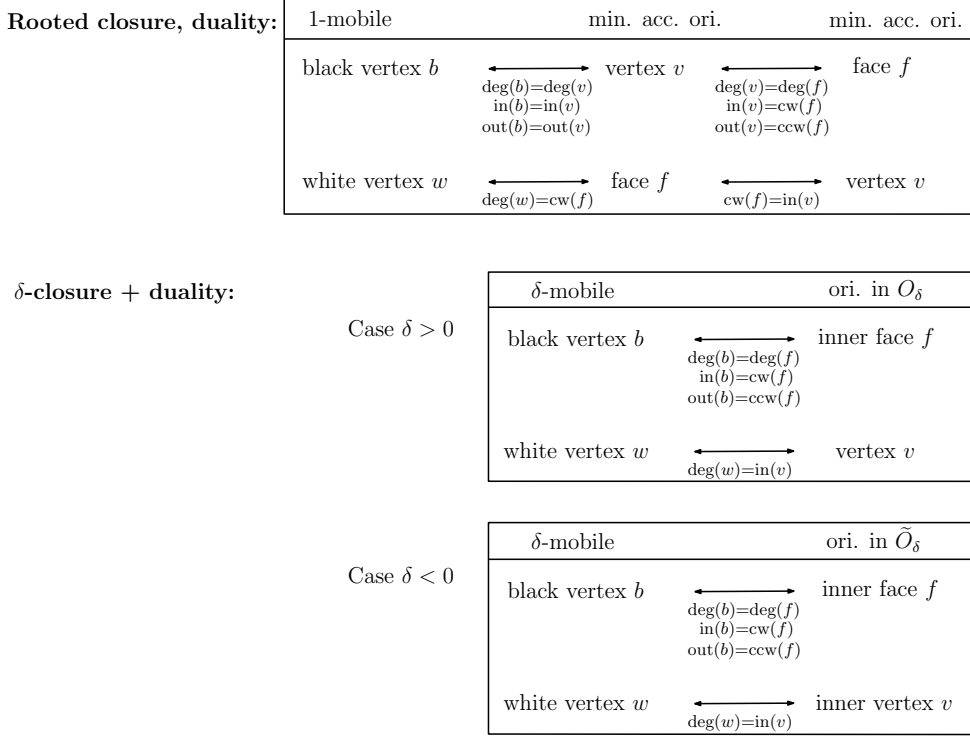
We now recall the bijection given in (Ber07) between 1-mobiles and minimal accessible orientations.

**Closure.** Let  $D$  be a decorated mobile with  $p$  edges and  $q$  outgoing stems (hence excess  $\delta = p - q$ ). The corresponding fully decorated mobile  $D'$  has  $p$  ingoing and  $q$  outgoing stems. A clockwise walk around  $D'$  (with the face area on the left of the walker) sees a succession of outgoing stems and ingoing stems. Associating an opening parenthesis to outgoing stems and a closing parenthesis to ingoing stems, one obtains thus a cyclic binary word with  $q$  opening and  $p$  closing parentheses. This yields in turn a matching of outgoing stems with ingoing stems, leaving  $|\delta|$  stems unmatched, which are ingoing if  $\delta > 0$  and outgoing if  $\delta < 0$ ; see Figure 2. The *partial closure*  $C$  of the decorated mobile  $D$  is obtained by forming a directed edge out of each matched pair, see Figure 2(a)-(b). We consider  $C$  as a planar map with two types of edges (those of the mobile, which are undirected, and the new formed edges, which are directed) and  $|\delta|$  stems. Note that, if  $\delta \geq 0$ , there are  $\delta$  white corners incident to the root-face of  $C$ , because initially the number of such corners is equal to the number of edges, and then each matched pair of stems decreases this number by 1. These corners, which stay incident to the root-face throughout the partial closure, are called *exposed white corners*.

The *rooted-closure* of the decorated mobile  $D$  is obtained from the partial closure  $C$  by erasing every white vertex and edge of the mobile (this might result in a disconnected embedded graph in general).

**Opening.** Let  $M$  be an oriented map (rooted or not) with vertex set  $V$  and face set  $F$ . The *partial opening* of  $M$  is the map  $C$  with two types of vertices (black vertices in  $V$  and white vertices in  $W = \{w_f, f \in F\}$ ) and two types of edges (directed and undirected) obtained as follows.

- Insert a white vertex  $w_f$  inside each face  $f$  of  $M$ .
- Draw an undirected edge between  $w_f$  and each corner incident to  $f$  which precedes an ingoing half-edge in clockwise order around its incident vertex. If  $M$  is corner-rooted, then the stem indicating the root-corner is interpreted as an ingoing half-edge and gives rise to an edge of  $C$ .



**Fig. 3:** The closure-bijections, with the parameter correspondences.

If  $M$  is a corner-rooted orientation, the *rooted-opening* of  $M$  is obtained from the partial opening  $C$  by erasing all the ingoing half-edges of  $M$ , thereby creating an undirected embedded bicolored graph with some outgoing stems incident to black corners.

We recall the result from (Ber07) (see also (BC)) that we shall generalize.

**Theorem 3** *The rooted closure is a bijection between decorated mobiles of excess  $\delta = 1$  and (corner-rooted) minimal accessible orientations. The rooted opening is the inverse mapping. Lastly, the parameter-correspondence is shown in Figure 3, top-part.*

### 3.2 Bijection for $\delta$ -mobiles

**$\delta$ -closure.** We now define the  $\delta$ -closure of a  $\delta$ -mobile (the definition depends on the sign of  $\delta$ ). Let  $D$  be a  $\delta$ -mobile and let  $C$  be the partial closure of  $D$ . The  $\delta$ -closure  $M$  of  $D$  is defined as follows.

- If  $\delta > 0$ , then  $C$  has  $\delta$  ingoing stems (incident to the root-face). The vertex-rooted orientation  $M$  is obtained from  $C$  by first creating a root-vertex  $v$  of  $M$  in the root-face of  $C$  and connecting it to each ingoing stem (stems thus become part of an edge of  $M$  directed away from  $v$ ); second erasing the edges and white vertices of the mobile.
- If  $\delta < 0$ , then  $C$  has  $\delta$  outgoing stems (incident to the root-face). The vertex-rooted orientation  $M$  is obtained from  $C$  by first creating a root-vertex  $v$  of  $M$  in the root-face of  $C$  and connecting it

to each outgoing stem and then reorienting these edges (stems thus become part of an edge of  $M$  directed away from  $v$ ); second erasing the edges and white vertices of the mobile.

- If  $\delta = 0$ , then  $M$  is the face-rooted orientation obtained from  $C$  by erasing the edges and white vertices of the mobile.

Actually, it is not obvious from our definitions that the  $\delta$ -closures give *connected* orientations but we prove this and more below.

**Theorem 4** *Let  $\delta$  be in  $\mathbb{Z}$ .*

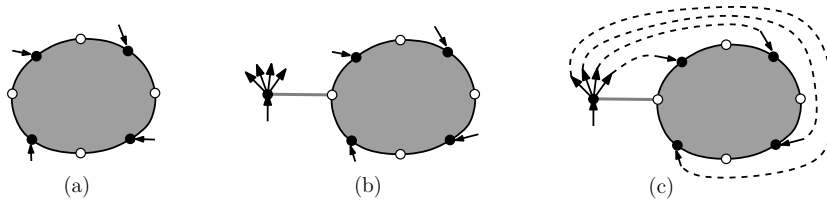
- For  $\delta > 0$ , the  $\delta$ -closure is a bijection between  $\delta$ -mobiles and the set  $\mathcal{S}_\delta$  of source-accessible minimal orientation. By duality  $\mathcal{S}_\delta$  is in bijection with the set  $\mathcal{O}_\delta$  of clockwise-minimal accessible orientations. The parameter-correspondence (inherited from the  $\delta$ -closure followed by duality) is shown in Figure 3 bottom-part.
- For  $\delta < 0$ , the  $\delta$ -closure is a bijection between  $\delta$ -mobiles and the subset  $\tilde{\mathcal{S}}_\delta$  of source-accessible minimal orientation such that every face incident to the root has clockwise degree 1. By duality  $\tilde{\mathcal{S}}_\delta$  is in bijection with the subset  $\tilde{\mathcal{O}}_\delta$  of clockwise-minimal accessible orientations. The parameter-correspondence is shown in Figure 3 bottom-part.
- For  $\delta = 0$ , the  $\delta$ -closure is a bijection between  $\delta$ -mobiles and minimal orientations.

The remaining of this section is devoted to the proof of Theorem 4 (the proof for  $\delta = 0$  is omitted).

**Case  $\delta > 0$ .** We first prove that the  $\delta$ -closure of a  $\delta$ -mobile is in  $\mathcal{S}_\delta$ . Let  $D$  be a  $\delta$ -mobile, let  $C$  be its partial closure and let  $M$  be its  $\delta$ -closure. As observed above, the mobile  $D$  has  $\delta > 0$  exposed white corners. Let  $D'$  be the decorated mobile obtained from  $D$  by creating a new black vertex  $b$ , joining  $b$  to an exposed white corner, and adding  $\delta$  outgoing stems to  $b$ . The excess of  $D'$  is 1, hence by Theorem 3 the rooted closure of  $D'$  gives a minimal accessible orientation  $M'$ . Moreover, it is easily seen (Figure 4) that the root-corner of  $M'$  is incident to the new vertex  $b$  (because the ingoing stem incident to  $b$  is not matched during the partial closure). Moreover (provided the ingoing root half-edge is not counted)  $b$  is a source of the orientation  $M'$ , and the vertex-rooted orientation  $M$  is induced by the corner-rooted orientation  $M'$ . Thus, the orientation  $M$  is in  $\mathcal{S}_\delta$ .

The following comment will be useful later (for the case  $\delta < 0$ ): the closure  $M$  of  $D$  is in  $\tilde{\mathcal{S}}_\delta$  if and only if each of the exposed white corners of  $D$  is incident to a (white) leaf of  $D$ . Indeed, a white vertex  $w_f$  of  $D$  has an exposed white corner if and only if it corresponds to a face  $f$  of  $M$  incident to the root-vertex  $b$ . Moreover, the clockwise degree of  $f$  is (as always) the degree of  $w_f$ .

We now prove that the  $\delta$ -closure is a bijection by defining the inverse mapping. Let  $M$  be a vertex-rooted orientation in  $\mathcal{S}_\delta$ . By applying the partial opening of  $M$  and then erasing every ingoing half-edge of  $M$ , one obtains an embedded graph with stems  $\hat{D}$ . The embedded graph  $\hat{D}$  is in fact disconnected since the root-vertex  $b$  of  $M$  is incident to no edge of  $\hat{D}$  (since  $b$  is a source of  $M$ ). The  $\delta$ -opening  $D$  of  $M$  is obtained from  $\hat{D}$  by erasing the vertex  $b$ . In order to prove that  $D$  is a decorated mobile (i.e. a tree with stems), we consider a minimal accessible orientation  $M'$  obtained from  $M$  by choosing a root-corner for  $M$  among the corners incident to the root-vertex  $b$ . By Theorem 3, the rooted opening of  $M'$  gives a decorated mobile  $D'$ . Clearly,  $D$  is obtained from  $D'$  by erasing the black vertex  $b$ . Moreover,  $b$  is a leaf of  $D'$  (since  $b$  is incident to no ingoing half-edge except the stem indicating the root-corner of  $O$ ), hence  $D$  is a mobile, and it has excess  $\delta$ . Lastly, since the rooted closure and rooted opening are inverse mappings, it is clear that  $\delta$ -closure and  $\delta$ -opening are inverse mappings, hence bijections.



**Fig. 4:** Formulation of the  $\delta$ -closure, for  $\delta > 0$ , as a reduction to the rooted closure. Figure (a) shows generically the partial closure of a  $\delta$ -mobile with  $\delta = 4$ , in (b) one creates a black vertex  $b$  with  $\delta$  outgoing stems, and connects it to an exposed white corner, in (c) one performs the remaining matchings of stems to complete the  $\delta$ -closure.

**Case  $\delta < 0$ .** We denote  $d = -\delta$ . Let  $D$  be a  $\delta$ -mobile. We associate to  $D$  a  $d$ -mobile  $\phi(D)$  obtained from  $D$  by transforming each of its  $d$  unmatched outgoing stems into an edge of  $\phi(D)$  connected to a newly created white leaf. Observe that the  $\delta$ -closure of  $D$  and the  $d$ -closure of  $\phi(D)$  coincide. Hence the  $\delta$ -closure is the composition of the mapping  $\phi$  and the  $d$ -closure. Moreover, the mapping  $\phi$  is a bijection between the set of  $\delta$ -mobiles and the set  $\mathcal{D}_d$  of  $d$ -mobiles such that every exposed white corner belongs to a leaf. Indeed,  $\phi(D)$  belongs to  $\mathcal{D}_d$  since the unique incident corner for each of the  $d$  newly created white leaves remains exposed during the partial closure; and the inverse mapping  $\phi^{-1}$  is obtained by replacing each edge incident to an exposed leaf by an outgoing stem. Lastly, by the observations above (case  $\delta > 0$ ), the  $d$ -closure induces a bijection between the set  $\mathcal{D}_d$  and the set  $\mathcal{S}_d$ . The inverse mapping to the  $\delta$ -closure, called the  $\delta$ -opening, is obtained as the composition of  $\phi^{-1}$  with the  $d$ -opening. This completes the proof of Theorem 4 (in the cases  $\delta \neq 0$ ).

## 4 Bijective counting of triangulations with boundaries

In this section we obtain bijections for simple triangulations (a.k.a. *3-connected triangulations*, *maximal planar graphs*) and for triangulations with boundaries. The bijections are obtained by specializing the closures defined in the previous section to certain classes of orientations characterizing simple triangulations.

Let  $T$  be a face-rooted triangulation. A *3-orientation* of  $T$  is an orientation such that inner vertices have indegree 3 and outer vertices have indegree 1. Schnyder proved in (Sch89) that any simple face-rooted triangulation admits a 3-orientation, that any 3-orientation is accessible from the outer vertices and that the root-face is always directed. Moreover, one easily checks (using Euler's relation) that loops and double edges are obstructions to the existence of a 3-orientation. Thus, a planar triangulation admits a 3-orientation if and only if it is simple. In the following we simply call *3-orientation* a 3-orientation of a face-rooted triangulation. From Lemma 1 one obtains:

**Lemma 5** *Face-rooted simple triangulations are in bijection with minimal 3-orientations. Such orientations are clockwise-minimal and accessible.*

Minimal 3-orientations are the orientations in  $\tilde{\mathcal{O}}_3$  such that all inner vertices have indegree 3 and all faces have degree 3. Thus, we can use the case  $\delta = -3$  of Theorem 4 to conclude that face-rooted simple triangulations are in bijection with  $(-3)$ -mobiles having every vertex of degree 3 (recall that outgoing stems count in the degree of a black vertex). In fact, the constraint that the excess is  $-3$  can be omitted, since it is a consequence of all vertices having degree 3 (as easily seen by induction on the number of vertices). Call *trivalent* the decorated mobiles with all vertices of degree 3. We obtain:



**Proposition 6 (Recovering (Sch98))** *The  $\delta$ -closure, case  $\delta = -3$  (together with duality) induces a bijection between face-rooted triangulations with  $n + 3$  vertices and trivalent mobiles with  $n$  white vertices.*

**Proposition 7 (Counting rooted simple triangulations)** *For  $n \geq 0$ , let  $t_n$  be the number of corner-rooted simple triangulations with  $n + 3$  vertices. The generating function  $T(x) = \sum_{n \geq 0} (2n + 1)t_n x^n$  satisfies*

$$T(x) = u^3, \text{ where } u = 1 + x u^4.$$

*Consequently, the Lagrange inversion formula gives:  $t_n = 2 \frac{(4n + 1)!}{(n + 1)!(3n + 2)!}$ .*

**Proof:** Euler relation easily implies that a triangulation with  $n + 3$  vertices has  $2n + 1$  non-root faces. Hence  $(2n + 1)t_n$  is the cardinality of the set  $\mathcal{H}_n$  of face-rooted triangulations with  $n$  inner vertices having an additional marked corner  $c$  not incident to the root-face (think of obtaining this map by first marking a corner and then a face). Marking the corner  $c$  is equivalent to marking a black corner of the associated mobile (since the black vertices are in correspondence to the triangular faces). In other words,  $\mathcal{H}_n$  is in bijection, via the  $(-3)$ -closure, with trivalent mobiles that have  $n$  white vertices and a marked black corner, and  $T(x)$  is the generating function of this class of mobiles. The rest of the proof is omitted.  $\square$

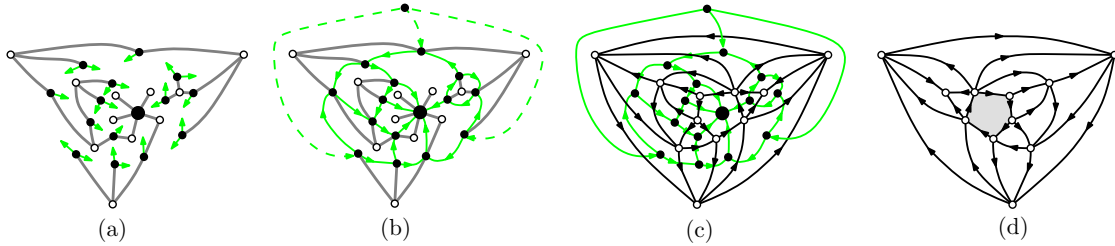
We now proceed to count bijectively the triangulations with boundaries. In the following,  $k$  is an integer greater than 3. A  $k$ -gonal triangulation is a map having one face of degree  $k$  which is *simple* (incident to  $k$  distinct vertices) and all other faces of degree 3. The  $k$ -gonal face is called *boundary face*, and the vertices are called *boundary* or *non-boundary* depending on whether they are incident to the boundary face. A *pseudo 3-orientation* of a  $k$ -gonal triangulation is an orientation such that all non-boundary vertices have indegree 3, and the boundary face is directed. A pseudo 3-orientation of a  $k$ -gonal triangulation is shown in Figure 5(d). By the Euler relation, an annular  $k$ -gonal triangulation with  $n$  non-boundary vertices has  $3n + 2k - 3$  edges. Hence, the sum of indegrees of the boundary vertices is  $2k - 3$ .

**Lemma 8** *A  $k$ -gonal triangulation admits a pseudo 3-orientation if and only if it is simple. Moreover this orientation is accessible from any boundary vertex.*

**Proof:** It is easy to see (by a counting argument using Euler relation) that loops and multiple edges are obstructions for pseudo 3-orientations. Now, let  $A$  be a simple  $k$ -gonal triangulation. From  $A$  we construct a simple triangulation  $T$  as follows. Place a triangle  $t$  inside the boundary face  $f$  of  $A$  and add edges between vertices of  $t$  and vertices of  $f$  so as to triangulate the interior of  $f$ . Then take  $t$  as a root-face of  $T$  and endow  $T$  with a 3-orientation  $O$ . Deleting the triangle  $t$  and the incident edges, and reorienting the boundary face to be directed, we obtain an orientation  $O'$  of  $A$  such that all non-boundary vertices have indegree 3, that is, a pseudo 3-orientation.

For the second assertion, remember that  $O$  is accessible from the vertices of the triangle  $t$ . Hence in  $O'$  each vertex  $v$  in  $A$  must be accessible from some boundary vertex. But since the boundary face is directed,  $v$  is in fact accessible from any boundary vertex.  $\square$

Following the terminology in (PS06), we call *annular  $k$ -gonal triangulation* a simple face-rooted  $k$ -gonal triangulation whose root-face is not the boundary face; see Figure 5(d). Let  $T$  be an annular  $k$ -gonal triangulation  $T$ , with root-face  $f$  and boundary face  $f'$ . A 3-cycle  $C$  of  $T$  is called *separating* if  $C$  is different from (the contour of) the root-face and has  $f$  on one side and  $f'$  on the other side. The annular triangulation  $T$  is said to be *non-separated* if it has no separating 3-cycle. We denote by  $\mathcal{T}_k$  set of annular  $k$ -gonal triangulations, and by  $\mathcal{N}_k$  the subset of non-separated ones.



**Fig. 5:** (a) A mobile in  $\mathcal{B}_k$ . (b) The 3-closure. (c) Duality. (d) The non-separated  $k$ -annular triangulation (with its minimal pseudo 3-orientation).

**Proposition 9** *Any annular  $k$ -gonal triangulation  $A$  admits a unique minimal pseudo 3-orientation. This orientation is accessible from all outer vertices if and only if  $A$  is non-separated. Moreover, the root-face is directed (in clockwise direction) in this case.*

*To summarize, the set  $\mathcal{N}_k$  of non-separated annular  $k$ -gonal triangulations is in bijection with the set of clockwise-accessible minimal pseudo 3-orientations.*

**Proof (sketch):** Let  $A$  be an annular  $k$ -gonal triangulation. We first prove the existence and uniqueness of the minimal pseudo 3-orientation. The rule that the boundary face has to be directed prevents us from using Lemma 1 directly. Instead, we apply this lemma to the map  $A'$  obtained from  $A$  by contracting all the edges of the boundary face; the boundary face is thereby contracted into a vertex  $b$ . Pseudo 3-orientations of  $A$  such that the boundary face is oriented clockwise correspond bijectively to  $\alpha$ -orientations of  $A'$  with  $\alpha(b) = k - 3$  and  $\alpha(v) = 3$  for  $v \neq b$  (recall that  $\alpha$  is the indegree). The existence of an  $\alpha$ -orientation of  $A'$  is granted by Lemma 8. Moreover, the bijection preserves minimality in both directions. Hence,  $A$  has a unique minimal pseudo 3-orientation by Lemma 1.

We now prove the statement concerning accessibility. Let  $A$  be an annular  $k$ -gonal triangulation with its minimal pseudo 3-orientation. Any simple cycle  $C$  of  $A$  separates the sphere in two regions; we call non-boundary the region not containing the boundary face. A simple counting argument shows that if  $C$  has length  $k$ , it is incident to exactly  $k - 3$  ingoing half-edges lying (strictly) inside the non-boundary region. This shows that a separating 3-cycle (incident to 0 ingoing half-edges) is an obstruction for accessibility from every outer vertex (since one of these vertices is inside the non-boundary region). To prove the converse implication, observe that non-accessibility from a vertex  $v$  is equivalent to the fact that  $v$  cannot reach one of the boundary vertices (because of Lemma 8). Moreover, it is not hard to prove that non-accessibility from a vertex  $v$  implies the existence of a cycle  $C$  containing  $v$  in its non-boundary region but incident to no ingoing half-edges from that region. Hence this cycle has length 3; and if  $v$  is an outer-vertex then  $C$  is a separating cycle.

We now consider a minimal pseudo 3-orientation  $O$  satisfying accessibility from every outer vertex and proceed to prove that the root-face is directed clockwise. The boundary vertices are reachable from any outer vertex (by a directed path) and any outer vertex is reachable from boundary vertices (by Lemma 8). Thus, any outer edge  $e$  is part of a circuit, hence part of a simple circuit. The simple circuit is clockwise (by minimality of  $O$ ), hence  $e$  has the root-face on its left-side.  $\square$

By definition, the clockwise-accessible minimal pseudo 3-orientations are the orientations in  $\mathcal{O}_3$  such that every vertex has indegree 3 and every face has degree 3 except for one *boundary* face  $b$  which has

degree  $k$  and is directed clockwise. Moreover, a counting argument (using Euler relation) shows that the indegrees of the boundary vertices must add up to  $2k - 3$ . Let  $D$  be the 3-mobile giving an orientation  $O \in \mathcal{N}_k$  (by the 3-closure followed by duality) and let  $v_b$  be the black vertex of  $D$  corresponding to the boundary face  $b$ . Since  $b$  is counterclockwise,  $v_b$  has no outgoing stem and has  $k$  white neighbors, which clearly (by definition of closure) corresponds to the boundary vertices; see Figure 5. Hence, the degree of these white vertices of the mobile must add up to  $2k - 3$ . Lastly, as in the case of triangulations without boundary, the condition of the excess being 3 is implied by the degree conditions on black and white vertices, so can be omitted. To conclude, by specialization of the 3-closure, Proposition 9 translates into:

**Theorem 10** *The family  $\mathcal{N}_k$  of non-separated annular  $k$ -gonal triangulations is in bijection with the family  $\mathcal{B}_k$  of decorated mobiles having every vertex of degree 3 except for one black vertex  $b$  of degree  $k$  carrying no outgoing stem and such that the degrees of its  $k$  (white) neighbors add up to  $2k - 3$ .*

**Theorem 11 (Counting rooted  $k$ -gonal triangulations)** *Let  $k > 3$ ,  $n \geq 0$ , and let  $t_{k,n}$  be the number of simple corner-rooted  $k$ -gonal triangulations with  $n + k$  vertices having the root-corner in the  $k$ -gonal face. The generating function  $T_k(x) = \sum_{n \geq 0} (2n + k - 2) t_{k,n} x^n$  satisfies*

$$T_k(x) = \binom{2k-4}{k-3} u^{2k-3}, \quad \text{where } u = 1 + x u^4.$$

Consequently, the Lagrange inversion formula gives:  $t_{k,n} = \frac{2(2k-3)!}{(k-1)!(k-3)!} \frac{(4n+2k-5)!}{n!(3n+2k-3)!}$ .

**Proof:** Let  $\vec{\mathcal{T}}_k$  be the set of annular  $k$ -gonal triangulations with a marked corner in the boundary face (equivalently, a marked boundary vertex). Let  $\vec{\mathcal{N}}_k$  be the subset of these annular triangulations that are non-separated. Let also  $\vec{\mathcal{T}}$  be the set of corner-rooted simple triangulations with a marked inner face. The separating 3-cycles of an annular triangulation are linearly ordered by inclusion of their boundary region (the region which contains the boundary face). Thus, there is a unique decomposition of annular triangulations  $A \in \vec{\mathcal{T}}_k$  into a pair  $(N, T) \in \vec{\mathcal{N}}_k \times \vec{\mathcal{T}}$ . This decomposition is a bijection and translates into the generating function equation  $T_k(x) = N_k(x)T(x)$ , where  $T_k(x), N_k(x), T(x)$  are respectively the generating function of the maps in  $\vec{\mathcal{T}}_k, \vec{\mathcal{N}}_k, \vec{\mathcal{T}}$  counted by number of non-boundary vertices. The generating function  $\vec{\mathcal{T}}_k(x)$  is  $\sum_{n \geq 0} (2n + k - 2) t_{k,n} x^n$  because maps in  $\vec{\mathcal{T}}_k$  with  $n$  non-boundary vertices have  $2n + k - 2$  non-boundary faces. Moreover, by Proposition 7,  $T(x) = u^3$ , where  $u = 1 + x u^4$ . It remains to express  $N_k(x)$  in terms of  $u$ . By Theorem 10 (and the fact that marking a corner in the boundary face accounts to marking a corner incident to the special black vertex in the associated mobile),  $N_k(x)$  is the generating function of corner-rooted mobile (counted by number of white vertices) such that the root-vertex is a black corner of degree  $k$  whose (white) neighbors have total degree  $2k - 3$ . The white vertices have a total of  $k - 3$  hanging subtrees, which are trivalent trees whose generating function is  $v = u^2$  (easy proof omitted). In addition, there are  $\binom{2k-4}{k-3}$  ways to distribute the  $k - 3$  hanging trees on the  $k$  white vertices. Hence,  $N_k(x) = \binom{2k-4}{k-3} u^{2k-6}$ .  $\square$

## 5 Bijective counting of quadrangulations with boundaries

In this section we reiterate the strategy used in previous section to the case of quadrangulations.

We call *2-orientation* a face-rooted orientation, in which faces have degree 4, inner vertices have indegree 2, and outer vertices have indegree 1. De Fraysseix *et al.* (dFOdM01) have shown that any simple face-rooted quadrangulation admits a 2-orientation, that any 2-orientation is accessible from the outer vertices and that the root-face is always directed. Moreover, one easily checks that a double edge is an obstruction for 2-orientation. Hence minimal 2-orientations are in bijection with simple face-rooted quadrangulations. By definition, minimal 2-orientations are the orientations in  $\tilde{\mathcal{O}}_4$  having faces of degree 4 and inner vertices of indegree 2. The  $(-4)$ -closure (followed by duality) gives a bijection between such orientations and  $(-4)$ -mobiles whose white vertices have degree 2 and whose black vertices have degree 4. Call these mobiles *tetravalent*. We obtain:

**Proposition 12 (Counting rooted simple quadrangulations)** *For  $n \geq 0$ , let  $q_n$  be the number of rooted simple quadrangulations with  $n + 4$  vertices. Then the generating function  $Q(x) = \sum_{n \geq 0} (n + 1)q_n x^n$  satisfies*

$$Q(x) = u^4, \quad \text{where } u = 1 + x u^3.$$

Consequently, the Lagrange inversion formula gives:  $q_n = 2 \frac{(3n + 3)!}{(n + 2)!(2n + 3)!}$ .

Call *k-gonal quadrangulation* a map with faces of degree 4 except for one face of degree  $k > 4$ . The strategy for counting simple *k-gonal quadrangulations* parallels the case of triangulations. One first defines an *annular k-gonal quadrangulation* to be a simple quadrangle-rooted *k-gonal quadrangulation*. One then proves that any such map admits a unique minimal *pseudo 2-orientation* (orientation such that the boundary face is clockwise and non-boundary vertices have indegree 2), and that this orientation is clockwise-accessible if and only if the map is *non-separated* (no 4-cycle separates the root-face from the boundary face). Any annular *k-gonal quadrangulation* decomposes uniquely into a pair made of a face-rooted quadrangulation with an additional marked face and a non-separated annular *k-gonal quadrangulation*. Moreover, the later maps are in bijection (via the 4-closure) with a family of pseudo-tetravalent decorated mobiles which is easy to enumerate. We obtain:

**Theorem 13 (Counting rooted simple 2k-gonal quadrangulations)** *Let  $k > 2$ ,  $n \geq 0$ , and let  $q_{k,n}$  be the number of rooted simple 2k-gonal quadrangulations with  $n + 2k$  vertices and with the root-corner incident to the 2k-gonal face. The generating function  $Q_k(x) = \sum_{n \geq 0} (n + k - 1)q_{k,n} x^n$  satisfies*

$$Q_k(x) = \binom{3k - 3}{k - 2} u^{3k - 2}, \quad \text{where } u = 1 + x u^3.$$

Consequently, the Lagrange inversion formula gives:  $q_{k,n} = \frac{3(3k - 2)!}{(k - 2)!(2k - 1)!} \frac{(3n + 3k - 4)!}{n!(2n + 3k - 2)!}$ .

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