BIJECTIVE COUNTING OF INVOLUTIVE BAXTER PERMUTATIONS

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Abstract. We enumerate bijectively the family of involutive Baxter permutations according to various parameters; in particular we obtain an elementary proof that the number of involutive Baxter permutations of size $2n$ with no fixed points is $\frac{3\cdot 2^{n-1}}{(n+1)(n+2)}\binom{2n}{n}$, a formula originally discovered by M. Bousquet-Mélou using generating functions. The same coefficient also enumerates planar maps with n edges, endowed with an acyclic orientation having a unique source, and such that the source and sinks are all incident to the outer face.

1. Introduction

Baxter permutations, named after Glen Baxter [2] who introduced them in an analysis context, are pattern-avoiding permutations (precisely the forbidden patterns are $2-41-3$ and $3-14-2$) with many nice combinatorial properties [17, 7, 16, 11]. Their counting coefficients appear recurrently in combinatorics; the so-called *Baxter number* (the number of Baxter permutations of size n)

$$B_n = \frac{2}{n(n+1)^2}\sum_{r=0}^{n-1}\binom{n+1}{r}\binom{n+1}{r+1}\binom{n+1}{r+2}$$

also counts plane bipolar orientations with n edges [3], certain rectangulations with n points on the diagonal [12, 1], certain Young tableaux with parity constraints [6], and so on. Subfamilies of Baxter permutations have also been considered: *alternating* and *doubly alternating* Baxter permutations have been enumerated in [8, 10, 14], and Baxter permutations of size n avoiding the pattern $2-4-1-3$ have been shown to be in bijection with rooted non-separable maps with $n+1$ edges [9, 4] (therefore there are $\frac{2(3n+3)!}{(2n+3)!(n+2)!}$ such permutations of size n). A permutation in $\mathfrak{S}_n$ is classically drawn as an $n\times n$ grid G of unit squares, with exactly one boxed square in each row and in each column. With this representation in mind, it is known (see e.g. [4]) that the set of Baxter permutations of size n is *globally invariant* by any of the 8 transformations of the dihedral group acting on the grid G. So it is a natural problem to try to count how many Baxter permutations are fixed by a given transformation of the dihedral group. In a previous paper [12] the case of the half-turn rotation was solved (whereas the case of rotations of order 4 is open); the idea is that a Baxter permutation can be encoded by a nonintersecting triple of paths, in a way that commutes with the half-turn transformation.

In this note we focus on the mirror reflection according to a diagonal, that is, we count *involutive* Baxter permutations. A difficulty is that the encoding of Baxter permutations by triples of paths does not commute in any sense with the diagonal mirror transformation (there is no nice transfer of mirror symmetry from the Baxter permutation to the associated triple of paths). An important ingredient here is the recent article [4], in which the authors establish a direct bijective correspondence between Baxter permutations and so-called *plane bipolar orientations*,

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Supported by the European Research Council under the European Community's 7th Framework Programme, ERC grant agreement no 208471 - ExploreMaps project.

which are acyclic orientations on embedded planar graphs (i.e., planar maps) with a unique source and a unique sink both lying in the outer face. Thanks to this correspondence, the successive combinatorial manipulations to encode involutive Baxter permutations can be carried out on (oriented) planar maps, which we find convenient to handle due to their more geometric flavor. As a consequence of our bijective encoding we obtain:

- a closed-form multivariate formula (in Theorem 2) for the number of involutive Baxter permutations according to the numbers of elements, descents (which are of two types, either crossing or not crossing the diagonal $\{x = y\}$ in the diagrammatic representation), and fixed points.
- a closed-form univariate formula (in Theorem 6) for the number b_n of involutive fixed-point free Baxter permutations of $2n$ elements:

$$b_n = \frac{3 \cdot 2^{n-1}}{(n+1)(n+2)} \binom{2n}{n}.$$

Note that b_n has surprisingly a simpler (summation-free) expression than the Baxter number B_n. The univariate formula for b_n (already announced in [4]) and a multivariate formula restricted to fixed-point free Baxter permutations have been discovered by M. Bousquet-Mélou [5] using generating functions and the so-called "obstinate" kernel method. As follows from the correspondence with plane orientations (to be described in Section 2) the number b_n also counts acyclic orientations on planar maps with n edges, a unique source, and all extremal vertices (source and sinks) lying in the outer face.

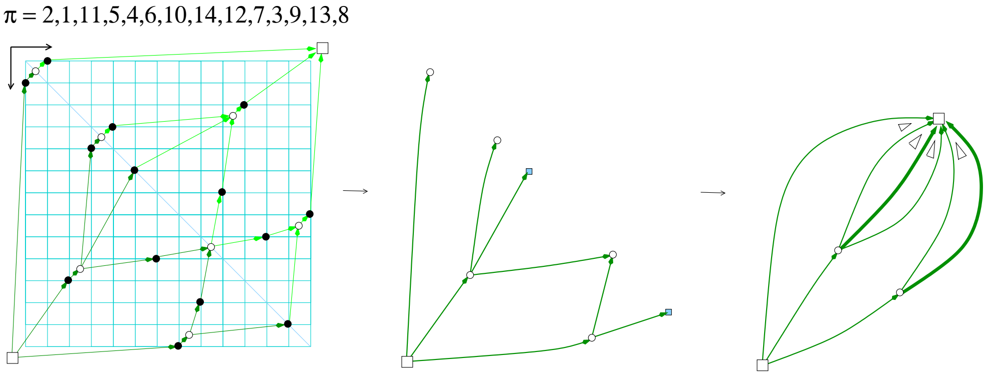
Outline. The main steps of our method are the following: (i) by a quotient-argument already outlined in [4], interpret the plane bipolar orientations corresponding to involutive Baxter permutations as certain plane bipolar orientations with decorations at the corners and edges incident to the sink, (ii) adapt the known bijective encoding of plane bipolar orientations by non-intersecting triples of paths to take account of the decorations, (iii) count the obtained non-intersecting triples of paths using the Lindström-Gessel-Viennot lemma (for the multivariate formula) or similar principles with small adjustments (for the univariate formula).

2. **Involutive Baxter permutations as decorated plane bipolar orientations**

Let $\mathcal{B}$ be the class of Baxter permutations and $\mathcal{O}$ the class of plane bipolar orientations. As shown in [4] (see Figure 1 for an example), one can construct from the diagrammatic representation of $\pi \in \mathcal{B}$ an embedded plane bipolar orientation $\phi(\pi)$ where black points of degree 2 correspond to edges (one such vertex on each edge) and white points correspond to vertices. The induced mapping Φ from $\mathcal{B}$ to $\mathcal{O}$ (where $\Phi(\pi)$ is the plane bipolar orientation induced by $\phi(\pi)$ after erasing the black vertices) is a bijection that satisfies several parameter correspondences (elements are mapped to edges, descents are mapped to non-pole vertices,...) and preserves many symmetries; in particular if π is involutive (i.e., $\pi^{-1} = \pi$) then $\phi(\pi)$ is fixed by the reflection according to the line $\{x = y\}$, see Figure 1(a).

A *planar map* is a connected graph planarly embedded in the plane (considered up to continuous deformation). Define a *monosource* orientation as an acyclic orientation O of a planar map with a unique source and such that the source and all sinks lie in the outer face. Additionally an arbitrary subset of the sinks of degree 1 are marked; these are called the *sailing sinks* of O (small shaded squares in Figure 1(b)). Edges incident to sailing sinks are called *sailing edges*.

As illustrated in Figure 1(a)-(b), there is a bijection between the class $\mathcal{I}$ of involutive Baxter permutations and the class $\mathcal{M}$ of monosource orientations which



(a) (b) (c)

FIGURE 1. (a) An involutive Baxter permutation π (black points give the diagrammatic representation of π) superimposed with the associated embedded plane bipolar orientation O (a black point in each edge of O, a white vertex for each vertex of O). The whole drawing is invariant by the reflexion according to the line $\{x = y\}$. (b) The monosource orientation M obtained by keeping the part of O below the line $\{x = y\}$, where sailing sinks are represented as small shaded squares, (c) the associated decorated plane bipolar orientation obtained by merging all the sinks of M. The marked sink-edges (which are drawn bolder) are those incident to sailing sinks in M, the marked sink-corners (indicated by the small triangles) are the newly created sink-corners.

transforms standard parameters as follows (a *descent* in a permutation $\pi \in \mathfrak{S}_n$ is an integer $i \in [1..n-1]$ such that $\pi(i) > \pi(i+1)$, and a descent is said to *cross the diagonal* if $\pi(i) > i$ and $\pi(i+1) < i+1$):

$$\begin{array}{rcl} 2n \text{ non-fixed points} & \leftrightarrow & n \text{ non-sailing edges,} \\ 2k \text{ descents not crossing the diagonal} & \leftrightarrow & k \text{ non-extremal vertices,} \\ p \text{ fixed points} & \leftrightarrow & p \text{ sailing sinks,} \\ r \text{ descents crossing the diagonal} & \leftrightarrow & r \text{ non-sailing sinks.} \end{array}$$

We now claim that orientations in $\mathcal{M}$ correspond to plane bipolar orientations with certain decorations. Given a plane bipolar orientation, the *sink-degree* is the degree of the sink, and a *sink-edge* is an edge incident to the sink. A *corner* of a planar map is an angular sector delimited by two consecutive edges around a vertex. For a plane bipolar orientation, a *sink-corner* is a corner incident to the sink but not in the outer face (note that the number of sink-corners is the sink-degree minus 1). Define a *decorated plane bipolar orientation* as a plane bipolar orientation where an arbitrary subset of the sink-corners are marked, and a subset of sink-edges are marked in such a way that the sink-corners incident to a marked sink-edge are marked. As shown in Figure 1(b)-(c), there is a bijection between the class $\mathcal{M}$ of monosource orientations and the class $\mathcal{D}$ of decorated plane bipolar orientations with the following parameter-correspondence:

$$\begin{array}{rcl} n \text{ non-sailing edges} & \leftrightarrow & n \text{ non-marked edges,} \\ k \text{ non-extremal vertices} & \leftrightarrow & k \text{ non-pole vertices,} \\ p \text{ sailing sinks} & \leftrightarrow & p \text{ marked sink-edges,} \\ r \text{ non-sailing sinks} & \leftrightarrow & p+r-1 \text{ marked sink-corners.} \end{array}$$

3. Encoding by paths

We now explain how to encode the sink-edges and sink-corners of a decorated plane bipolar orientation O, this is illustrated in Figure 2. Let $i+1$ be the sink-degree (so there are i sink-corners), p the number of marked sink-edges and q the number of marked sink-corners. By a *binary walk* we will mean an oriented walk in $\mathbb{Z}^2$ having steps East $(+1,0)$ and North $(0,+1)$. First, encode the marked sink-corners by a binary walk obtained by reading the sink-corners from left to right, writing an East-step if the sink-corner is marked and a North-step otherwise. Then append an East-step to both the beginning and end of the binary walk; we thus have a binary walk with $q+2$ East-steps and $i-q$ North-steps. Note that the sink-edges that are allowed to be marked correspond to points of the walk preceded and followed by an East-step (the ones corresponding to marked sink-edges are surrounded in Figure 2). Delete these vertices, and then renormalize the path to have only steps of length 1. Above each East-step of the renormalized path write the number of points that have been deleted in the corresponding horizontal portion (before renormalization); finally delete the last (East-) step of the walk. The finally obtained walk W starts with an East-step if not empty (it is empty iff all sink-edges are marked); it has length $i+1-p$, $i-q$ North-steps, $q+1-p$ East-steps, and is accompanied by a sequence S of $q+2-p$ nonnegative numbers adding up to p. The pair (W,S) is called the *sink-code* for O.

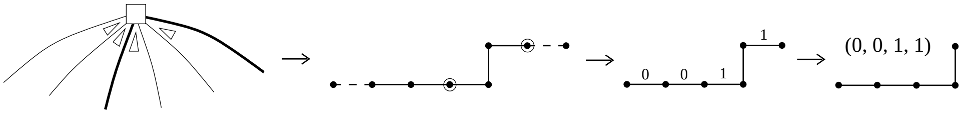


FIGURE 2. Encoding the marked sink-corners (indicated as small triangles) and sink-edges (indicated as bolder edges) by a binary walk and a sequence of weights.

Let $\mathcal{T}$ be the class of non-intersecting triples W_1, W_2, W_3 of finite binary walks having starting points $(-1,1)$, $(0,0)$ and $(0,-1)$, same numbers of North-steps, and with $\text{length}(W_1) = \text{length}(W_2) = \text{length}(W_3) - 1$. As described in [4] (the original bijection, between Baxter permutations and triples of paths, is due to Dulucq and Guibert [11]) there is a bijection between the class $\mathcal{O}$ of plane bipolar orientations and the class $\mathcal{T}$, with following parameter-correspondence (see also Figure 3(a)):

$$\begin{array}{rcl} n \text{ edges} & \leftrightarrow & \text{length}(W_1) = n-1, \\ k \text{ non-pole vertices} & \leftrightarrow & W_1 \text{ has } k \text{ East-steps}, \\ i+1 \text{ sink-edges} & \leftrightarrow & W_3 \text{ ends with an East-step followed by } i \text{ North-steps.} \end{array}$$

Now define $\mathcal{E}$ as the class of 4-tuples (W_1, W_2, W_3, S), where (W_1, W_2, W_3) is a non-intersecting triple of binary walks starting respectively from the points $(-1,1)$, $(0,0)$ and $(0,-1)$; and S is a sequence of non-negative integers such that:

- The walks W_1 and W_2 have same lengths and same numbers of East-steps.
- Denoting by (x_2, y_2) and (x_3, y_3) the coordinates of the respective endpoints of W_2 and W_3, we have $x_3 \geq x_2$ and $x_2 + y_2 \geq x_3 + y_3$. Let $a := x_3 - x_2$ and $b := x_2 + y_2 - x_3 - y_3$.
- The sequence S is made of $a+1$ non-negative integers that add up to b.

Let O be a decorated plane bipolar orientation with sink-degree $i+1$. Let $(W_1, W_2, W_3) \in \mathcal{T}$ be the triple of walks associated with O (without the decorations), and (W,S) the sink-code for O. Define W_3' as W_3 where the suffix East North^i is replaced by W, see Figure 3. Then it is easily checked that $E := (W_1, W_2, W_3', S) \in \mathcal{E}$. Note that one can recover W_3 from E (W_3 is the unique

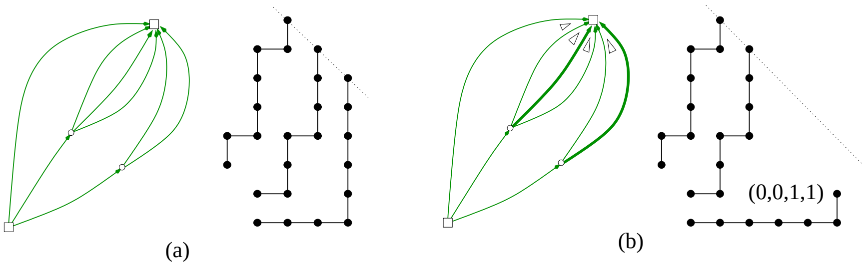


FIGURE 3. (a) A plane bipolar orientation is encoded by a non-intersecting triple of binary walks. (b) The encoding of a decorated plane bipolar orientation.

binary walk ending at (x_2+1, y_2-1) and equal to W_3' in the area $\{x \leq x_2\}$), hence one can recover O. Moreover if O has p marked sink-edges and q marked sink-corners, then with the notations above, $b = p$ and $a = q+1-p$.

Overall, we obtain a bijection between the class $\mathcal{D}$ of decorated plane bipolar orientations and the class $\mathcal{E}$, with the following parameter-correspondence:

$$\begin{array}{rcl} m \text{ edges} & \leftrightarrow & W_1 \text{ and } W_2 \text{ have length } m-1, \\ k \text{ non-pole vertices} & \leftrightarrow & W_1 \text{ and } W_2 \text{ have } k \text{ East-steps}, \\ p \text{ marked sink-edges} & \leftrightarrow & W_3 \text{ has length } m-p, \\ q \text{ marked sink-corners} & \leftrightarrow & W_3 \text{ has } k+q-p+1 \text{ East-steps}. \end{array}$$

Composing the bijection between $\mathcal{I}$ and $\mathcal{D}$ with the bijection between $\mathcal{D}$ and $\mathcal{E}$ we finally obtain (taking $m = n+p$ and $q = p+r-1$ in the correspondence above):

Theorem 1. *There is a bijection between involutive Baxter permutations with* $2n$ *non-fixed points,* $2k$ *descents not crossing the diagonal,* p *fixed points,* r *descents crossing the diagonal; and* 4*-tuples of the form* (W_1, W_2, W_3, S) *where* (W_1, W_2, W_3) *is a non-intersecting triple of binary walks with starting points* $(-1,1)$, $(0,0)$, $(0,-1)$, *end-points* $(k-1, n+p-k)$, $(k, n+p-k-1)$, $(k+r, n-k-r-1)$, *and where* S *is a sequence of* $r+1$ *nonnegative numbers adding up to* p.

4. Counting

Let (A_1, A_2, A_3) and (B_1, B_2, B_3) be the starting points and end-points in Theorem 1. By the Lindström-Gessel-Viennot lemma [13], the number of non-intersecting triples of binary walks with starting points A_1, A_2, A_3 and end-points B_1, B_2, B_3 is the determinant of the 3×3 matrix $(m_{i,j})$, with $m_{i,j}$ the number of binary walks from A_i to B_j (note that each entry $m_{i,j}$ is an explicit binomial coefficient). The number of choices for the sequence S in Theorem 1 is clearly equal to $\binom{p+r}{r}$. We obtain (taking out of the determinant a common binomial factor for each column):

Theorem 2 (multivariate enumeration formula)**.** *For* $n > 0$, *and* k, p, r *nonnegative integers, the number* $a_{n,k,p,r}$ *of involutive Baxter permutations with* $2n$ *non-fixed points,* $2k$ *descents not crossing the diagonal,* p *fixed points,* r *descents crossing the diagonal is given by*

$$a_{n,k,p,r} = \frac{\binom{p+r}{r}\binom{n+p-1}{k}^2\binom{n}{t}}{nq^2(q+1)(k+1)(t+1)} \cdot \begin{vmatrix} q(q+1) & q(q-1) & s(s-1) \\ k(q+1) & (k+1)q & s(t+1) \\ k(k-1) & k(k+1) & t(t+1) \end{vmatrix} \qquad (1)$$

where $q := n+p-k$, $s := n-k-r$, $t := k+r$.

An equivalent multivariate formula for $a_{n,k,0,r}$ has been obtained by Bousquet-Mélou [5] using the "obstinate" kernel method. Note that, by the correspondence of Section 2, the number $a_{n,k,0,r}$ counts monosource orientations (without taking sailing sinks into account) with n edges, r sinks, and k non-extremal vertices.

We now prove that the number b_n of involutive Baxter permutations with no fixed point and $2n$ elements satisfies $b_n = \frac{3\cdot 2^{n-1}}{(n+1)(n+2)}\binom{2n}{n}$ (even though $b_n = \sum_{k,r} a_{n,k,0,r}$, our proof does not exploit the multivariate formula of Theorem 2). Let $\mathcal{F}_n$ be the set of involutive Baxter permutations of size $2n$ with no fixed point. In this part it is convenient to rotate the encoding triples of paths by 45 degrees counter-clockwise, to delete the first (East-) step in the third path of the triple, and to rescale by $\sqrt{2}$. This way, the binary walks considered in the previous section become paths having steps $(-1/2,+1/2)$ (annotated $\nwarrow$) or $(+1/2,+1/2)$ (annotated $\nearrow$) [1], and the bijection of Theorem 1 specializes as follows:

Claim 3. *For $n \geq 1$, $\mathcal{F}_n$ is in bijection with the set $\mathcal{R}_n$ of non-intersecting triples of paths, each with $n-1$ steps either $\nwarrow$ or $\nearrow$, with starting points $(-1,0)$, $(0,0)$, $(1,0)$ and such that the endpoints of the first two paths are at distance 1 on the line $\{y = \frac{n-1}{2}\}$.*

Proof. By Theorem 1 $\mathcal{F}_n$ is in bijection with the set of non-intersecting triples of finite binary walks (W_1, W_2, W_3) with starting points $(-1,1)$, $(0,0)$, and $(0,-1)$, and end-points on the line $\{x+y=n-1\}$ such that the end-points of W_1 and W_2 are at distance $\sqrt{2}$. Since the 3 walks do not intersect, the first step of the third walk is always an East-step, hence can be removed with no loss of information. This way the 3 walks start on the line $\{x+y=0\}$ with W_1 at distance $\sqrt{2}$ from W_2 itself at distance $\sqrt{2}$ from W_3. Hence, rotating the figure counter-clockwise by $\pi/4$ and rescaling by $\sqrt{2}$, one has a triple of paths in $\mathcal{R}_n$. ∎

To enumerate the triples in $\mathcal{R}_n$, we inject $\mathcal{R}_n$ in the bigger set $\mathcal{U}_n$ defined the same way, except that the third path is allowed to intersect the two other paths. Let $\mathcal{S}_n$ be the subset of objects in $\mathcal{U}_n$ where the third path meets the second path (and possibly also the first path); so we have $\mathcal{U}_n = \mathcal{R}_n + \mathcal{S}_n$. Let u_n, r_n, s_n be the cardinalities of $\mathcal{U}_n$, $\mathcal{R}_n$, and $\mathcal{S}_n$, respectively. We have $u_n = r_n + s_n$, so that

$$b_n = r_n = u_n - s_n.$$

We now state a basic lemma [15] which we will use to obtain formulas for u_n and s_n (the case $k=1$, which gives the Catalan numbers, is illustrated in Figure 4(a)):

Lemma 4 (folklore)**.** *For n and k positive integers, let $a_n^{(k)}$ be the number of non-intersecting pairs (P_1, P_2) of paths (counted up to horizontal translation) each with $n-1$ steps either $\nwarrow$ or $\nearrow$, starting at distance k on the line $\{y=0\}$ and ending at distance 1 on the line $\{y=(n-1)/2\}$. Then*

$$a_n^{(k)} = \frac{2k(2n-1)!}{(n-k)!(n+k)!}.$$

The lemma directly yields a formula for u_n; indeed the first two paths of a triple in $\mathcal{U}_n$ are non-intersecting, start at distance 1 on $\{y=0\}$ and end at distance 1 on $\{y=\frac{n-1}{2}\}$, and the third path is unconstrained. Hence

$$u_n = 2^{n-1}a_n^{(1)} = 2^{n-1}\frac{(2n)!}{n!(n+1)!}.$$

To obtain a formula for s_n, we "double" the set $\mathcal{S}_n$. For a triple $\gamma \in \mathcal{S}_n$, the *mirror* of γ is obtained by applying a vertical mirror (reflexion according to a vertical line,

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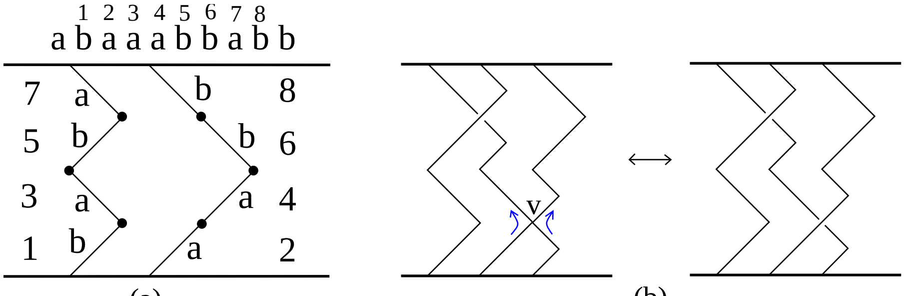
[1] We talk about *walks* when steps are {East,North} and *paths* when steps are $\{\nwarrow, \nearrow\}$.



(a) (b)

FIGURE 4. (a) A non-intersecting pair of paths with starting points and end-points at distance 1 can be encoded by a Dyck word, (b) bijection between $\mathcal{V}_n$ and $\mathcal{S}_n + \mathrm{mir}(\mathcal{S}_n)$.

up to a global translation) to γ. Denote by $\mathrm{mir}(\mathcal{S}_n)$ the set of mirrors of triples in $\mathcal{S}$. Note that $\mathcal{S}_n \cap \mathrm{mir}(\mathcal{S}_n)$ is empty (indeed, in $\mathcal{S}_n$, the middle-starting path intersects only the right-starting path; whereas in $\mathrm{mir}(\mathcal{S}_n)$ the middle-starting path intersects only the left-starting path). We now establish a bijection between $\mathcal{S}_n + \mathrm{mir}(\mathcal{S}_n)$ and a set easy to count. Let $\mathcal{V}_n$ be the set of triples of paths, each with $n-1$ steps either $\nwarrow$ or $\nearrow$, with starting points $(-1,0)$, $(0,0)$, $(1,0)$, and such that the left-starting path and the right-starting path do not intersect and end at distance 1 on the line $\{y = \frac{n-1}{2}\}$.

Claim 5. *The set $\mathcal{V}_n$ is in bijection with $\mathcal{S}_n + \mathrm{mir}(\mathcal{S}_n)$.*

Proof. The bijection relies on a simple argument akin to the Gessel-Viennot lemma. First we describe the mapping from $\mathcal{V}_n$ to $\mathcal{S}_n + \mathrm{mir}(\mathcal{S}_n)$. Consider a triple of paths from $\mathcal{V}_n$, and denote by P_ℓ, P_m, P_r the paths starting from $(-1,0)$, $(0,0)$, and $(1,0)$, respectively. Since the end-points of P_ℓ and P_r are at distance 1 (i.e., consecutive) on $\{y = \frac{n-1}{2}\}$, the path P_m has to intersect $P_\ell \cup P_r$. Let v be the first intersection of P_m with $P_\ell \cup P_r$ (note that v can not be on both P_ℓ and on P_r, see Figure 4(b)). If $v \in P_r$ we exchange the parts of P_r and P_m after v, this yields a triple of paths in $\mathcal{S}$ (see Figure 4(b)); if $v \in P_\ell$ we exchange the parts of P_ℓ and P_m after v, this yields a triple of paths in $\mathrm{mir}(\mathcal{S})$. It is now straightforward to get the inverse mapping, from $\mathcal{S}_n + \mathrm{mir}(\mathcal{S}_n)$ to $\mathcal{V}_n$. Let $\gamma \in \mathcal{S}_n + \mathrm{mir}(\mathcal{S}_n)$, and denote again by P_ℓ, P_m, and P_r the paths starting from $(-1,0)$, $(0,0)$, and $(1,0)$, respectively. If $\gamma \in \mathcal{S}$ let v be the first intersection of P_m and P_r; exchange the portions of P_m and P_r after v. If $\gamma \in \mathrm{mir}(\mathcal{S})$ let v be the first intersection of P_ℓ and P_m; exchange the portions of P_ℓ and P_m after v. ∎

Let v_n be the cardinality of $\mathcal{V}_n$. Claim 5 implies that $v_n = 2s_n$. Now v_n is easy to obtain. Indeed, for a triple in $\mathcal{V}_n$, the left-starting path and right-starting path start at distance 2 on $\{y = 0\}$, end at distance 1 on $\{y = \frac{n-1}{2}\}$, and are non-intersecting; and the middle-starting path is unconstrained. Hence, with the notation of Lemma 4,

$$v_n = 2^{n-1} a_n^{(2)} = 2^{n+1} \frac{(2n-1)!}{(n-2)!(n+2)!},$$

so that $s_n = v_n/2 = 2^n \frac{(2n-1)!}{(n-2)!(n+2)!}$. From $b_n = u_n - s_n$ and the expressions of u_n and s_n we obtain:

Theorem 6 (univariate formula, recovers [5] in a bijective way)**.** *The number* b_n *of involutive Baxter permutations with no fixed point and* $2n$ *elements is*

$$b_n = \frac{3\cdot 2^{n-1}}{(n+1)(n+2)}\binom{2n}{n}. \tag{2}$$

Note that, by the correspondence of Section 2, b_n is the number of monosource orientations (without taking sailing sinks into account) with n edges.

Acknowledgement. I am very grateful to Mireille Bousquet-Mélou and Nicolas Bonichon for helfpul discussions and corrections on a preliminary version. I also thank Mireille for communicating to me the counting formulas for involutive Baxter permutations.

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