EL-Shellability of Generalized Noncrossing Partitions Associated to Well-Generated Complex Reflection Groups

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Introduction

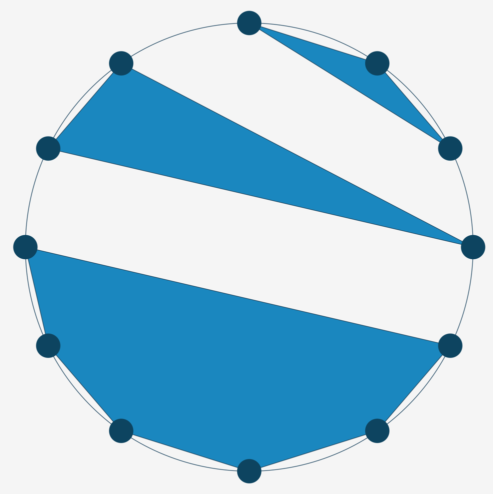
The poset of (classical) m-divisible noncrossing partitions was introduced by Kreweras and Edelman. It can be realized as an interval in the absolute order on the symmetric group. In this spirit, there is a natural generalization to all well-generated complex reflection groups. It was shown in [1] and [2] that these posets are EL-shellable if associated to a real reflection group.

Question

Can this property be generalized to all well-generated complex reflection groups?

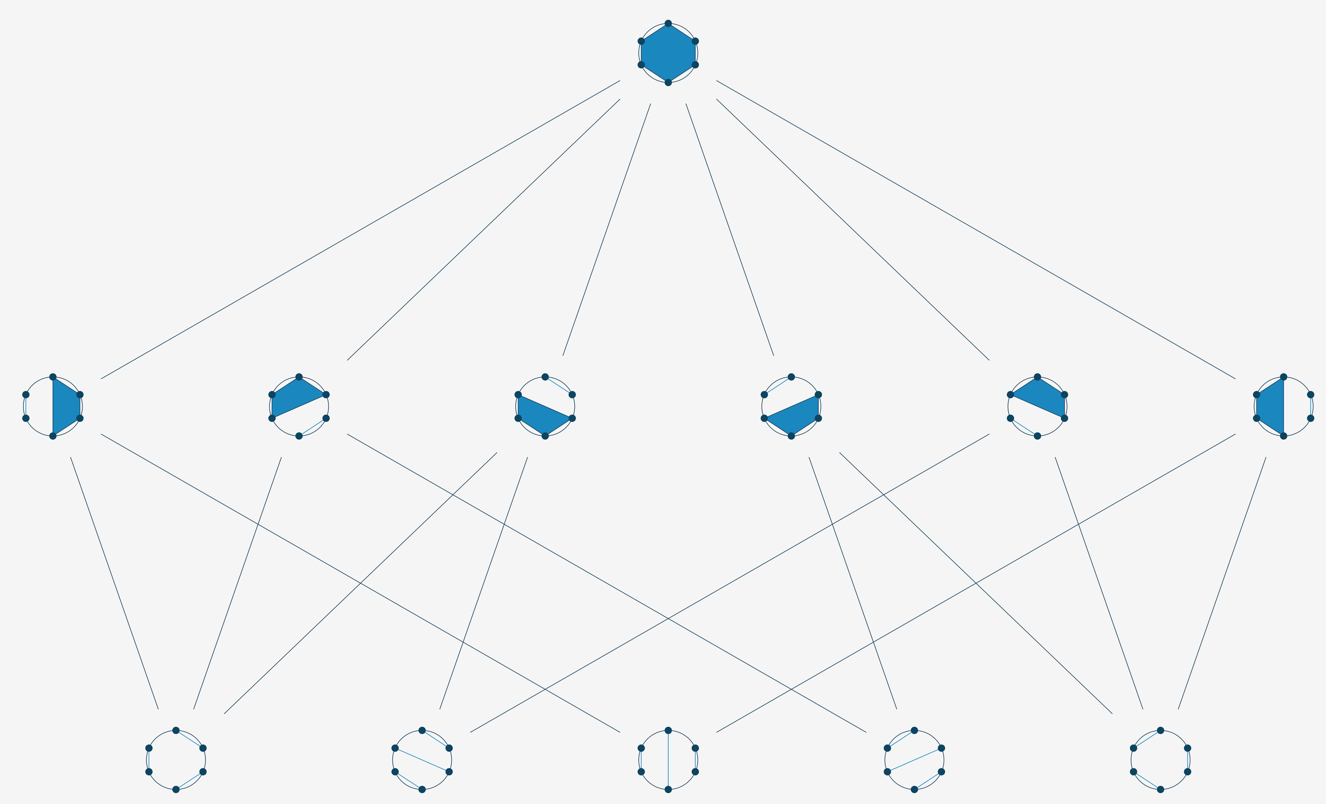
Example

A 3-divisible noncrossing partition of $\{1, 2, \ldots, 12\}$.



Example

The poset of 2-divisible noncrossing partitions of $\{1, 2, \ldots, 6\}$.



Main Result

Let $NC_W^{(m)}$ denote the poset of m-divisible noncrossing partitions associated to a well-generated complex reflection group W.

Theorem

The poset $NC_W^{(m)}$ is EL-shellable for all well-generated complex reflection groups W.

Complex Reflection Groups

There is one infinite family, $G(d, e, n)$, of irreducible complex reflection groups, and 34 exceptional groups. Groups of type $G(d, e, n)$ can be realized as groups of monomial $(n \times n)$-matrices, where the non-zero entries are d-th roots of unity and the product of all non-zero entries is a $\frac{d}{e}$-th root of unity.

A complex reflection group of rank n is called well-generated if it can be generated by n reflections. These are the groups of type $G(d, 1, n)$ and $G(d, d, n)$, and 26 of exceptional type.

Open Cases

The groups $G(1, 1, n)$, $G(2, 1, n)$ and $G(2, 2, n)$ are real reflection groups. Moreover, Bessis and Corran showed that $NC^{(1)}_{G(d,1,n)} \cong NC^{(1)}_{G(2,1,n)}$ for $d \geq 2$, [3].

Only groups of type $G(d, d, n)$, where $d > 2$, and the 20 exceptional groups, which are no real reflection groups, need to be considered.

Noncrossing Partitions

Let W be a well-generated complex reflection group, and let T denote the set of all reflections of W. For $w \in W$, let $\ell_T(w)$ be the minimal number of reflections needed to form w. Define $u \leq_T v$ if and only if $\ell_T(v) = \ell_T(u) + \ell_T(u^{-1}v)$. Let $\varepsilon \in W$ denote the identity element, and let $\gamma \in W$ be a Coxeter element. The interval $NC_W^{(1)} = [\varepsilon, \gamma]$ in $(W, \leq_T)$ is called lattice of noncrossing partitions associated to W.

Define $NC_W^{(m)} = \left\{(w_0; w_1, \ldots, w_m) \in \left(NC_W^{(1)}\right)^{m+1} \mid \gamma = w_0 w_1 \cdots w_m, \sum_{i=0}^{m} \ell_T(w_0) = \ell_T(\gamma)\right\}$, and $(u_0; u_1, \ldots, u_m) \leq (v_0; v_1, \ldots, v_m)$ if and only if $u_i \geq_T v_i$ for all $1 \leq i \leq m$. The poset $(NC_W^{(m)}, \leq)$ is called poset of generalized noncrossing partitions associated to W.

EL-Shellability

Let P be a graded poset, with a unique bottom and a unique top element. An edge-labeling of P is a function assigning to each cover relation of P a unique value. A chain in P is called rising if the associated sequence of edge-labels is strictly increasing. An edge-labeling is called EL-labeling if for every interval of P, there exists a unique maximal rising chain, which is lexicographically first among all maximal chains in this interval. P is called EL-shellable if it admits an EL-labeling.

Idea

Find an EL-labeling for the 1-divisible noncrossing partitions associated to the remaining groups. Then, construct an EL-labeling for the corresponding m-divisible noncrossing partitions out of it, analogously to [1].

Strategy

Let T denote the set of reflections of W. Use the natural labeling $\lambda : \mathcal{E}(NC_W^{(1)}) \to T, (u, v) \mapsto u^{-1}v$.

Find a linear order of T such that λ becomes an EL-labeling!

γ-compatible Reflection Ordering

A linear order $\prec$ of T is called γ-compatible if and only if the following is satisfied: if $t_1, t_2 \in T$ are noncommuting reflections such that the induced interval $I(t_1, t_2) = [\varepsilon, t_1 \vee t_2]$ in $NC_W^{(1)}$ has rank 2, then there exist exactly two reflections $\tilde{t}_1, \tilde{t}_2 \in T \cap I(t_1, t_2)$ such that $\tilde{t}_1\tilde{t}_2 \leq_T \gamma$ implies $\tilde{t}_1 \prec \tilde{t}_2$.

Theorem

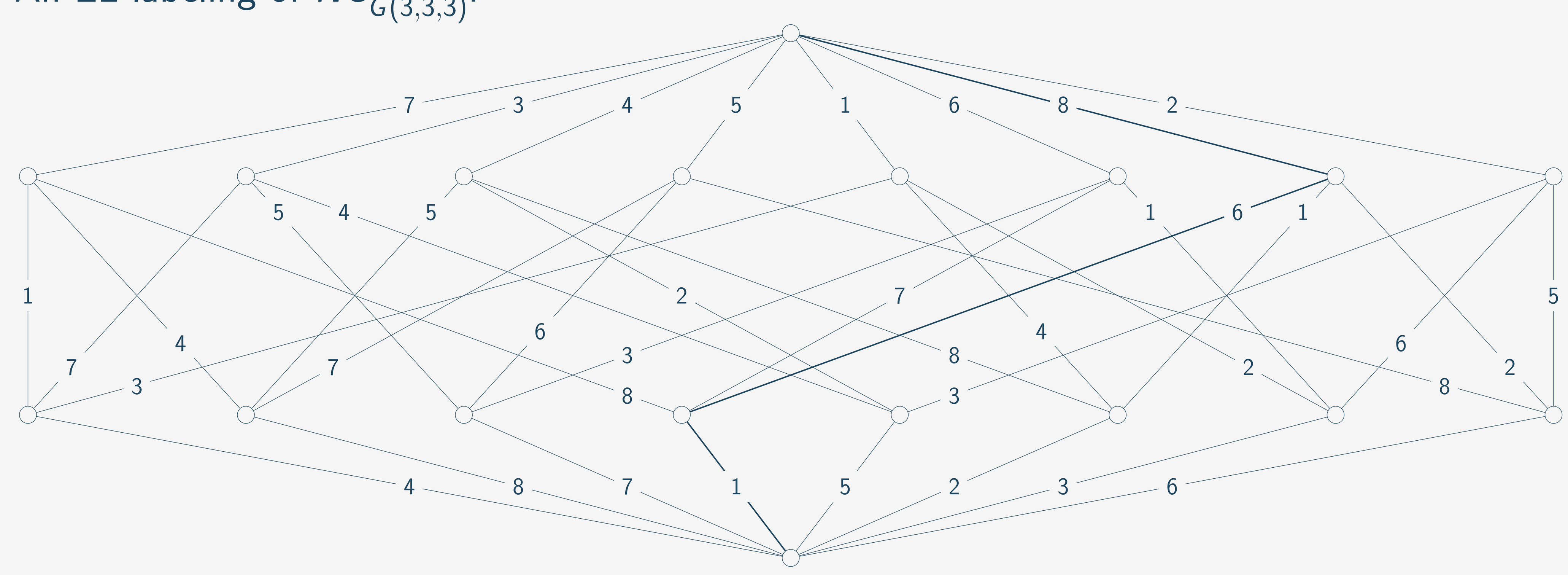
For every well-generated complex reflection group, and every Coxeter element γ, there exists a γ-compatible reflection ordering.

Theorem

If T is ordered by a γ-compatible reflection ordering, then λ is an EL-labeling of $NC_W^{(1)}$.

Example

An EL-labeling of $NC^{(1)}_{G(3,3,3)}$.



Application

Denote by $d_1 < d_2 < \cdots < d_n$ the degrees of W, and define $\mathrm{Cat}_W^{(m)} = \prod_{i=1}^{n} \frac{md_n + d_i}{d_i}$.

Corollary

The order complex of the poset $NC_W^{(m)}$ with maximal and minimal elements removed is homotopy equivalent to a wedge of $\left(Cat^{(-m-1)}(W) - Cat^{(-m)}(W)\right)$-many $(n-2)$-spheres.

References

[1] D. Armstrong. Generalized Noncrossing Partitions and Combinatorics of Coxeter Groups. *Mem. Amer. Math. Soc.*, 202, 2009.

[2] C. A. Athanasiadis, T. Brady, and C. Watt. Shellability of Noncrossing Partition Lattices. *Proc. Amer. Math. Soc.*, 135:939–949, 2007.

[3] D. Bessis and R. Corran. Non-crossing Partitions of Type (e, e, r). *Adv. Math.*, 202:1–49, 2006.

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