Enumeration of permutations sorted with two passes through a stack and D_8 symmetries

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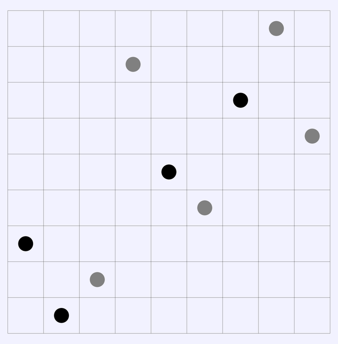
Definitions: Permutation Patterns, Symmetries, Stack Sorting, Permutation Statistics

Classical patterns

$\pi \in \mathfrak{S}_k$ is a pattern of $\sigma \in \mathfrak{S}_n$ if $\exists\, i_1 < \ldots < i_k$ such that $\sigma_{i_1} \ldots \sigma_{i_k}$ is order isomorphic to π



$\sigma_{i_1} \ldots \sigma_{i_k}$ is an occurrence of π

Generalized patterns

Dashed: Add adjacency constraints between some elements $\sigma_{i_1}, \ldots, \sigma_{i_k}$

Example: $\sigma_{i_1}\sigma_{i_2}\sigma_{i_3}\sigma_{i_4}$ is an occurrence of 2-41-3 $\Rightarrow i_3 = i_2 + 1$

Barred: Add some absence constraints

Example: Occurrence of $3\bar{5}241$ = occurrence of 3241 that cannot be extended to an occurrence of 35241

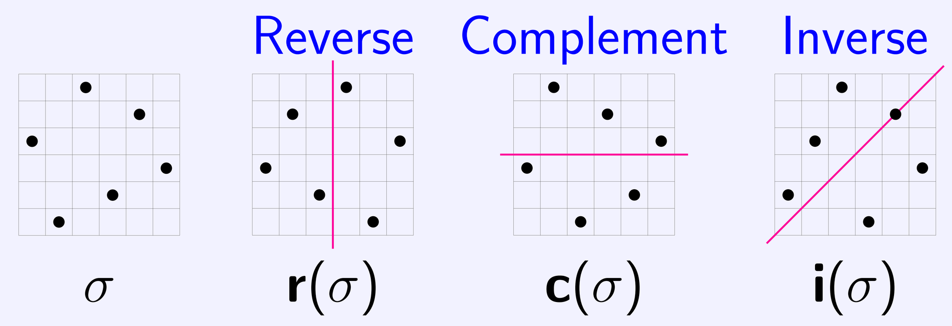
Mesh pattern: Stretched diagram with shaded cells

An occurrence of a mesh pattern is a set of points matching the diagram while leaving zones empty.

Example: μ = [diagram] is a pattern of σ = [diagram].

Symmetries

Symmetries of the square transform permutations *via* their diagrams



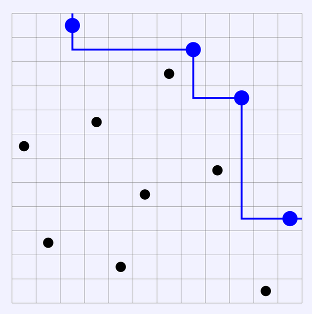
These operators generate an 8-element group:
$D_8 = \{\mathbf{id}, \mathbf{r}, \mathbf{c}, \mathbf{i}, \mathbf{r}\circ\mathbf{c}, \mathbf{i}\circ\mathbf{r}, \mathbf{i}\circ\mathbf{c}, \mathbf{i}\circ\mathbf{c}\circ\mathbf{r}\}$

Pattern avoidance

$\mathrm{Av}(\pi, \tau, \ldots)$ is the set of permutations that do not contain any occurrence of the (generalized) patterns $\pi, \tau, \ldots$

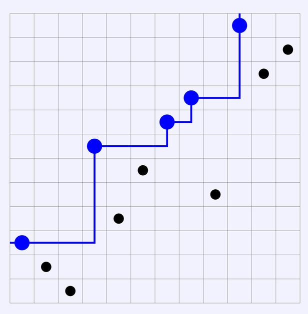
Some permutation statistics

Number of RtoL-max



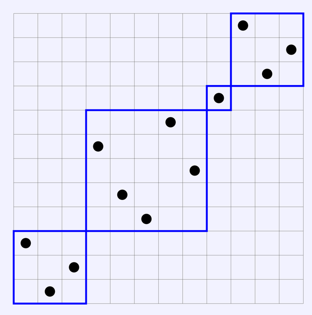
$\mathrm{rmax}(\sigma) = 4$

Number of LtoR-max



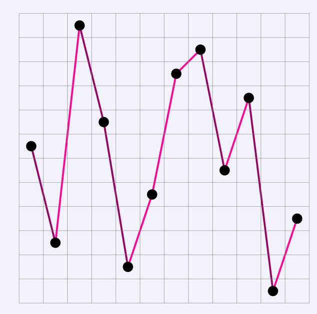
$\mathrm{lmax}(\sigma) = 5$

Number of components



$\mathrm{comp}(\sigma) = 4$

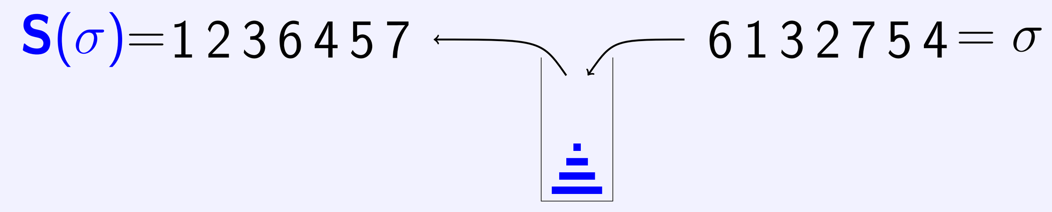
Up-down word



$\mathrm{udword}(\sigma) = dudduuudu$

Stack sorting

Algorithmic description: Try to sort with a stack satisfying the Hanoi condition



Equivalent recursive description:

$$\begin{cases} \mathbf{S}(\varepsilon) = \varepsilon \\ \mathbf{S}(LnR) = \mathbf{S}(L)\mathbf{S}(R)n \text{ where } n = \max(LnR) \end{cases}$$

Some previous results: about Stack Sorting, Permutation Patterns and Enumeration

Notation

For any sorting operator **Sort**, $\mathrm{Id}(\mathbf{Sort})$ denotes the set of permutations sorted by **Sort**

One-stack sortable permutations

$\mathrm{Id}(\mathbf{S}) = \mathrm{Av}(231)$

Enumeration by Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$ [Knuth 1973]

(West-)two-stack sortable permutations

$\mathrm{Id}(\mathbf{S}\circ\mathbf{S}) = \mathrm{Av}(2341, 3\bar{5}241)$

Enumeration by $\frac{2(3n)!}{(n+1)!(2n+1)!}$ [West 1993, Zeilberger 1992, ...]

First results: Characterization and Enumeration of Permutations Sorted by $\mathbf{S}\circ\alpha\circ\mathbf{S}$ for any $\alpha \in D_8$

Characterization with excluded patterns

$\mathrm{Id}(\mathbf{S}\circ\mathbf{S}) = \mathrm{Id}(\mathbf{S}\circ\mathbf{i}\circ\mathbf{c}\circ\mathbf{r}\circ\mathbf{S}) = \mathrm{Av}(2341, 3\bar{5}241)$

$\mathrm{Id}(\mathbf{S}\circ\mathbf{c}\circ\mathbf{S}) = \mathrm{Id}(\mathbf{S}\circ\mathbf{i}\circ\mathbf{r}\circ\mathbf{S}) = \mathrm{Av}(231)$

$\mathrm{Id}(\mathbf{S}\circ\mathbf{r}\circ\mathbf{S}) = \mathrm{Id}(\mathbf{S}\circ\mathbf{i}\circ\mathbf{c}\circ\mathbf{S}) = \mathrm{Av}(1342, 31\text{-}4\text{-}2)$

$\mathrm{Id}(\mathbf{S}\circ\mathbf{i}\circ\mathbf{S}) = \mathrm{Id}(\mathbf{S}\circ\mathbf{r}\circ\mathbf{c}\circ\mathbf{S}) = \mathrm{Av}(3412, 3\text{-}4\text{-}21)$

Enumeration of $\mathrm{Id}(\mathbf{S}\circ\mathbf{r}\circ\mathbf{S})$

The sets $\mathrm{Id}(\mathbf{S}\circ\mathbf{S})$ and $\mathrm{Id}(\mathbf{S}\circ\mathbf{r}\circ\mathbf{S})$ are enumerated by the same sequence $\left(\frac{2(3n)!}{(n+1)!(2n+1)!}\right)_n$

Enumeration of $\mathrm{Id}(\mathbf{S}\circ\mathbf{i}\circ\mathbf{S})$

The sets $\mathrm{Id}(\mathbf{S}\circ\mathbf{i}\circ\mathbf{S})$, Bax and TBax are enumerated by the Baxter numbers.

Bax = Av(2-41-3, 3-14-2) and TBax = Av(2-41-3, 3-41-2)

Enumerated by $Bax_n = \frac{2}{n(n+1)^2}\sum_{k=1}^{n}\binom{n+1}{k-1}\binom{n+1}{k}\binom{n+1}{k+1}$

Method of proof: Generating trees and rewriting systems

Generating tree for $\mathrm{Av}(\pi, \tau, \ldots)$

Infinite tree where

- Vertices at level n are permutations of $\mathfrak{S}_n$ avoiding $\pi, \tau, \ldots$
- Children are obtained by insertion of a new element in an active site

Sites are on one of the four sides of the diagram

Active site: when insertion does not create a pattern π or τ ...

Fact

Two classes having isomorphic generating trees are in bijection.

Rewriting system for $\mathrm{Av}(\pi, \tau, \ldots)$

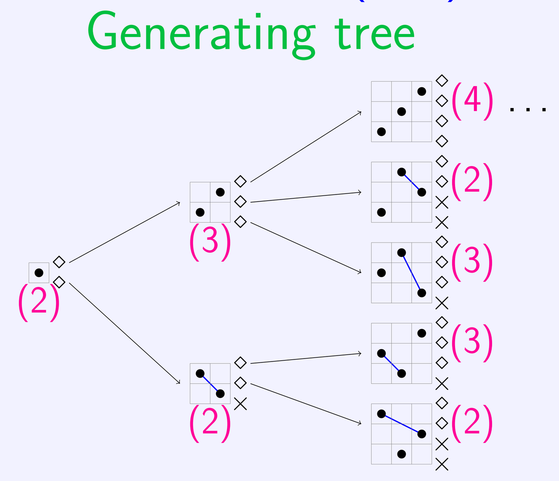
- Associate labels to permutations (e.g. number of active sites)
- Find a rule that describes the labels of the children of σ from the label of σ

Rewriting system encoding the tree =

- Label of permutation 1
- Succession rule(s) for the labels of the children

Example: Av(321) with insertion on the right

Generating tree



Active sites ◇ are the ones above all the inversions

Rewriting system

$$\begin{cases} (2) \\ (k) \rightsquigarrow (k+1)(2)(3)\ldots(k) \end{cases}$$

Proof:

- Labels record the number of sites above all the inversions
- Insertion in the topmost site creates one new active site
- Insertion in any other site creates an inversion with $\max(\sigma)$

Generating tree and rewriting system for $\mathrm{Id}(\mathbf{S}\circ\mathbf{r}\circ\mathbf{S})$

Common rewriting system for $\mathrm{Id}(\mathbf{S}\circ\mathbf{S})$ and $\mathrm{Id}(\mathbf{S}\circ\mathbf{r}\circ\mathbf{S})$

$$\mathcal{R}_\Phi \begin{cases} (2,1,(1)) \\ (x,k,(p_1,\ldots,p_k)) \rightsquigarrow (2+p_j, j, (p_1,\ldots,p_{j-1},i)) \text{ for } 1\le j\le k \text{ and } p_{j-1} < i \le p_j \\ \qquad\qquad (x+1, k+1, (p_1,\ldots,p_k,i)) \text{ for } p_k < i \le x \end{cases}$$

x = the number of active sites of σ, $\quad k$ = the number of RtoL-max in σ

p_ℓ = the number of active sites above the ℓ-th RtoL-max in σ

Generating tree and rewriting system for $\mathrm{Id}(\mathbf{S}\circ\mathbf{i}\circ\mathbf{S})$

Common rewriting system for $\mathrm{Id}(\mathbf{S}\circ\mathbf{i}\circ\mathbf{S})$ and TBax

$$\mathcal{R}_\Psi \begin{cases} (2,0) \\ (q,r) \rightsquigarrow (i+1, q+r-i) \text{ for } 1\le i\le q \\ \qquad\quad (q, r-j) \text{ for } 1\le j\le r \end{cases}$$

Elements are inserted below and to the right of the diagram respectively.

$q + r$ = the number of active sites of σ

Further results: Refined Enumeration According to Permutation Statistics

Statistics preserved by the bijection between $\mathrm{Id}(\mathbf{S}\circ\mathbf{r}\circ\mathbf{S})$ and $\mathrm{Id}(\mathbf{S}\circ\mathbf{S})$

Bijection Φ between $\mathrm{Id}(\mathbf{S}\circ\mathbf{r}\circ\mathbf{S})$ and $\mathrm{Id}(\mathbf{S}\circ\mathbf{S})$ preserves the statistics udword, rmax, lmax, zeil, indmax, slmax and slmax ∘r. Consequently, asc, des, maj, maj ∘r, maj ∘c, maj ∘rc, valley, peak, ddes, dasc, rir, rdr, lir, ldr are also preserved.

Hence, these statistics are all jointly equidistributed.

Proof: Plug each of these statistics in the common rewriting system

Statistics preserved by the bijection between $\mathrm{Id}(\mathbf{S}\circ\mathbf{i}\circ\mathbf{S})$ and Bax

des, lmax and comp are jointly equidistributed on $\mathrm{Id}(\mathbf{S}\circ\mathbf{i}\circ\mathbf{S})$ and on Bax.

$\mathrm{Id}(\mathbf{S}\circ\mathbf{i}\circ\mathbf{S})$	$\overset{\Psi}{\longleftrightarrow}$	TBax	$\overset{\text{[Giraudo 2011]}}{\longleftrightarrow}$	Pairs of twin binary trees	$\overset{\text{[Giraudo 2011]}}{\longleftrightarrow}$	Bax
lmax	$\longleftrightarrow$	lmax	$\longleftrightarrow$	length of rightmost branch	$\longleftrightarrow$	lmax
des	$\longleftrightarrow$	occ_μ	$\longleftrightarrow$	number of left edges	$\longleftrightarrow$	des
comp	$\longleftrightarrow$	comp	$\longleftrightarrow$	?	$\longleftrightarrow$	comp