Minimal transitive factorizations of a permutation of type (p, q)

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Given $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell) \vdash n$,

$$\alpha_\lambda = (1 \dots \lambda_1)(\lambda_1 + 1 \dots \lambda_1 + \lambda_2) \dots (n - \lambda_\ell + 1 \dots n).$$

$\mathcal{F}_\lambda :=$ the set of all $(n + \ell - 2)$-tuples $(\eta_1, \dots, \eta_{n+\ell-2})$ of transpositions such that

1. $\eta_1 \cdots \eta_{n+\ell-2} = \alpha_\lambda$
2. $\langle \eta_1, \dots, \eta_{n+\ell-2} \rangle = \mathcal{S}_n$.

Such tuples are called minimal transitive factorizations of α_λ, which are related to the branched covers of the sphere suggested by Hurwitz.

$(1\,4)(1\,3)(1\,2)$	$(2\,3)(1\,4)(1\,3)$
$(1\,4)(1\,2)(2\,3)$	$(2\,4)(1\,4)(2\,3)$
$(1\,3)(1\,2)(3\,4)$	$(2\,3)(1\,3)(3\,4)$
$(1\,4)(2\,3)(1\,3)$	$(2\,4)(2\,3)(1\,4)$
$(1\,2)(2\,4)(2\,3)$	$(2\,3)(3\,4)(1\,4)$
$(1\,2)(2\,3)(3\,4)$	$(3\,4)(1\,4)(1\,2)$
$(1\,3)(3\,4)(1\,2)$	$(3\,4)(1\,2)(2\,4)$
$(1\,2)(3\,4)(2\,4)$	$(3\,4)(2\,4)(1\,4)$

Table: The elements of $\mathcal{F}_{(4)}$ where $\alpha_{(4)} = (1\ 2\ 3\ 4)$

Question.

Find the cardinality of $\mathcal{F}_\lambda$.

1. Goulden and Jackson (1997) proved that if $(\lambda_1, \ldots, \lambda_\ell) \vdash n$,

$$|\mathcal{F}_{(\lambda_1,\ldots,\lambda_\ell)}| = (n+\ell-2)!\, n^{\ell-3} \prod_{i=1}^{\ell} \frac{\lambda_i^{\lambda_i}}{(\lambda_i-1)!}.$$

   Their proof is done for arbitrary λ, but this is not bijective.

2. Bousquet-Mélou and Schaeffer (2000) obtained the above formula by using the inclusion-exclusion principle.

Known bijective proofs

λ	$\lvert\mathcal{F}_\lambda\rvert$	bijective proof
(n)	n^{n-2}	Biane et al.
$(1, n-1)$	$(n-1)^n$	Kim-Seo (2003)
$(2, n-2)$	$4(n-1)(n-2)^{n-1}$	Seo(2004), Rattan (2006)
$(3, n-3)$	$\frac{27}{2}(n-1)(n-2)(n-3)^{n-2}$	Rattan (2006)

Enumeration of the Case $\lambda = (p, q)$

Recall that Goulden and Jackson's formula is

$$|\mathcal{F}_{(\lambda_1,\dots,\lambda_\ell)}| = (n+\ell-2)!\, n^{\ell-3} \prod_{i=1}^{\ell} \frac{\lambda_i^{\lambda_i}}{(\lambda_i-1)!}.$$

In case of $\lambda = (p, q)$,

$$|\mathcal{F}_{(p,q)}| = \frac{pq}{p+q}\binom{p+q}{p} p^p q^q.$$

Signed permutation

1. A signed permutation is a permutation σ on $\{\pm 1, \dots, \pm n\}$ satisfying $\sigma(-i) = -\sigma(i)$ for all $i \in \{1, \dots, n\}$.
2. The hyperoctohedral group B_n is the group of signed permutations on $\{\pm 1, \dots, \pm n\}$.
3. We will use the two notations

$$[a_1\ a_2 \dots a_k] = (a_1\ a_2 \dots a_k\ \ -a_1\ \ -a_2 \dots -a_k), \text{ zero cycle}$$
$$((a_1\ a_2 \dots a_k)) = (a_1\ a_2 \dots a_k)(-a_1\ \ -a_2 \dots -a_k), \text{ paired nonzero cycle}$$

4. $\epsilon_i := [i] = (i\ \ -i)$ and $((i\, j))$, transpositions of type B, satisfies

$$\epsilon_i((i\, j)) = ((i\, j))\epsilon_j = ((i\ \ -j))\epsilon_i = \epsilon_j((i\ \ -j))$$

1. The absolute order on B_n is

$$\pi \leq \sigma \quad \stackrel{def}{\Longleftrightarrow} \quad \ell(\sigma) = \ell(\pi) + \ell(\pi^{-1}\sigma),$$

   where $\ell(\pi)$ is the *absolute length* for $\pi \in B_n$.

2. The poset $\mathcal{S}_{\text{nc}}^{B}(p,q)$ of annular noncrossing permutations of type B is defined by the interval poset of B_{p+q} as follows:

$$\mathcal{S}_{\text{nc}}^{B}(p,q) := [\epsilon, \gamma_{p,q}] = \{\sigma \in B_{p+q} : \epsilon \leq \sigma \leq \gamma_{p,q}\} \subseteq B_{p+q},$$

   where ϵ is the identity and $\gamma_{p,q} = [1 \ldots p][p+1 \ldots p+q]$.

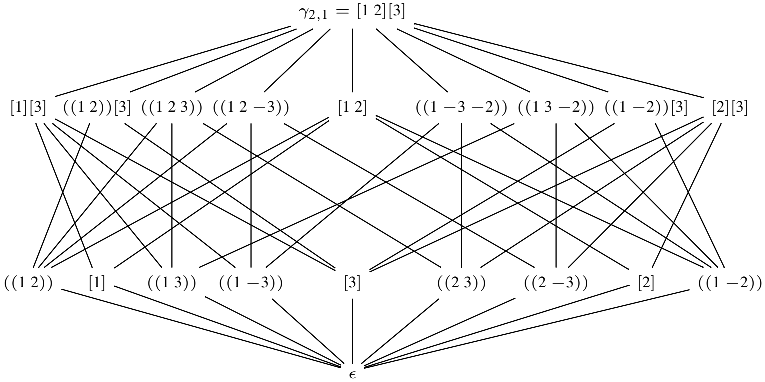
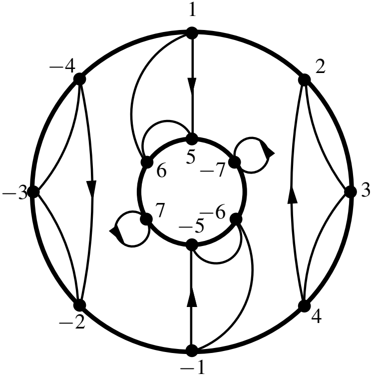


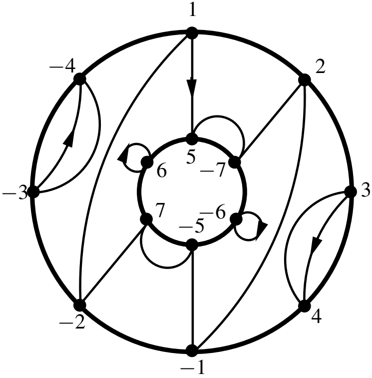
Figure: The Hasse diagram for $\mathcal{S}_{\mathrm{nc}}^{B}(2,1)$.

Drawing permutations on annulus with noncrossing arrows

$\pi = ((1\ 5\ 6))((2\ 3\ 4)) \in \mathcal{S}_{\mathrm{nc}}^{B}(4,3)$



$\sigma = [1\ 5\ {-7}\ 2]((3\ 4)) \in \mathcal{S}_{\mathrm{nc}}^{B}(4,3)$



1. Nica and Oancea (2009) showed that $\mathcal{S}^B_{\mathrm{nc}}(p,q)$ is poset-isomorphic to $NC^{(B)}(p,q)$ of annular noncrossing partitions of type B.
2. Goulden-Nica-Oancea (2011) showed that the number of maximal chains in the poset $NC^{(B)}(p,q)$ is

$$\binom{p+q}{q} p^p q^q + \sum_{c\geq 1} 2c \binom{p+q}{p-c} p^{p-c} q^{q+c}.$$

3. It turns out that half of the 2nd term is equal to $\left|\mathcal{F}_{(p,q)}\right|$.

$$\sum_{c\geq 1} c \binom{p+q}{p-c} p^{p-c} q^{q+c} = \frac{pq}{p+q} \binom{p+q}{q} p^p q^q.$$

Connectivity

1. A paired nonzero cycle $((a_1\ a_2 \dots a_k))$ touching both the interier and exterior circles is called connected.
2. A signed permutation with at least one connected paired nonzero cycle is called connected.
3. A maximal chain of $\mathcal{S}_{\mathrm{nc}}^{B}(p,q)$ with at least one connected signed permutation is called connected.

Proposition

The number of disconnected maximal chains of $\mathcal{S}_{\mathrm{nc}}^{B}(p,q)$ is equal to

$$\binom{p+q}{q} p^p q^q$$

and the number of connected maximal chains of $\mathcal{S}_{\mathrm{nc}}^{B}(p,q)$ is equal to

$$2\frac{pq}{p+q}\binom{p+q}{q} p^p q^q.$$

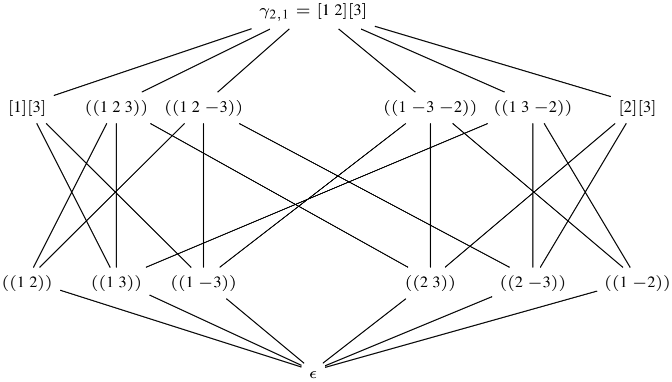


Figure: Connected maximal chains in $\mathcal{S}^{B}_{\mathrm{nc}}(2,1)$.

Theorem (Kim-Seo-Shin, 2012)

There is a 2-1 map from the set $\mathcal{CM}(\mathcal{S}_{\mathrm{nc}}^{B}(p,q))$ *of connected maximal chains in* $\mathcal{S}_{\mathrm{nc}}^{B}(p,q)$ *to the set* $\mathcal{F}_{(p,q)}$ *of minimal transitive factorizations of* $\alpha_{p,q}$*.*

Proof.

The composition of three maps $|\varphi^{+}| := |\cdot| \circ (\cdot)^{+} \circ \varphi$

$$\mathcal{CM}(\mathcal{S}_{\mathrm{nc}}^{B}(p,q)) \xrightarrow[\text{1-1}]{\varphi} \mathcal{F}_{(p,q)}^{(B)} \xrightarrow[\text{2-1}]{(\cdot)^{+}} \mathcal{F}_{(p,q)}^{+} \xrightarrow[\text{1-1}]{|\cdot|} \mathcal{F}_{(p,q)}$$

is the desired 2-1 map. □

Description of maps

$$
\begin{array}{ccc}
\{\epsilon < ((1\ 3)) < ((1\ 3\ -2)) < [1\ 2][3]\} & \in & \mathcal{CM}(\mathcal{S}^B_{\text{nc}}(2,1)) \\
\Big\downarrow {\scriptstyle \tau_i = {\pi_{i-1}}^{-1}\pi_i} & & \Big\downarrow {\scriptstyle \varphi} \\
(\ ((1\ 3)), ((2\ \ -3)), ((1\ \ -3))\ ) & \in & \mathcal{F}^{(B)}_{(2,1)} \\
\Big\downarrow {\scriptstyle \sigma_i = {\tau_i}^{+}} & & \Big\downarrow {\scriptstyle (\cdot)^{+}} \\
(\ ((1\ 3)), ((2\ 3)), ((1\ 3))\ ) & \in & \mathcal{F}^{+}_{(2,1)} \\
\Big\downarrow {\scriptstyle \eta_i = |\sigma_i|} & & \Big\downarrow {\scriptstyle |\cdot|} \\
(\ (1\ 3), (2\ 3), (1\ 3)\ ) & \in & \mathcal{F}_{(2,1)}
\end{array}
$$

Why is the map $(\cdot)^+$ surjective?

Given a minimal transitive factorization of $\beta_{3,2} = ((1\ 2\ 3))((4\ 5))$

$$(\ ((1\ 2)), ((2\ 5)), ((2\ 3)), ((4\ 5)), ((3\ 4))\ ) \in \mathcal{F}^{+}_{(3,2)},$$

since $\gamma_{3,2} = [1\ 2\ 3][4\ 5] = \epsilon_4\ \epsilon_1\ \beta_{3,2}$,

$$\begin{aligned}
\gamma_{3,2} &= \epsilon_4\ \epsilon_1\ ((1\ 2))\ ((2\ 5))\ ((2\ 3))\ ((4\ 5))\ ((3\ 4)) \\
&= \epsilon_4\ \epsilon_2\ ((1\ \ -2))\ ((2\ 5))\ ((2\ 3))\ ((4\ 5))\ ((3\ 4)) \\
&= \epsilon_4\ \epsilon_3\ ((1\ 2))\ ((2\ \ -5))\ ((2\ \ -3))\ ((4\ 5))\ ((3\ 4)) \\
&= \epsilon_4\ \epsilon_4\ ((1\ 2))\ ((2\ \ -5))\ ((2\ 3))\ ((4\ \ -5))\ ((3\ \ -4)) \\
&= ((1\ 2))\ ((2\ \ -5))\ ((2\ 3))\ ((4\ \ -5))\ ((3\ \ -4)),
\end{aligned}$$

we have one minimal transitive factorization of $\gamma_{3,2}$

$$(\ ((1\ 2)), ((2\ \ -5)), ((2\ 3)), ((4\ \ -5)), ((3\ \ -4))\ ) \in \mathcal{F}^{(B)}_{(3,2)}.$$

Why is the map $(\cdot)^+$ two-to-one?

$$(\tau_1, \tau_2, \tau_3, \tau_4, \tau_5) \;=\; (\;((1\ 2)), ((2\ \ -5)), ((2\ 3)), ((4\ \ -5)), ((3\ \ -4))\;) \in \mathcal{F}^{(B)}_{(3,2)}$$

$$\tau_i' = \begin{cases} \tau_i & \text{if } \tau_i \text{ is disconnected} \\ ((a\ \ -b)) & \text{if } \tau_i = ((a\ b)) \text{ is connected.} \end{cases}$$

$$(\tau_1', \tau_2', \tau_3', \tau_4', \tau_5') \;=\; (\;((1\ 2)), ((2\ 5)), ((2\ 3)), ((4\ \ -5)), ((3\ 4))\;) \in \mathcal{F}^{(B)}_{(3,2)}$$

It satisfies

$$(\tau_1, \tau_2, \tau_3, \tau_4, \tau_5)^+ = (\tau_1', \tau_2', \tau_3', \tau_4', \tau_5')^+ \in \mathcal{F}^+_{(3,2)}.$$

Summary

λ	$\lvert\mathcal{F}_\lambda\rvert$	bijective proof
(n)	n^{n-2}	Dénes et al.
$(1, n-1)$	$(n-1)^n$	Kim-Seo (2003)
$(2, n-2)$	$4(n-1)(n-2)^{n-1}$	Seo(2004), Rattan (2006)
$(3, n-3)$	$\frac{27}{2}(n-1)(n-2)(n-3)^{n-2}$	Rattan (2006)
(p, q)	$\frac{pq}{p+q}\binom{p+q}{p}p^p q^q$	Kim-Seo-Shin (2012)
(p, q, r)	$(p+q+r+1)pqr\binom{p+q+r}{p,q,r}p^p q^q r^r$	open
(1^n)	$(2n-2)!n^{n-3}$	open