Proofs of two conjectures of Kenyon and Wilson on Dyck tilings

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- **matching** on $\{1, 2, \ldots, 2n\}$

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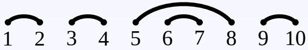

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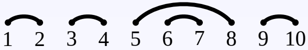
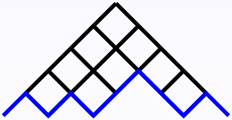


- **Dyck path** of length $2n \leftrightarrow$ **partition** contained in $(n-1, n-2, \ldots, 1)$

$\leftrightarrow$



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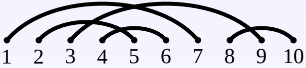

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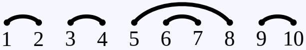

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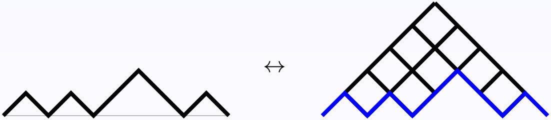

- These objects are counted by Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.

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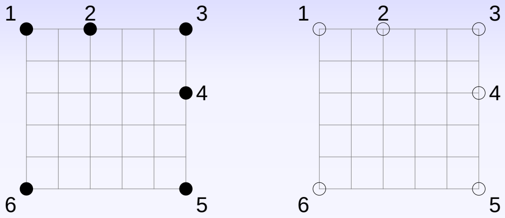

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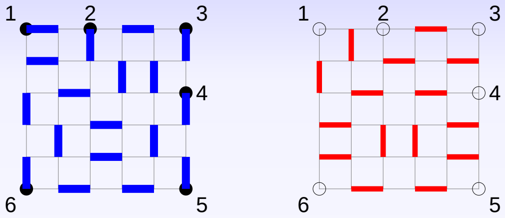

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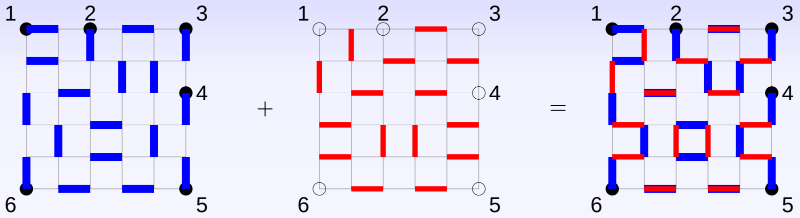

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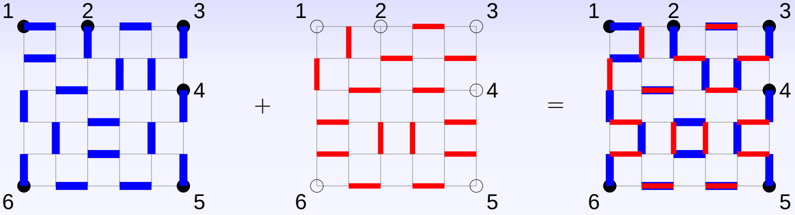

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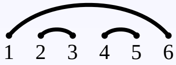

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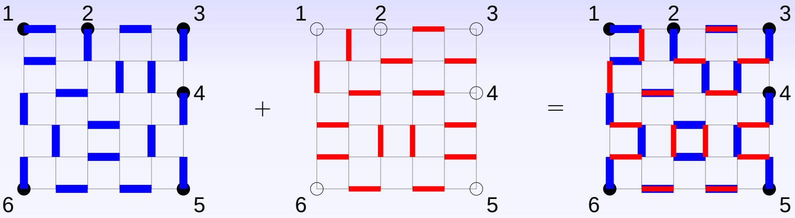

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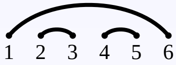

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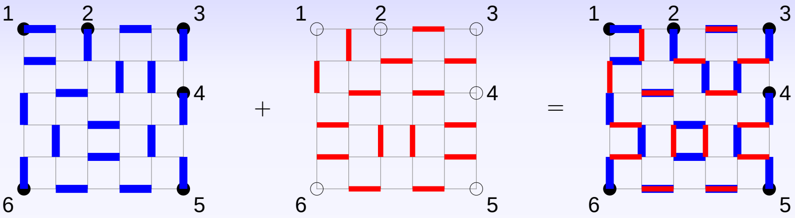

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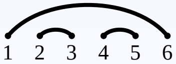

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- M: matrix whose rows and columns are indexed by Dyck paths of length $2n$:

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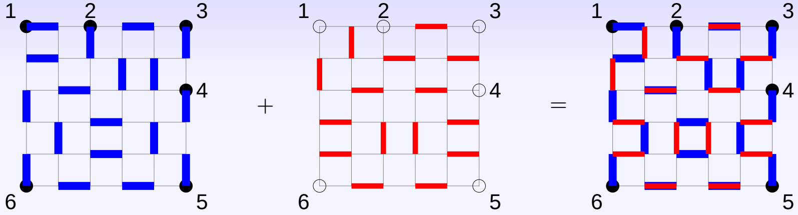

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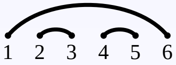

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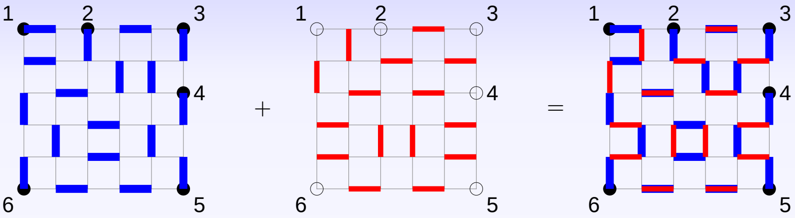

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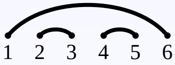

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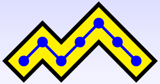
Theorem (Kenyon and Wilson, 2010)

$$(M^{-1})_{\lambda,\mu} = (-1)^{|\lambda/\mu|} \times (\text{\# \textit{cover-inclusive Dyck tilings of shape} } \lambda/\mu)$$

(cover-inclusive) Dyck tilings

- Dyck tile 

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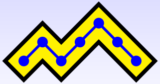
- Dyck tile 
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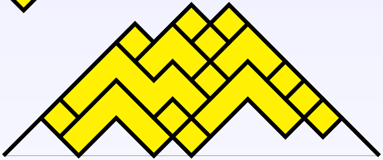

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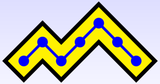
- Dyck tile 
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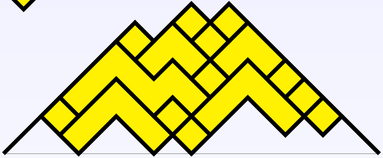

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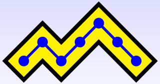
- Dyck tile 
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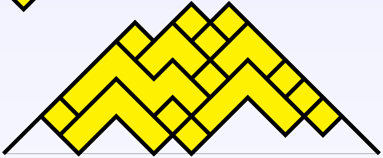
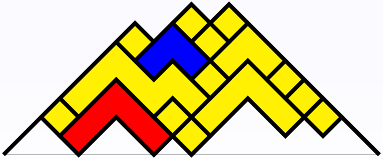
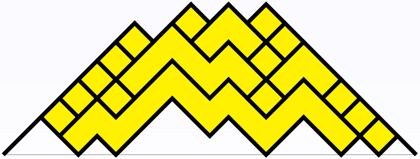


- Cell A is **covered** by cell B:

- cover-inclusive Dyck tiling: if tile A has a cell covered by a cell of tile B, then A is **completely covered** by B.





Notations

- $\mathcal{D}(\lambda/\mu)$: set of Dyck tilings of shape λ/μ

$$\mathcal{D}(\lambda/*) = \bigcup_{\nu \in \mathrm{Dyck}(2n)} \mathcal{D}(\lambda/\nu),$$
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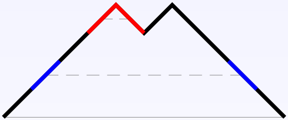
- Find q-analogs of $|\mathcal{D}(\lambda/*)|$ and $|\mathcal{D}(*/\mu)|$: **Kenyon and Wilson's conjectures**

Chords of Dyck paths

- A **chord** is a matching pair of up step and down step



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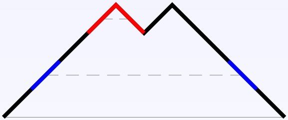
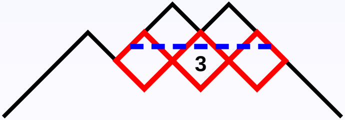
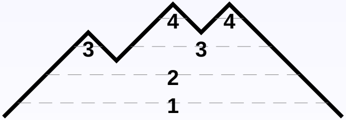


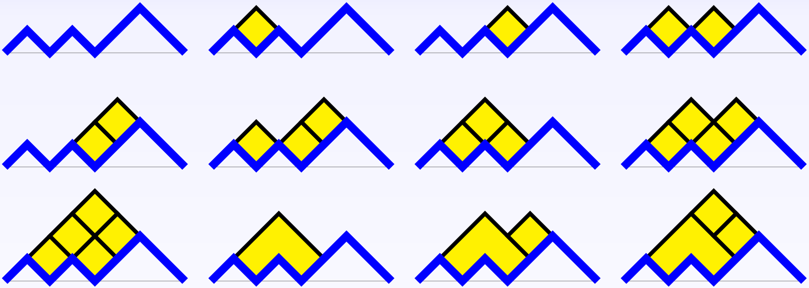
- The **length** $|c|$ and the **height** $\mathrm{ht}(c)$ are defined as follows:





$\mathcal{D}(\lambda/*)$: fixed lower path

- There are 12 Dyck tilings with fixed lower path



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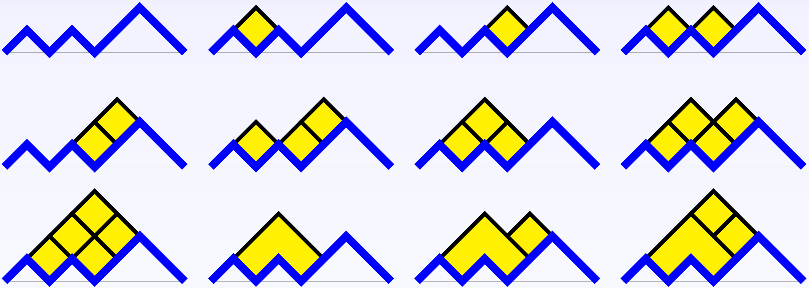
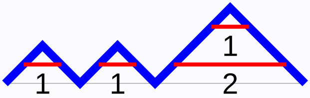


- The fixed lower path has half length $n = 4$ with chords of length 1,1,1,2



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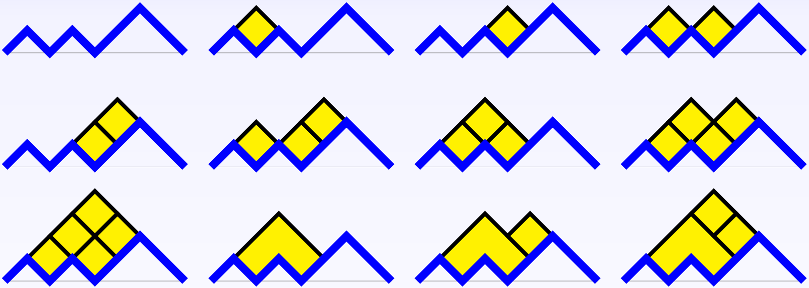

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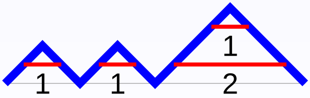

- $12 = \dfrac{4!}{1 \cdot 1 \cdot 1 \cdot 2} = \dfrac{n!}{\prod_{c \in \mathrm{Chord}(\lambda)} |c|}$

(weak) Conjecture 1 of KW

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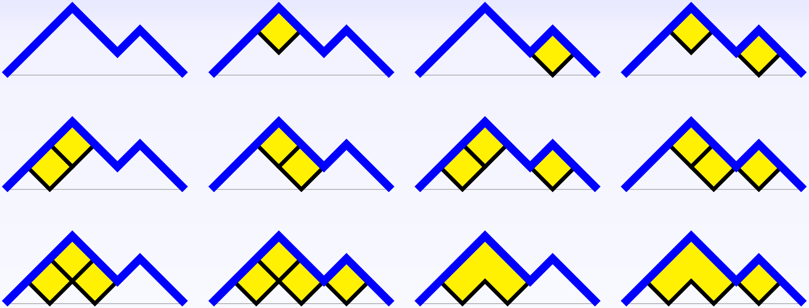

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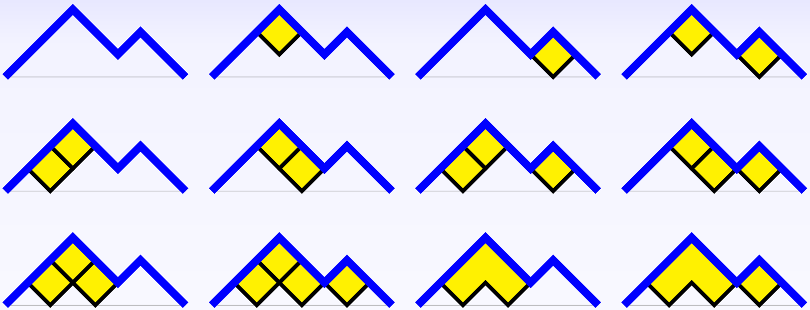

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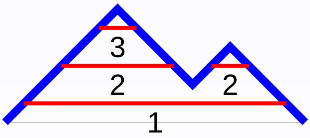

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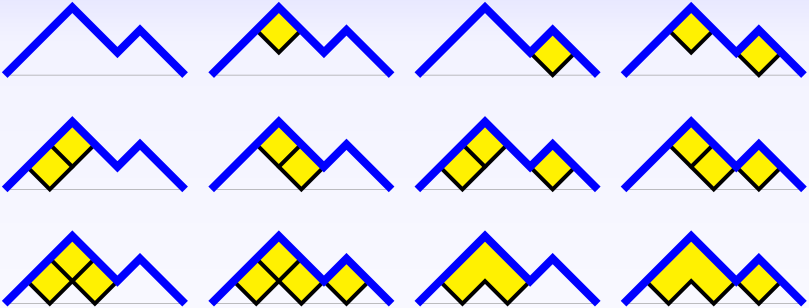

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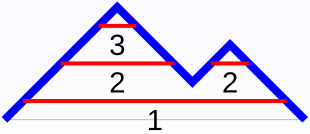

- $12 = 3 \cdot 2 \cdot 2 \cdot 1 = \prod_{c \in \mathrm{Chord}(\mu)} \mathrm{ht}(c)$ **(weak) Conjecture 2 of KW**

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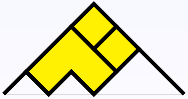

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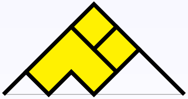

- The usual q-integers, q-factorials:

$$[n]_q = 1 + q + \cdots + q^{n-1}, \qquad [n]_q! = [1]_q\,[2]_q \cdots [n]_q\,.$$

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- Conjecture 2 has been proved bijectively by Kim and Konvalinka independently.

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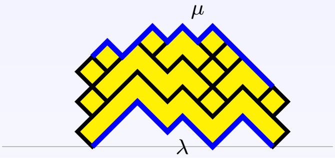

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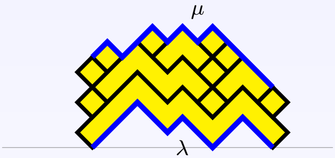

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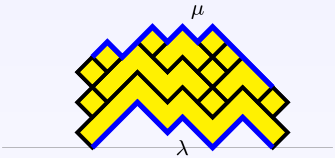

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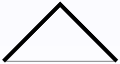
- Δ_n: the highest path = 

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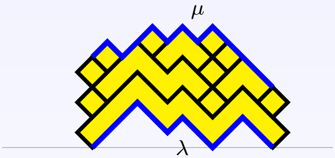

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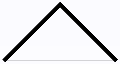
- Δ_n: the highest path = 
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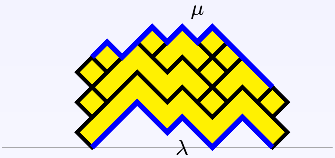

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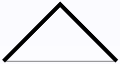
- Δ_n: the highest path = 

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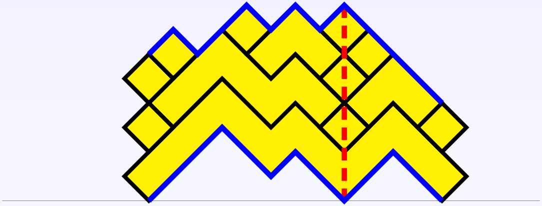
Why are the generalized Dyck tilings easier?



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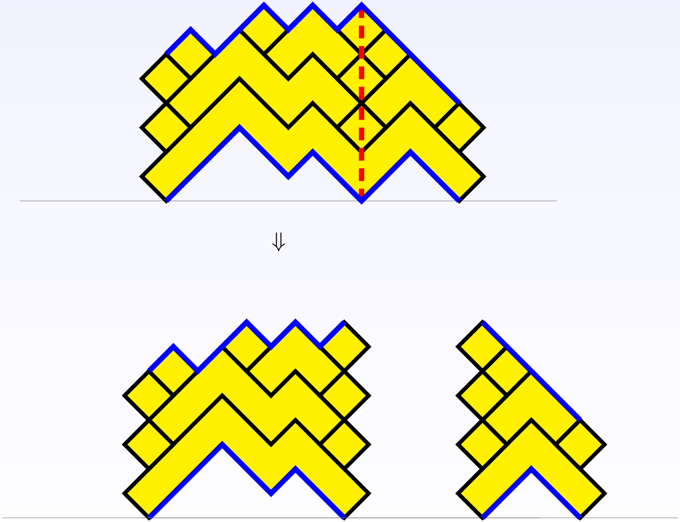

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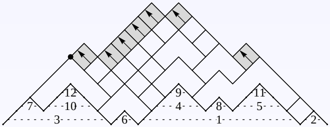

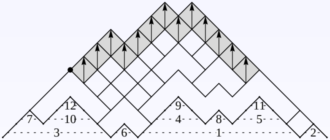

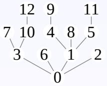

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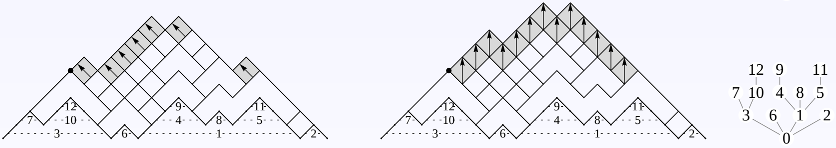

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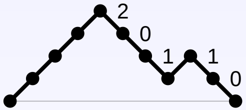

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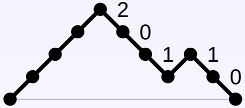

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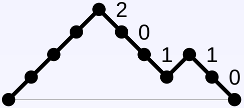

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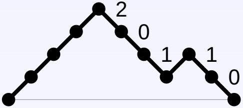

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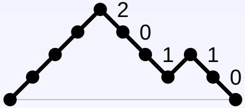

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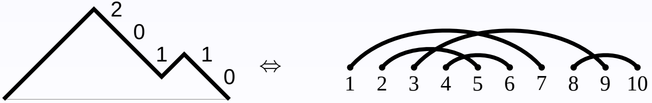

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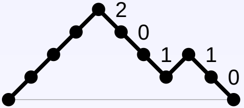

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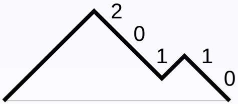

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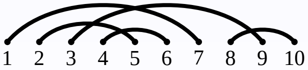

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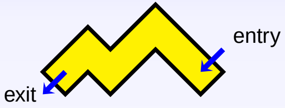
A bijection between Dyck tilings and Hermite histories

- The **entry** and the **exit** of a Dyck tile are defined:



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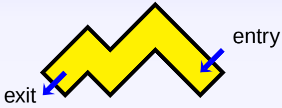

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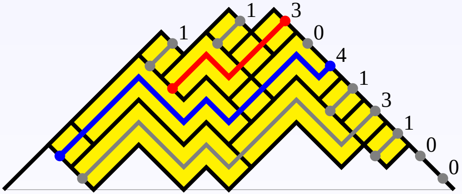

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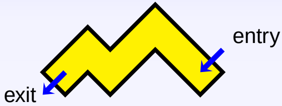

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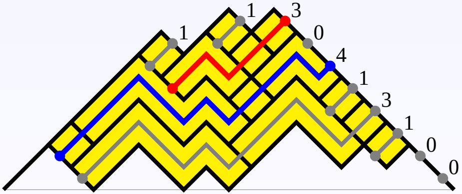

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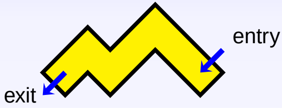

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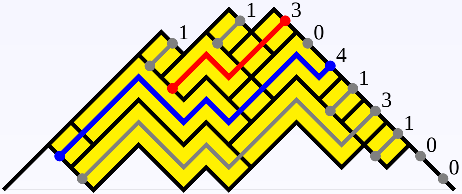

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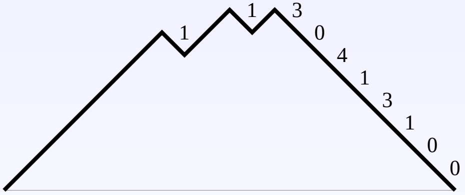

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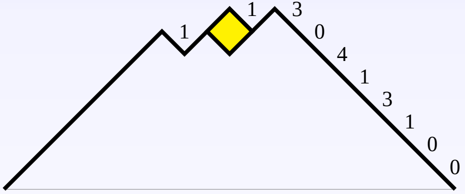

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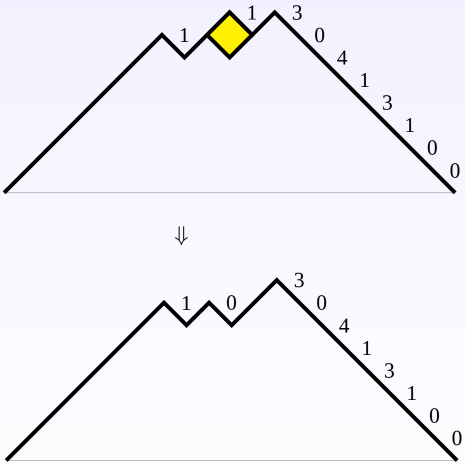

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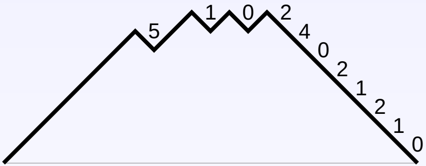

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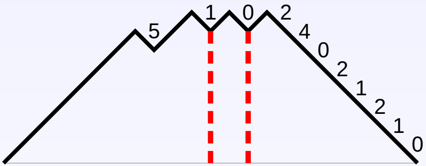

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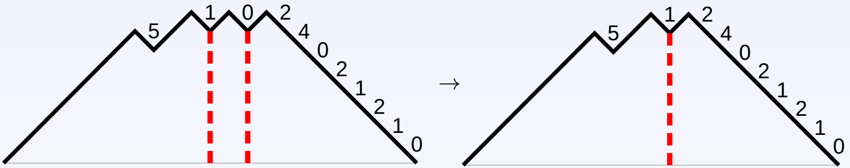

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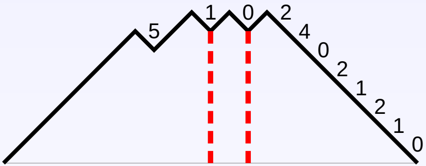

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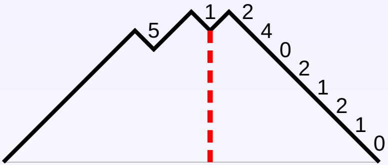

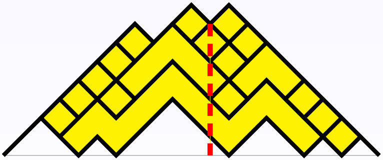

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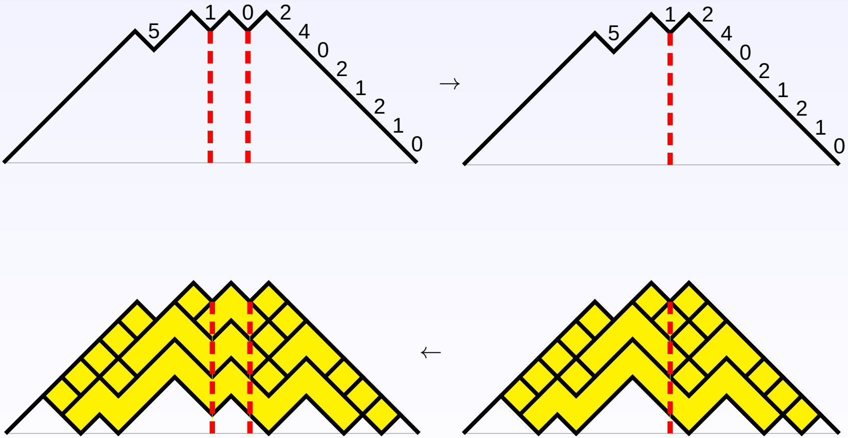

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- Bij_0 sends **upper path** μ to shape (of matching), and **tiles** to **crossings**

$$\sum_{T \in \mathcal{D}(*/\mu)} q^{\mathrm{tiles}(T)} = \prod_{c \in \mathrm{Chord}(\mu)} [\mathrm{ht}(c)]_q$$

- Bij_1 sends **lower path** λ to shape (of plane tree), and **art** to **inv**

$$\sum_{T \in \mathcal{D}(\lambda/*)} q^{\mathrm{art}(T)} = \frac{[n]_q!}{\prod_{c \in \mathrm{Chord}(\lambda)} [|c|]_q}$$

- When lower path $\lambda = \mathrm{zigzag}_\mathrm{n}$
  - Dyck tilings = Dyck tableaux (Aval, Boussicault, Dasse-Hartaut)
  - increasing ordered trees = permutations
  - Bij_1 sends **art** to **inv**

Summery

- There are three bijections on Dyck tilings.
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Thank you for your attention!