From KL polynomials to Khovanov algebras

Catharina Stroppel

(Bonn/Chicago)

From KL polynomials to Khovanov algebras

Catharina Stroppel

(Bonn/Chicago)

What are (parabolic) Kazhdan-Lusztig polynomials?

$$x, y \in S_n \rightsquigarrow p_{x,y} \in \mathbb{Z}[q, q^{-1}]$$

What are (parabolic) Kazhdan-Lusztig polynomials?

$$x, y \in S_n \rightsquigarrow p_{x,y} \in \mathbb{Z}[q, q^{-1}]$$

entries of the (inverse) base change matrix in $\mathbb{C}[q, q^{-1}][S_n]$

standard basis

What are (parabolic) Kazhdan-Lusztig polynomials?

$$x, y \in S_n \rightsquigarrow p_{x,y} \in \mathbb{Z}[q, q^{-1}]$$

entries of the (inverse) base change matrix in $\mathbb{C}[q, q^{-1}][S_n]$

standard basis $\rightsquigarrow$ $KL - basis$

What are (parabolic) Kazhdan-Lusztig polynomials?

$$x, y \in S_n \rightsquigarrow p_{x,y} \in \mathbb{Z}[q, q^{-1}]$$

entries of the (inverse) base change matrix in $\mathbb{C}[q, q^{-1}][S_n]$

standard basis $\rightsquigarrow$ $KL - basis$

$w \in S_n$

What are (parabolic) Kazhdan-Lusztig polynomials?

$$x, y \in S_n \rightsquigarrow p_{x,y} \in \mathbb{Z}[q, q^{-1}]$$

entries of the (inverse) base change matrix in $\mathbb{C}[q, q^{-1}][S_n]$

standard basis	$\rightsquigarrow$	$KL - basis$
$w \in S_n$		$B_w, w \in S_n$

What are (parabolic) Kazhdan-Lusztig polynomials?

$$x, y \in S_n \rightsquigarrow p_{x,y} \in \mathbb{Z}[q, q^{-1}]$$

entries of the (inverse) base change matrix in $\mathbb{C}[q, q^{-1}][S_n]$

standard basis	$\rightsquigarrow$	$KL - basis$
$w \in S_n$		$B_w, w \in S_n$

- Defined inductively starting from $B_e = e$,

What are (parabolic) Kazhdan-Lusztig polynomials?

$$x, y \in S_n \rightsquigarrow p_{x,y} \in \mathbb{Z}[q, q^{-1}]$$

entries of the (inverse) base change matrix in $\mathbb{C}[q, q^{-1}][S_n]$

$$\begin{array}{ccc} \text{standard basis} & \rightsquigarrow & KL-basis \\ w \in S_n & & B_w, w \in S_n \end{array}$$

- Defined inductively starting from $B_e = e$,
- Special case: $B_s = s + q$ for any simple transposition $s = (i, i+1)$,

What are (parabolic) Kazhdan-Lusztig polynomials?

$$x, y \in S_n \rightsquigarrow p_{x,y} \in \mathbb{Z}[q, q^{-1}]$$

entries of the (inverse) base change matrix in $\mathbb{C}[q, q^{-1}][S_n]$

$$\begin{array}{ccc} \text{standard basis} & \rightsquigarrow & KL-basis \\ w \in S_n & & B_w, w \in S_n \end{array}$$

- Defined inductively starting from $B_e = e$,
- Special case: $B_s = s + q$ for any simple transposition $s = (i, i+1)$,
- using the formulas

What are (parabolic) Kazhdan-Lusztig polynomials?

$$x, y \in S_n \rightsquigarrow p_{x,y} \in \mathbb{Z}[q, q^{-1}]$$

entries of the (inverse) base change matrix in $\mathbb{C}[q, q^{-1}][S_n]$

standard basis	$\rightsquigarrow$	$KL - basis$
$w \in S_n$		$B_w, w \in S_n$

- Defined inductively starting from $B_e = e$,
- Special case: $B_s = s + q$ for any simple transposition $s = (i, i+1)$,
- using the formulas

$$wB_s = \begin{cases} ws + qw & \text{if } \ell(ws) > \ell(w) \\ ws + q^{-1}w & \text{if } \ell(ws) < \ell(w) \end{cases} \tag{1}$$

What are (parabolic) Kazhdan-Lusztig polynomials?

$$x, y \in S_n \rightsquigarrow p_{x,y} \in \mathbb{Z}[q, q^{-1}]$$

entries of the (inverse) base change matrix in $\mathbb{C}[q, q^{-1}][S_n]$

standard basis	$\rightsquigarrow$	$KL - basis$
$w \in S_n$		$B_w, w \in S_n$

- Defined inductively starting from $B_e = e$,
- Special case: $B_s = s + q$ for any simple transposition $s = (i, i+1)$,
- using the formulas

$$wB_s = \begin{cases} ws + qw & \text{if } \ell(ws) > \ell(w) \\ ws + q^{-1}w & \text{if } \ell(ws) < \ell(w) \end{cases} \qquad (1)$$

and subtract possibly already constructed basis elements...

The example S_3

KL polynomials (in basis $e, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1$)

$$\begin{pmatrix} 1 & q & q & q^2 & q^2 & q^3 \\ 0 & 1 & 0 & q & q & q^2 \\ 0 & 0 & 1 & q & q & q^2 \\ 0 & 0 & 0 & 1 & 0 & q \\ 0 & 0 & 0 & 0 & 1 & q \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

In fact: KL-polys are in $\mathbb{N}[q]$

Lie theoretic explanation/origin: They describe multiplicities of simple highest weight modules in Jordan-Hölder series of Verma modules $M(x) = U(\mathfrak{gl}_3) \otimes_{U(\text{upper triang. matrices})} \mathbb{C}_{x(3,2,1)}$

$$[M(y) : L(x)\langle i\rangle] = \text{coefficient of } q^i \text{ in } p_{x,y}.$$

Parabolic versions

Choose a Young subgroup $W = S_{i_1} \times S_{i_2} \times \cdots \times S_{i_r}$ of S_n (generated by some subset of simple transpositions)
$x, y \in W \backslash S_n$ representatives of minimal length
$\rightsquigarrow$ parabolic KL-polynomial $p_{x,y}$ with the slightly adjusted rules

$$wB_s = \begin{cases} ws + qw & \text{if } l(ws) > l(w) \\ ws + q^{-1w} & \text{if } l(ws) < l(w) \\ 0 & \text{if } ws \text{ not a minimal length representative} \end{cases}$$

Example: $B_e = e$, $B_{s_1} = s_1 + qe$, $B_{s_1 s_2} = s_1 s_2 + q s_1$.

From now on: special case: $W = S_i \times S_{n-i}$

- Closed formulas and well-studied
  (Boe, Brenti - Dyck paths, Billey/Warrington - 321-avoiding)
- Overview article by Shigechi/Zinn-Justin
- Appear in many different contexts (e.g. Lie theory, geometry and algebra)
- All described by a certain family of diagrammatical algebras
  (”generalized Khovanov algebra”)
  developed in joint work with Brundan

Jordan-Hölder multiplicities for parabolic Verma modules $[M^{\mathfrak{p}}(\lambda) : L(\mu)\langle i\rangle]$

Our special **KL polys** control:

Jordan-Hölder multiplicities for finite dimensional modules for Lie superalgebras $\mathfrak{gl}(a|b)$

Finite dimensional modules for the Walled Brauer algebra $\mathrm{Br}_{r,s}(\delta)$

Geometry of the Grassmannian $Gr(i,n)$ (category of perverse sheaves)

Geometry of Springer fibers: $N : \mathbb{C}^n \to \mathbb{C}^n$ nilpotent map with two Jordan blocks of size $(i, n-i)$ $X = \{F_1 \subset F_2 \subset \cdots \subset F_n = \mathbb{C}^n\}$ full flags Y =subset of flags satisfying $NF_i \subset F_{i-1}$

Diagrammatical description

Fix bijection

$$S_i \times S_{n-i} \backslash S_n \quad \leftrightarrow \quad \{\wedge, \vee\text{-sequences of length } n \text{ with } i \wedge\text{'s}\}$$
$$e \quad \mapsto \quad \wedge \wedge \cdots \wedge \vee \vee \cdots \vee$$

To each such element we assign a cup diagram by connecting $\vee$'s with neighboured $\wedge$'s to the right, eg.

$\wedge \wedge \vee \vee \quad \wedge \vee \wedge \vee \quad \vee \wedge \wedge \vee \quad \wedge \vee \vee \wedge \quad \vee \wedge \vee \wedge \quad \vee \vee \wedge \wedge$

We can put the $\wedge\vee$-sequence for x on top of the cup diagram for y. The result $x\,\mathrm{c}(y)$ is *oriented* if each cup is oriented and there is no ray oriented $\vee$ to the left of a ray labeled $\wedge$.

Theorem (Brundan-S.)

The parabolic KL-polynomial is given by

$$p_{x,y}(q) \quad = \quad \begin{cases} q^{\text{clockwise cups}} & \text{if } x\,\mathrm{c}(y) \text{ is oriented} \\ 0 & \text{otherwise.} \end{cases}$$

Combinatorics labeling the irreducible representations

- Parabolic Verma modules are indexed by highest weights $\lambda = (\lambda_1 > \lambda_2 > \ldots > \lambda_i, \lambda_{i+1} > \ldots > \lambda_n)$
- Schubert cells in Grassmannian are indexed by partitions fitting into an $i, (n-i)$-box. Boundary path gives an $\wedge\vee$-sequence.
- Irreducible modules for walled Brauer algebra $\mathrm{Br}_{r,s}(\delta)$ are labeled by certain bipartitions $\lambda = (\lambda^L, \lambda^R)$

$$I_\wedge(\lambda) := \{\lambda_1^{\mathrm{L}}, \lambda_2^{\mathrm{L}} - 1, \lambda_3^{\mathrm{L}} - 2, \ldots\}$$
$$I_\vee(\lambda) := \{1 - \delta - \lambda_1^{\mathrm{R}}, 2 - \delta - \lambda_2^{\mathrm{R}}, 3 - \delta - \lambda_3^{\mathrm{R}}, \ldots\}$$

- Finite dimensional $\mathfrak{gl}(a|b)$-modules are indexed by highest weights $\lambda = (\lambda_1 > \lambda_2 > \ldots > \lambda_a, \lambda_{a+1} < \ldots < \lambda_{a+b})$

$$I_\wedge(\lambda) := \{x \in \mathbb{Z} \mid x \neq \lambda_k, \forall k\}$$
$$I_\vee(\lambda) := \{x \in \mathbb{Z} \mid x = \lambda_k, x = \lambda_k' \text{ for some } k \neq k'\}$$

In each case we can also label with some $\wedge\vee$-sequence instead!

Topological description of Springer fibres

N Jordan type $(i, n-i)$, $Y = \{F_1 \subset F_2 \subset \cdots \subset F_n = \mathbb{C}^n \mid NF_i \subset F_{i-1}\}$
$x \in S_i \times S_{n-i} \rightsquigarrow$ cup diagram $\mathrm{c}(x)$
$\rightsquigarrow$ "incidence subset" $Y(x) = \{(x_1, x_2, \ldots, x_{2n}) \mid x_i \in \mathbb{S}^2\} \subset (\mathbb{S}^2)^{2n}$
where we require $x_i = x_j$ if there is a cup from i to j and $x_i = p$ if there is a ray oriented $\wedge$ at i and $x_i = p'$ if there is a ray oriented $\vee$.

$$\boxed{Y \cong \bigcup_{x \in S_i \times S_{n-i}} Y(x)}$$

Theorem (S.-Webster)

1. *The $Y(x)$ form a cell decomposition.*
2. *The $\overline{Y(x)}$ with $\mathrm{c}(x)$ maximal number of cups form the irred. components of Y (cf. Spaltenstein-Vargas classification)*
3. *The graded space $\oplus_{(x,y)} H^*(\overline{Y(x)} \cap \overline{Y(y)})$ has a diagrammatical description!*

The diagrammatical algebra K_Λ

Fix $\Lambda = S_i \times S_{n-i} \backslash S_n$.

$$\text{Basis of } K_\Lambda\text{:} \quad c(x)^* y c(z) \qquad x, y, z \in \Lambda$$

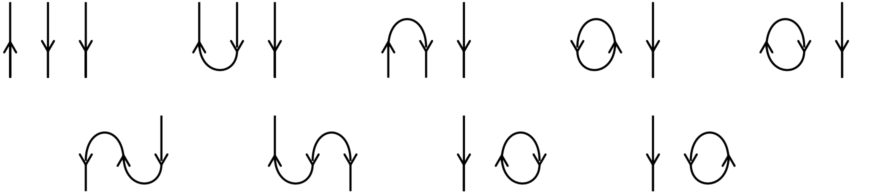
where $yc(x)$, $yc(z)$ are oriented, $c(x)^*$ the horizontally reflected $c(x)$

The diagrammatical algebra K_Λ

Fix $\Lambda = S_i \times S_{n-i} \backslash S_n$.

Basis of K_Λ: $c(x)^* y c(z) \quad x, y, z \in \Lambda$

where $yc(x)$, $yc(z)$ are oriented, $c(x)^*$ the horizontally reflected $c(x)$
For example: $i = 1$, $n = 3$:



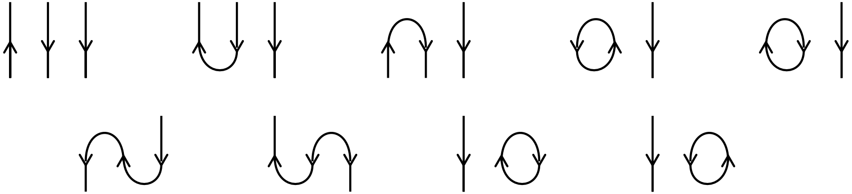
The diagrammatical algebra K_Λ

Fix $\Lambda = S_i \times S_{n-i} \backslash S_n$.

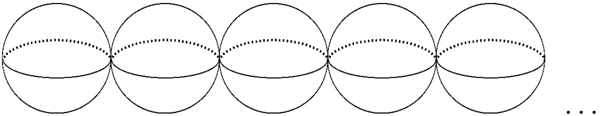
Basis of K_Λ: $c(x)^* y c(z) \quad x, y, z \in \Lambda$

where $yc(x)$, $yc(z)$ are oriented, $c(x)^*$ the horizontally reflected $c(x)$

For example: $i = 1$, $n = 3$:



Similarly for arbitrary n. Geometry for $n = 5$:



The diagrammatical algebra K_Λ and generalizations

Theorem (Brundan,S.)

1. *If we set* $\deg(c(x)^* y c(z)) = \#\,\text{clockwise cups} + \#\,\text{clockwise caps}$ *then the graded dimension equals* $\sum_{x,y,z} p_{x,z} p_{z,y} \in \mathbb{N}[q]$.
2. *There is an explicit diagrammatically defined associative graded algebra structure on* K_Λ.

- Nice algebras (e.g. Koszul, quasi-hereditary)
- Construction makes sense for any set Λ of $\wedge\vee$-sequences with a finite number of $\vee' s$.

Walled Brauer algebra $\mathrm{Br}_{r,s}(\delta)$ (Turaev, Koike)

- Generalizes the symmetric group S_n

Walled Brauer algebra $\mathrm{Br}_{r,s}(\delta)$ (Turaev, Koike)

- Generalizes the symmetric group S_n
- Classical Schur-Weyl duality (1937), $V = \mathbb{C}^n$

$$\mathfrak{gl}_n \curvearrowright V^{\otimes d} \curvearrowleft \mathbb{C}[S_n] \tag{2}$$

Walled Brauer algebra $\mathrm{Br}_{r,s}(\delta)$ (Turaev, Koike)

- Generalizes the symmetric group S_n
- Classical Schur-Weyl duality (1937), $V = \mathbb{C}^n$

$$\mathfrak{gl}_n \curvearrowright V^{\otimes d} \curvearrowleft \mathbb{C}[S_n] \tag{2}$$

- Mixed Schur-Weyl duality (Nikhitin 1980s), $W = V^*$

$$\mathfrak{gl}_n \curvearrowright V^{\otimes r} \otimes W^{\otimes s} \curvearrowleft \mathrm{Br}(n) \tag{3}$$

Walled Brauer algebra $\mathrm{Br}_{r,s}(\delta)$ (Turaev, Koike)

- Generalizes the symmetric group S_n
- Classical Schur-Weyl duality (1937), $V = \mathbb{C}^n$

$$\mathfrak{gl}_n \curvearrowright V^{\otimes d} \curvearrowleft \mathbb{C}[S_n] \tag{2}$$

- Mixed Schur-Weyl duality (Nikhitin 1980s), $W = V^*$

$$\mathfrak{gl}_n \curvearrowright V^{\otimes r} \otimes W^{\otimes s} \curvearrowleft \mathrm{Br}(n) \tag{3}$$

- Mixed super Schur-Weyl duality (Brundan-S.), equip V with a $\mathbb{Z}_2$-grading $V = V_0 \oplus V_1$ with $\mathrm{dim V_0} - \dim V_1 = m - n = \delta$

$$\mathfrak{gl}(m|n) \curvearrowright V^{\otimes r} \otimes W^{\otimes s} \curvearrowleft \mathrm{Br}(\delta) \tag{4}$$

Diagrammatical description of $\mathrm{Br}(\delta)$

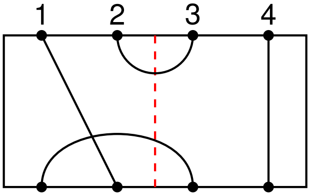
basis: isotopy classes of walled Brauer diagrams

- diagrams drawn in a rectangle with $(r+s)$ vertices numbered $1, \ldots, r$ and, separated by a wall, $r+1, \ldots, r+s$ on top and bottom
- Each vertex must be connected to exactly one other vertex by a smooth curve drawn in the interior of the rectangle; curves can cross transversally, no triple intersections.
- Horizontal edges must cross the wall, vertical edges must not.

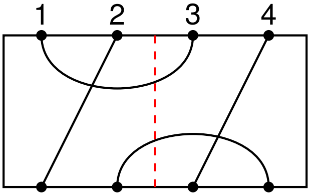
Multiplication: concatenation and replacing internal circles by δ.

Example from $\mathrm{Br}_{2,2}(\delta)$

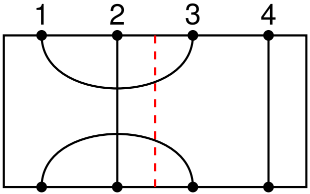
$\sigma =$



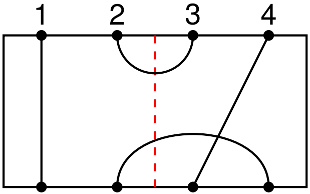
$\tau =$



$\tau\sigma =$



$\sigma\tau = \delta$



Basic properties

- $\dim \mathrm{Br}(\delta) = ?$

Basic properties

- $\dim \mathrm{Br}(\delta) = ?$ **Answer:** $\dim \mathrm{Br}_{r,s}(\delta) = (r+s)!$

Basic properties

- $\dim \mathrm{Br}(\delta) = ?$ Answer: $\dim \mathrm{Br}_{r,s}(\delta) = (r+s)!$
- **Combinatorics of the symmetric group:**

$$\begin{aligned} \{\text{irreducible } S_n\text{-modules}\} &\leftrightarrow \{\text{partitions of } n\} \\ S(\lambda) &\leftrightarrow \lambda \\ \{\text{basis of } S(\lambda)\} &\leftrightarrow \{\text{standard tableaux of shape } \lambda\} \end{aligned}$$

Basic properties

- $\dim \mathrm{Br}(\delta) =?$ Answer: $\dim \mathrm{Br}_{r,s}(\delta) = (r+s)!$
- **Combinatorics of the symmetric group:**

$$\begin{aligned} \{\text{irreducible } S_n\text{-modules}\} &\leftrightarrow \{\text{partitions of } n\} \\ S(\lambda) &\leftrightarrow \lambda \\ \{\text{basis of } S(\lambda)\} &\leftrightarrow \{\text{standard tableaux of shape } \lambda\} \end{aligned}$$

- **Combinatorics of the walled Brauer algebra ($\delta \neq 0$):**

$$\begin{aligned} \{\text{irred. } \mathrm{Br}_{r,s}\text{-modules}\} &\leftrightarrow \{\text{bipartitions } \lambda = (\lambda^L, \lambda^R) \text{ with (5)}\} \\ S(\lambda) &\leftrightarrow \lambda \\ \{\text{basis of } S(\lambda)\} &\leftrightarrow \{\text{(r,s)-up-down tableaux of shape } \lambda\} \end{aligned}$$

$$|\lambda^{\mathrm{L}}| = r - t, |\lambda^{\mathrm{R}}| = s - t \text{ for } 0 \leq t \leq \min(r, s) \tag{5}$$

but: the concrete representation depends on δ.

General Theorem

Given one of our geometric, Lie theoretic or algebraic categories $\mathcal{C}$ let

- Λ be the corresponding set of $\wedge\vee$-sequences and
- K_Λ the associated graded algebra

Theorem (Brundan, S.)

There is an equivalence of categories

$$\mathcal{C} \cong K_\Lambda - \mathrm{mod}$$

$\Rightarrow$ elementary, mostly combinatorial, description of $\mathcal{C}$!

Main References:
Series of paper with Jon Brundan (starting from scratch)

Thanks very much for your attention!