CONJECTURED COMBINATORIAL INTERPRETATION OF IWAHORI-HECKE ALGEBRA CHARACTERS

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Outline

(1) The symmetric group and Iwahori-Hecke algebra
(2) Representations and characters
(3) Descending star networks
(4) Conjectured formulas for evaluating characters

The symmetric group S_n and group algebra $\mathbb{C}[S_n]$

Generators: $s_1, \ldots, s_{n-1}$.

Relations:

$$
\begin{aligned}
s_i^2 &= e && \text{for } i = 1, \ldots, n-1, \\
s_i s_j s_i &= s_j s_i s_j && \text{for } |i-j| = 1, \\
s_i s_j &= s_j s_i && \text{for } |i-j| \geq 2.
\end{aligned}
$$

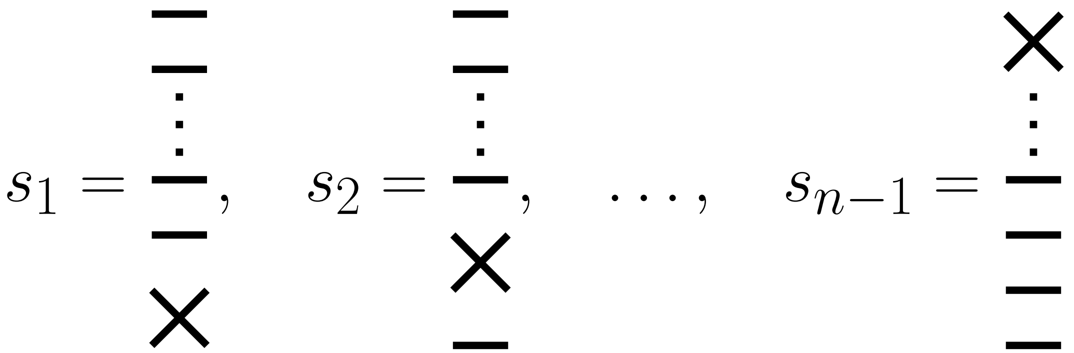
Call $s_{i_1} \cdots s_{i_\ell}$ *reduced* if it is equal to no shorter product; call ℓ the *length* of this element of S_n.

$\mathbb{C}[S_n] = \mathbb{C}$-linear combinations of S_n-elements.

Call a homomorphism $\rho : \mathbb{C}[S_n] \to \mathrm{Mat}_{d \times d}(\mathbb{C})$ a ($\mathbb{C}$-) *representation* of $\mathbb{C}[S_n]$ (of degree d).

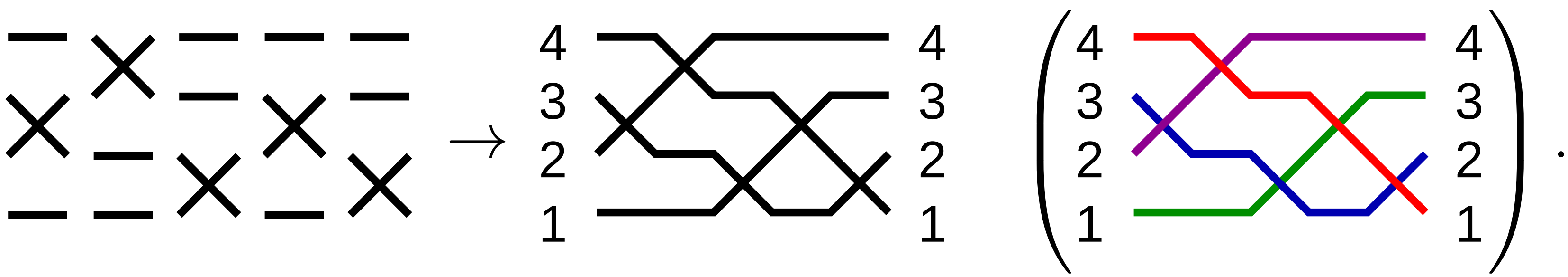
Wiring diagrams, one-line, two-line notation

Multiply generators by concatenating graphs



Define one-line and two-line notation by following wires.

Example: The wiring diagram of $s_2 s_3 s_1 s_2 s_1$ in S_4 is



Two-line notation is $\binom{1234}{3421}$; one-line notation is 3421.

The Iwahori-Hecke algebra $H_n(q)$

Generators over $\mathbb{C}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$: $T_{s_1}, \ldots, T_{s_{n-1}}$.

Relations:

$$\begin{aligned} T_{s_i}^2 &= (q-1)T_{s_i} + qT_e && \text{for } i = 1, \ldots, n-1, \\ T_{s_i}T_{s_j}T_{s_i} &= T_{s_j}T_{s_i}T_{s_j} && \text{for } |i-j| = 1, \\ T_{s_i}T_{s_j} &= T_{s_j}T_{s_i} && \text{for } |i-j| \geq 2. \end{aligned}$$

Natural basis: $\{T_w \,|\, w \in S_n\}$,

$$\begin{aligned} T_w &= T_{s_{i_1}} \cdots T_{s_{i_\ell}}, && (w = s_{i_1} \cdots s_{i_\ell} \text{ reduced}), \\ T_e &= \text{multiplicative identity.} \end{aligned}$$

Call a homomorphism $\rho_q : H_n(q) \to \mathrm{Mat}_{d \times d}(\mathbb{C}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}])$ a ($\mathbb{C}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-) *representation* of $H_n(q)$ (of degree d).

Partitions and characters

A *partition* of n is a weakly decreasing nonnegative integer sequence $\lambda = (\lambda_1, \ldots, \lambda_k)$ summing to n.
Write $\lambda \vdash n$ for "λ is a partition of n".

Each degree-d representation of $\mathbb{C}[S_n]$ or $H_n(q)$ can be described in terms of certain *irreducible representations*

$$\{\rho^\lambda \,|\, \lambda \vdash n\}, \qquad \{\rho_q^\lambda \,|\, \lambda \vdash n\},$$

or corresponding functions called *irreducible characters*

$$\{\chi^\lambda \,|\, \lambda \vdash n\}, \qquad \{\chi_q^\lambda \,|\, \lambda \vdash n\},$$

where

$$\chi^\lambda : \mathbb{C}[S_n] \to \mathbb{C} \qquad \chi_q^\lambda : H_n(q) \to \mathbb{C}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$$

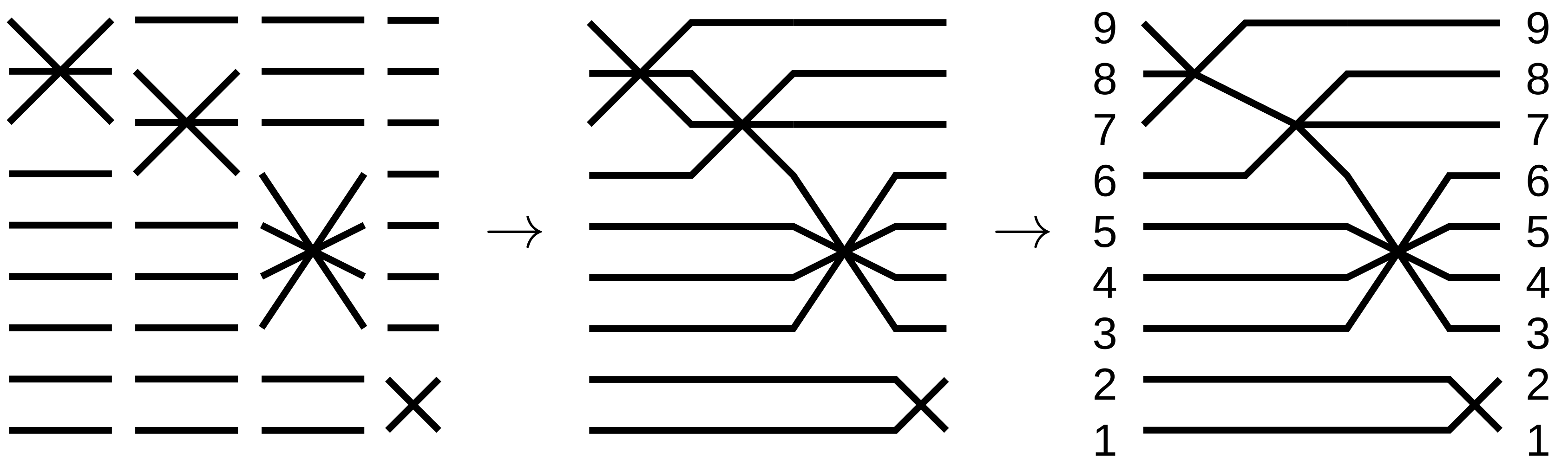
$$w \mapsto \operatorname{tr}(\rho^\lambda(w)), \qquad T_w \mapsto \operatorname{tr}(\rho_q^\lambda(T_w)).$$

Open problems and a strategy for progress

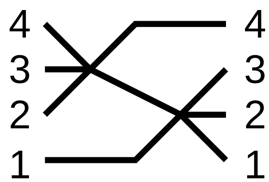
Problem: State a nice formula for $\chi^\lambda(w)$ or $\chi_q^\lambda(T_w)$.

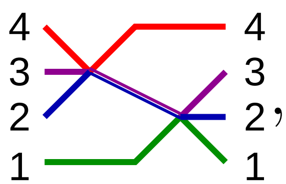
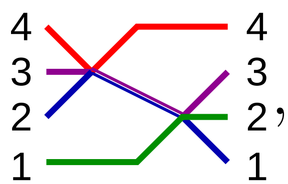
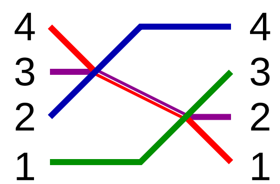
Idea: (G-J, G, S-S, H '92-'93) Choose a strategic subset $Q \subseteq S_n$ and state a formula for $\chi^\lambda(\sum_{w\in Q} w)$ or $\chi_q^\lambda(\sum_{w\in Q} T_w)$.

Strategy: Let $Q = Q(F)$ be the set of permutations covering a *descending star network* F.



Descending star networks

Example: $F =$ .

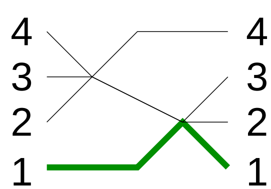
$Q = \Big\{$ , , , $\dots \Big\}$

$$= \left\{ \binom{1234}{1234}, \quad \binom{1234}{2134}, \quad \binom{1234}{3421}, \dots \right\}.$$

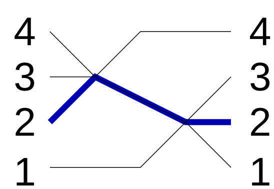
Fact: Q always contains $\binom{1 \cdots n}{1 \cdots n}$.

Let π_j denote the unique j-to-j path in F.

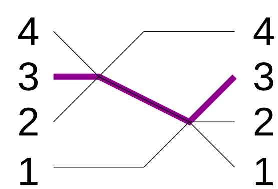
Example:

,

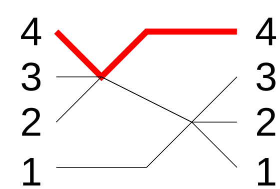
π_1

,

π_2

,

π_3

.

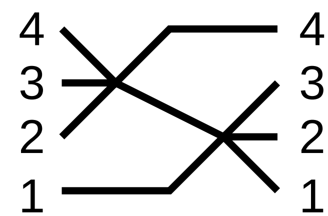
π_4

F-tableaux

Define an *F-tableau* of shape $\lambda \vdash n$ to be an arrangement of $\pi_1, \ldots, \pi_n$ into left-justified rows, with λ_i paths in row i.

Call an F-tableau *semistandard* (SS) if

$\begin{array}{|c|}\hline \pi_j \\ \hline \pi_i \\ \hline \end{array} \Rightarrow$ π_i lies entirely below π_j, $\begin{array}{|c|c|}\hline \pi_i & \pi_j \\ \hline \end{array} \Rightarrow$ π_i intersects or lies entirely below π_j.

Example: Semistandard F-tableaux for $F =$ 

are

(shape 31) $\begin{array}{|c|c|c|}\hline \pi_4 \\ \cline{1-1} \pi_1 & \pi_2 & \pi_3 \\ \hline \end{array}$, $\begin{array}{|c|c|c|}\hline \pi_4 \\ \cline{1-1} \pi_1 & \pi_3 & \pi_2 \\ \hline \end{array}$,

(shape 4) $\begin{array}{|c|c|c|c|}\hline \pi_1 & \pi_2 & \pi_3 & \pi_4 \\ \hline \end{array}$, $\begin{array}{|c|c|c|c|}\hline \pi_2 & \pi_1 & \pi_3 & \pi_4 \\ \hline \end{array}$, $\begin{array}{|c|c|c|c|}\hline \pi_3 & \pi_4 & \pi_2 & \pi_1 \\ \hline \end{array}$, $\ldots$

(none of shapes 22, 211, 1111.)

Gasharov's interpretation

If permutations Q cover descending star network F, define

$$\beta(F) = \sum_{w \in Q} w, \qquad \beta_q(F) = \sum_{w \in Q} T_w.$$

Theorem: (G '96) $\chi^\lambda(\beta(F)) = \#$ SS F-tableaux of shape λ.

Example: For previous network F, we have

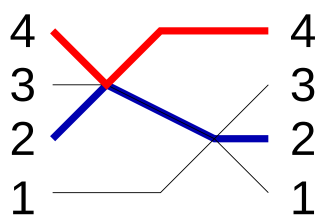
$$\chi^{31}(\beta(F)) = 2, \quad \chi^{4}(\beta(F)) = 18,$$
$$\chi^{22}(\beta(F)) = \chi^{211}(\beta(F)) = \chi^{1111}(\beta(F)) = 0.$$

Question: What is $\chi_q^\lambda(\beta_q(F))$?

Inversions in F-tableaux

Call intersecting paths (π_j, π_i) an *inversion* in an F-tableau if $j > i$ and π_j appears in an earlier column than π_i.

Define $\textsc{inv}(U) = \#$ inversions in U.

Example: For $F =$ 

, we have

$$\textsc{inv}\begin{pmatrix} \begin{array}{|c|cc} \hline \color{red}{\pi_4} & & \\ \hline \pi_1 & \color{blue}{\pi_2} & \multicolumn{1}{|c|}{\pi_3} \\ \hline \end{array} \end{pmatrix} = 2, \qquad \textsc{inv}\left(\begin{array}{|c|c|c|c|} \hline \pi_1 & \pi_2 & \pi_3 & \pi_4 \\ \hline \end{array} \right) = 0,$$

$$\textsc{inv}\left(\begin{array}{|c|c|c|c|} \hline \pi_2 & \pi_1 & \pi_3 & \pi_4 \\ \hline \end{array} \right) = 1,$$

$$\textsc{inv}\begin{pmatrix} \begin{array}{|c|cc} \hline \color{red}{\pi_4} & & \\ \hline \pi_1 & \pi_3 & \multicolumn{1}{|c|}{\color{blue}{\pi_2}} \\ \hline \end{array} \end{pmatrix} = 3, \qquad \textsc{inv}\left(\begin{array}{|c|c|c|c|} \hline \pi_3 & \pi_4 & \pi_2 & \pi_1 \\ \hline \end{array} \right) = 4, \ldots$$

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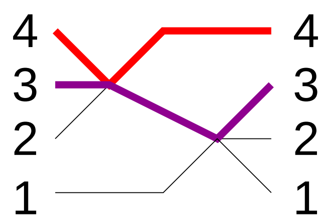
Example: For $F =$ 

, we have

$$\text{INV}\left(\begin{array}{|c|cc}\hline \color{red}{\pi_4} & & \\ \hline \pi_1 & \multicolumn{1}{c|}{\pi_2} & \multicolumn{1}{c|}{\color{purple}{\pi_3}} \\ \hline\end{array}\right) = 2, \qquad \text{INV}\left(\begin{array}{|c|c|c|c|}\hline \pi_1 & \pi_2 & \pi_3 & \pi_4 \\ \hline\end{array}\right) = 0,$$

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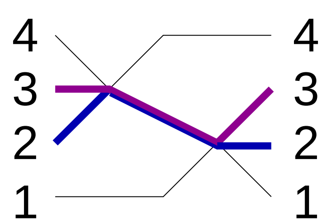
Example: For $F =$ 

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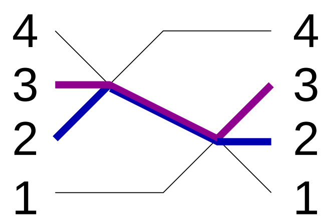
Example: For $F =$ 

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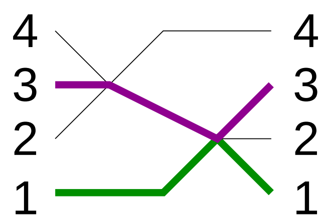
Example: For $F =$ 

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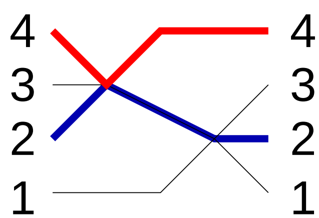
Example: For $F =$ 

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$$\text{INV}\begin{pmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 \end{pmatrix} = 0,$$

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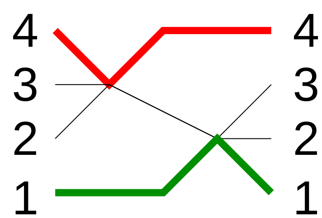
Example: For $F =$ 

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No inversion

Inversions in F-tableaux

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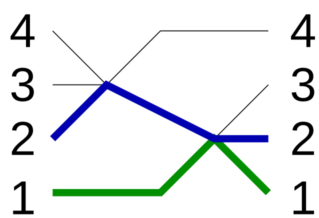
Example: For $F =$ 

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Conjectured interpretation of $\chi_q^\lambda(\beta_q(F))$

Conjecture: (Shelton '11) We have

$$\chi_q^\lambda(\beta_q(F)) = \sum_U q^{\mathrm{INV}(U)},$$

where the sum is over all SS F-tableaux of shape λ.

Example: For previous network F, we have

$$\chi_q^{31}(\beta_q(F)) = q^2 + q^3,$$

$$\chi_q^{4}(\beta_q(F)) = 1 + 3q + 5q^2 + 5q^3 + 3q^4 + q^5,$$

$$\chi_q^{22}(\beta_q(F)) = \chi_q^{211}(\beta_q(F)) = \chi_q^{1111}(\beta_q(F)) = 0.$$