Crystal energy via charge in types A and C

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based on arXiv:1107.4169 (Math. Zeitschrift)
and work joint with Naito, Sagaki, Shimozono (in progress)

Outline

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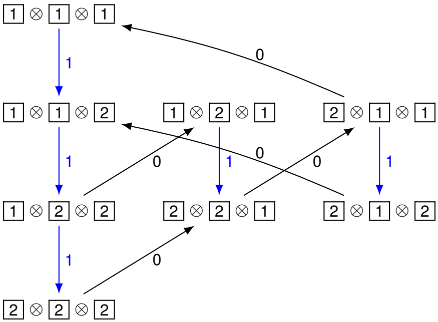
Crystal graph



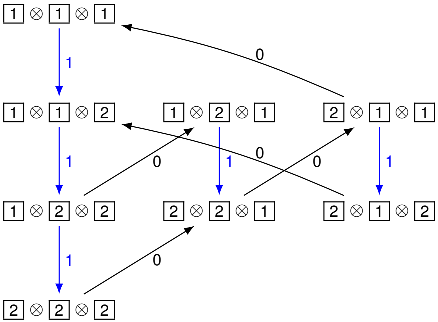
A $U_q(\mathfrak{g})$-crystal is a nonempty set B with maps

$$\mathrm{wt}\colon B \to P$$

$$e_i, f_i\colon B \to B \cup \{\emptyset\} \quad \text{for all } i \in I$$

Write $b \xrightarrow{\ i\ } b'$ for $b' = f_i(b)$

Crystal graph



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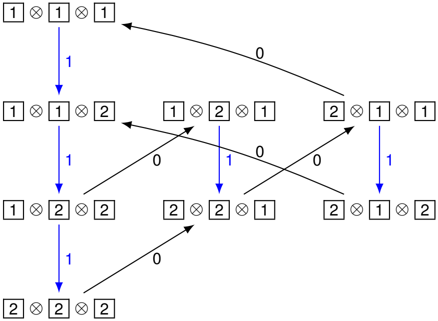

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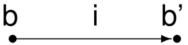
Write 

for $b' = f_i(b)$

Kashiwara–Nakashima tableaux

embed $B(1^N) \hookrightarrow B(\square)^{\otimes|\lambda|}$

Type A_r: $\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{r-1} \boxed{r} \xrightarrow{r} \boxed{r+1}$

Example

Type A_3

$$\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline 4 \\ \hline \end{array} \mapsto \boxed{4} \otimes \boxed{3} \otimes \boxed{1}$$

- strictly increasing in columns

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Type C_3 $\quad \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline \bar{3} \\ \hline \end{array} \mapsto \boxed{\bar{3}} \otimes \boxed{3} \otimes \boxed{1}$

- alphabet $[\bar{r}] := \{1 < 2 < \ldots < r < \bar{r} < \overline{r-1} < \ldots < \bar{1}\}$
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- for column $b = b(k) \ldots b(1)$ there is no pair $(z, \bar{z})$ s.t.:

$$z = b(p)\,, \qquad \bar{z} = b(q)\,, \qquad q - p \leq k - z\,.$$

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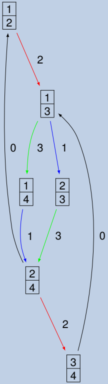
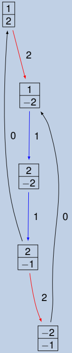
Column KR crystals for types $A_n^{(1)}$ and $C_n^{(1)}$

Example

$B^{2,1}$ of type $A_3^{(1)}$



$B^{2,1}$ of type $C_2^{(1)}$



Outline

Energy function

$B := B_\mu = B^{\mu'_1,1} \otimes B^{\mu'_2,1} \otimes \cdots$, connected by f_0 arrows.

The energy $D : B \to \mathbb{Z}$ originates from exactly solvable lattice models (computed via local energies and the combinatorial R-matrix).

Alternative construction (S., Tingley) as affine grading on B :

- constant on classical components (f_0 arrows removed)
- increases by 1 along f_0 arrows which are not at the end of a 0-string (Demazure arrows)

Remark

In most cases, B is still connected upon removal of non-Demazure f_0 arrows.
$\Rightarrow$ D is well-defined up to constant.
Notable exception: type C.

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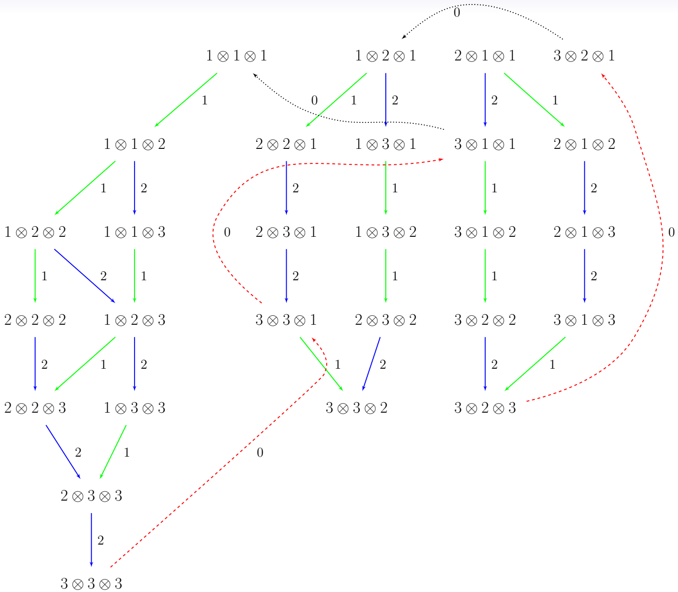

1 ⊗ 1 ⊗ 1
1 ⊗ 2 ⊗ 1
2 ⊗ 1 ⊗ 1
3 ⊗ 2 ⊗ 1
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Outline

Charge type A

Charge à la Lascoux and Schützenberger:
w word of partition content μ

Example

$\mu = (3, 3, 3, 1)$

1132214323

$\mathrm{charge}(1132214323) = 1 + 2 + 3 = 6$

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Charge on KN tableaux - type A

$$B_\mu := \bigotimes_{i=1}^{\mu_1} B^{\mu'_i,1}$$

circular order $\prec_i$: $i \prec_i i+1 \prec_i \cdots \prec_i n \prec_i 1 \prec_i \cdots \prec_i i-1$

construct reordered c from $b \in B_\mu$

Example

$$b = \begin{array}{|c|c|c|c|}\hline 3 & 2 & 1 & 2 \\ \hline 5 & 3 & 2 & \\ \cline{1-3} 6 & 4 & 4 & \\ \cline{1-3}\end{array} \quad \text{and} \quad c = \begin{array}{|c|c|c|c|}\hline 3 & 3 & 4 & 2 \\ \hline 5 & 2 & 2 & \\ \cline{1-3} 6 & 4 & 1 & \\ \cline{1-3}\end{array}$$

$$\mathrm{cw}(b) = \begin{pmatrix} 6 & 5 & 4 & 4 & 3 & 3 & 2 & 2 & 2 & 1 \\ 1 & 1 & 3 & 2 & 2 & 1 & 4 & 3 & 2 & 3 \end{pmatrix}$$

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$$\sum_{\gamma \in \mathrm{Des}(c)} \mathrm{arm}(\gamma) = \mathrm{charge}(\mathrm{cw}_2(b))$$

Remark

A similar construction works in type C.

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Relation between charge and energy

Theorem (Lenart, S. 2011)

$B = B^{r_N,1} \otimes \cdots \otimes B^{r_1,1}$ *of type* $A_n^{(1)}$ *or type* $C_n^{(1)}$
Then for $b \in B$

$$D(b) = \mathrm{charge}(b)$$

Idea of proof: Verify that charge satisfies the recursive relations of the energy function.

Relation between charge and energy

Theorem (Lenart, S. 2011)

$B = B^{r_N,1} \otimes \cdots \otimes B^{r_1,1}$ of type $A_n^{(1)}$ or type $C_n^{(1)}$
Then for $b \in B$

$$D(b) = \mathrm{charge}(b)$$

Idea of proof: Verify that charge satisfies the recursive relations of the energy function.

Outline

Crystals

Energy function

Charge

Arbitrary type

Generalizing the charge to arbitrary root systems

Key concept: quantum Bruhat graph (QBG).

In type A_{n-1}, it is the graph on S_n with directed edges

$$w \longrightarrow wt_{ij}\,,$$

where

$$\ell(wt_{ij}) = \ell(w) + 1 \quad \text{(Bruhat graph)}\,, \qquad \text{or}$$
$$\ell(wt_{ij}) = \ell(w) - \ell(t_{ij}) = \ell(w) - 2(j-i) + 1\,.$$

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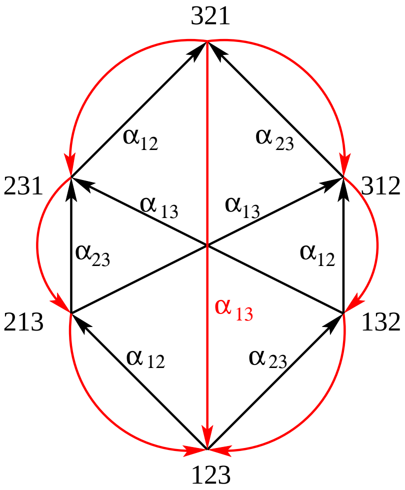
Quantum Bruhat graph for S_3:



The key ingredient

Fact. Fix two column strict fillings (in type A)

$$\begin{array}{|c|}\hline a_1 \\ \hline a_2 \\ \hline \vdots \\ \hline a_k \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|}\hline b_1 \\ \hline b_2 \\ \hline \vdots \\ \hline b_k \\ \hline \end{array},$$

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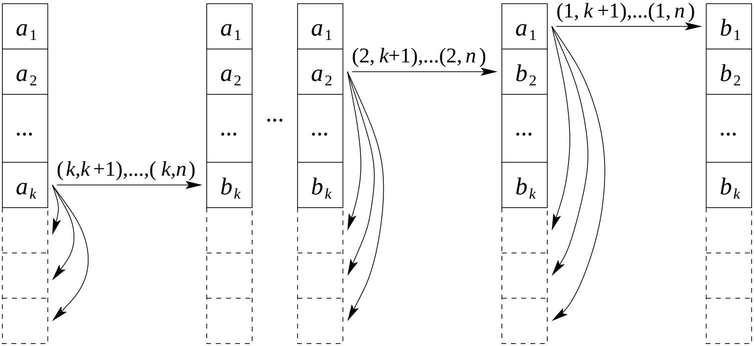
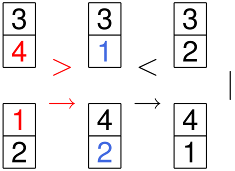

a_1
a_2
...
a_k
$(k,k+1),\dots,(k,n)$
a_1
a_2
...
b_k
...
a_1
a_2
...
b_k
$(2,k+1),\dots(2,n)$
a_1
b_2
...
b_k
$(1,k+1),\dots(1,n)$
b_1
b_2
...
b_k

Fillings as chains of permutations

$$b = \begin{array}{|c|c|c|c|}\hline 3 & 2 & 1 & 2 \\ \hline 4 & 3 \\ \cline{1-2}\end{array} \quad \mapsto \quad c = \begin{array}{|c|c|c|c|}\hline 3 & 3 & 1 & 2 \\ \hline 4 & 2 \\ \cline{1-2}\end{array} \quad \mapsto \quad \Pi = (\pi_1, \pi_2, \ldots)\,.$$



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$$\begin{array}{ccccc} \begin{array}{|c|}\hline 3 \\ \hline 4 \\ \hline\end{array} & > & \begin{array}{|c|}\hline 3 \\ \hline 1 \\ \hline\end{array} & < & \begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline\end{array} \\ & \rightarrow & & \rightarrow & & \Big| \\ \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} & & \begin{array}{|c|}\hline 4 \\ \hline 2 \\ \hline\end{array} & & \begin{array}{|c|}\hline 4 \\ \hline 1 \\ \hline\end{array} \end{array}$$

$$((2,3),(2,4),(1,3),(1,4)\ |$$

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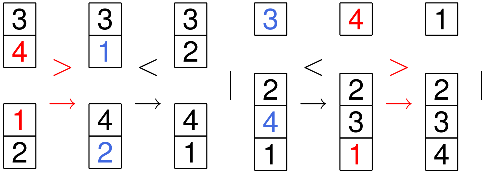

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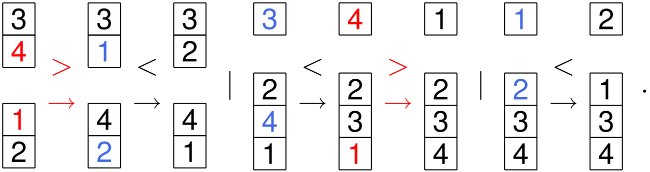

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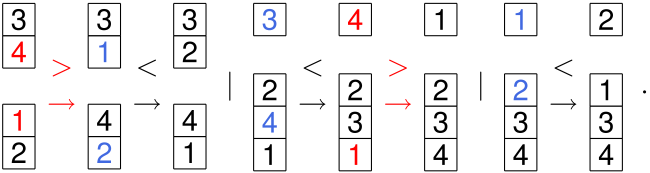

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l_r 1 1 3 3 2 2 2 1 1 1

3	3	1	2
4	2_*	–	–
	*	–	–

3	3	1_*	2_*
4	2		
		*	*

$l_r = \mathrm{arm}(\mathrm{descent})$

$$\mathrm{charge}(b) = \sum_{\gamma \in \mathrm{Des}(c)} \mathrm{arm}(\gamma) = \sum_{\pi_r > \pi_{r+1}} l_r =: \mathrm{level}(\Pi)\,.$$

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$$\begin{matrix} \begin{smallmatrix}3\\4\end{smallmatrix} & > & \begin{smallmatrix}3\\1\end{smallmatrix} & < & \begin{smallmatrix}3\\2\end{smallmatrix} & \mid & 3 & < & 4 & > & 1 & \mid & 1 & < & 2 \\ \begin{smallmatrix}1\\2\end{smallmatrix} & \to & \begin{smallmatrix}4\\2\end{smallmatrix} & \to & \begin{smallmatrix}4\\1\end{smallmatrix} & & \begin{smallmatrix}2\\4\\1\end{smallmatrix} & \to & \begin{smallmatrix}2\\3\\1\end{smallmatrix} & \to & \begin{smallmatrix}2\\3\\4\end{smallmatrix} & & \begin{smallmatrix}2\\3\\4\end{smallmatrix} & \to & \begin{smallmatrix}1\\3\\4\end{smallmatrix} \end{matrix}\ .$$

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Construction of level statistic

Step 1. Fix a partition μ.

Step 2. Associate with μ a sequence (μ-chain) Γ of pairs (i_r, j_r) (i.e., roots in type A) – several choices possible, but not explained.

Example. For $\mu = (4, 2, 0)$, we considered

$$\Gamma = ((2,3),(2,4),(1,3),(1,4)|(1,2),(1,3),(1,4)|(1,2),(1,3),(1,4))\,.$$

Step 3. Define $l_r = \#\{s \geq r \,:\, (i_s, j_s) = (i_r, j_r)\}\,.$

Step 4. Define admissible subsets:

$$\mathcal{A}(\Gamma) = \mathcal{A}(\mu) = \#\{\text{subsets } \Pi \text{ of } \Gamma \text{ giving rise to paths in the QBG}\}\,.$$

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Remarks.

1. The above construction works for any finite root system, as all the ingredients apply to the general case.

2. The level statistic originates in the Ram-Yip formula for Macdonald polynomials of arbitrary type:

$$(*) \qquad P_\mu(x;q,0) = \sum_{\Pi \in \mathcal{A}(\mu)} q^{\mathrm{level}(\Pi)}\, x^{\mathrm{weight}(\Pi)}\,.$$

In fact, we can rewrite (*) via the bijection between $\mathcal{A}(\mu)$ and fillings explained before (which also works in type C).

Theorem (L.)

In types A and C, we have

$$P_\mu(x;q,0) = \sum_{b \in B^{\mu'_1,1} \otimes B^{\mu'_2,1} \otimes \ldots} q^{\mathrm{charge}(b)}\, x^{\mathrm{weight}(b)}\,.$$

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Construction. (L. and Lubovsky) *On $\mathcal{A}(\mu)$ was defined the structure of an affine crystal (purely combinatorially) – the quantum alcove model.*

Conjecture. (L. and Lubovsky)

1. *There is a bijection between $\mathcal{A}(\mu)$ in type X_n and the KR crystal $B_\mu := B^{\mu'_1,1} \otimes B^{\mu'_2,1} \otimes \ldots$ of type $X_n^{(1)}$ under which the arrows of $\mathcal{A}(\mu)$ correspond to arrows of B_μ.*
2. *If $\Pi \in \mathcal{A}(\mu) \leftrightarrow b \in B_\mu$ under this bijection, then*

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- The KR crystal and its energy function are realized in terms of quantum Lakshmibai-Seshadri (LS) paths.
- For μ regular (in type A: partitions with distinct parts), the quantum LS paths are in bijection with $\mathcal{A}(\Gamma)$ for a special μ-chain Γ. The conjecture is verified in this case.
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