A generalization of the alcove model and its applications

Arthur Lubovsky

State University of New York at Albany

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Joint work with Cristian Lenart.

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This action is by partial permutations, and is represented as a colored directed graph

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for $i = 1, \ldots, r = \mathrm{rank}(\mathfrak{g})$, and $b, b' \in B$.
f_i are called crystal operators.

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Indexed by $r \times s$ rectangles and denoted $B^{r,s}$. We only consider columns $B^{r,1}$.

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Note: Existing models are type specific, work mostly in classical Lie types $A - D$, and increase in complexity beyond type A.

Tensor products of KR crystals in type A_{n-1}

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Example

Let $\mu = (3, 2, 2, 1)$, $n = 5$.

$$\begin{array}{|c|}\hline 2\\\hline 3\\\hline 4\\\hline 5\\\hline\end{array} \otimes \begin{array}{|c|}\hline 1\\\hline 2\\\hline 4\\\hline\end{array} \otimes \begin{array}{|c|}\hline 2\\\hline\end{array} \longleftrightarrow \begin{array}{|c|c|c|}\hline 2&1&2\\\hline 3&2\\\cline{1-2} 4&4\\\cline{1-2} 5\\\cline{1-1}\end{array}$$

Crystal operators on $B^{\otimes\mu}$ in type A

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Action of f_1 on b:

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- Obtain 1-signature

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$f_1(b) =$

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Note: f_i is defined by similar procedure on $i, i+1$, for $i \neq 0$ and f_0 is defined by similar procedure on $n, 1$.

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$B^{r,1}$ is realised by Kashiwara-Nakashima columns.

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The quantum Bruhat graph on W is the directed graph with labeled edges

$$w \xrightarrow{\alpha} ws_\alpha \,, \quad \text{where}$$

$$\ell(ws_\alpha) = \ell(w) + 1 \quad \text{(Bruhat graph)}\,, \qquad \text{or}$$
$$\ell(ws_\alpha) = \ell(w) - 2\mathrm{ht}(\alpha^\vee) + 1 \,.$$

Example type A_{n-1}

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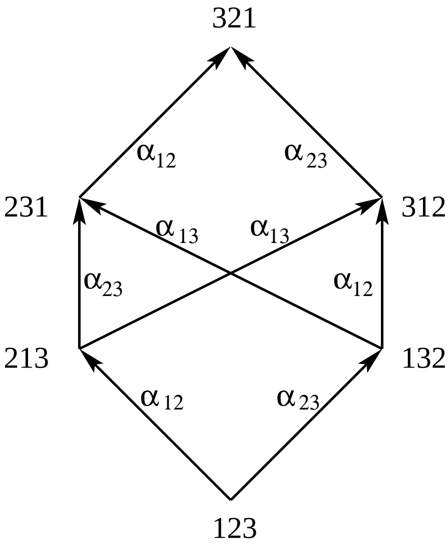
Identify: (i, j) with α_{ij} and $s_{\alpha_{ij}}$.
$s_{\alpha_{ij}}$ is realized as the transposition of i and j.

Bruhat graph for S_3



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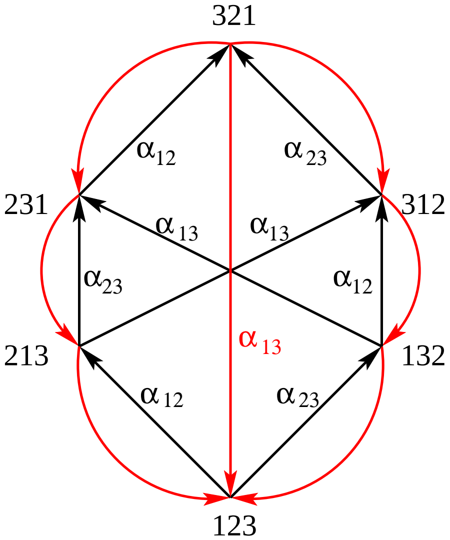

Quantum alcove model

Given a dominant weight μ, we associate with it a sequence of roots, called a μ-chain (several choices possible, but not explained):

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Let $r_i = s_{\beta_i}$, $w_i = r_{j_1} \ldots r_{j_i}$. J is admissible if

$$Id = w_0 \xrightarrow{\beta_{j_1}} w_1 \xrightarrow{\beta_{j_2}} \ldots \xrightarrow{\beta_{j_s}} w_s$$

is a path in the quantum Bruhat graph.

Quantum alcove model (cont.)

Construction: (Lenart and L.) *Combinatorial crystal operators* $f_1, \ldots, f_r$ *and* f_0 *on the collection* $\mathcal{A}(\mu)$ *of admissible subsets by analogy with the bracketing procedure for words.*

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Remark: The restriction of the non-affine combinatorial crystal operators $f_1, \ldots, f_r$ to admissible subsets corresponding to paths in the Bruhat graph is the classical alcove model of Lenart-Postnikov (a discrete counterpart of the Littelmann path model).

The quantum alcove model in type A_{n-1}

Let $\Gamma(k)$ be the chain of roots:

$$\begin{array}{llll} (\,(k,k+1), & (k,k+2), & \ldots, & (k,n)\,, \\ & & \ldots & \\ \;(2,k+1), & (2,k+2), & \ldots, & (2,n)\,, \\ \;(1,k+1), & (1,k+2), & \ldots, & (1,n)\;)\,. \end{array}$$

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Example

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$\Gamma(2) =$

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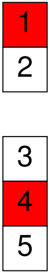

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Recall: μ is a partition, μ'_i is the height of column i.

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A μ-chain Γ is constructed by concatenating $\Gamma(k)$ chains for $k = \mu'_1, \mu'_2, \ldots$.

Crystal operators

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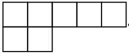
Let $n = 3\,,\quad \mu =$ . $\Gamma = \Gamma(2)\Gamma(2)\Gamma(1)\Gamma(1)\Gamma(1) =$

$$((2,3),(1,3)\mid(2,3),(1,3)\mid(1,2),(1,3)\mid(1,2),(1,3)\mid(1,2),(1,3))\,.$$

Let $J = \{1,4,7,8\}$.

Note: J is admissible: corresponds to a path in the quantum Bruhat graph.

Crystal operators

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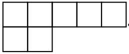
Let $n = 3\,,\ \ \mu =$ . $\Gamma = \Gamma(2)\Gamma(2)\Gamma(1)\Gamma(1)\Gamma(1) =$

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Note: J is admissible: corresponds to a path in the quantum Bruhat graph.

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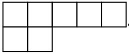
Let $n = 3$, $\mu =$ . $\Gamma = \Gamma(2)\Gamma(2)\Gamma(1)\Gamma(1)\Gamma(1) =$

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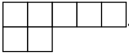
Let $n = 3$, $\mu =$ . $\Gamma = \Gamma(2)\Gamma(2)\Gamma(1)\Gamma(1)\Gamma(1) =$

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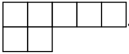
Let $n = 3\,,\quad \mu =$ . $\Gamma = \Gamma(2)\Gamma(2)\Gamma(1)\Gamma(1)\Gamma(1) =$

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Crystal operators

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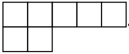
Let $n = 3\,, \quad \mu =$ . $\Gamma = \Gamma(2)\Gamma(2)\Gamma(1)\Gamma(1)\Gamma(1) =$

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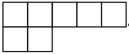
Let $n = 3\,, \quad \mu =$ . $\Gamma = \Gamma(2)\Gamma(2)\Gamma(1)\Gamma(1)\Gamma(1) =$

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$$\Gamma(J) = (\underline{(2,3)},(1,2)\,|\,(3,2),\underline{(1,2)}\,|\,(2,3),(2,1)\,|\,\underline{(2,3)},\underline{(3,1)}\,|\,(1,2),(1,3))$$

Crystal operators (cont.)

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$1\underline{1}21$

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- For f_1 only look at $(1,2)$, $(2,1)$ in $\Gamma(J)$.
- Concatenate first letters to make a word.
- Ignore underlined letters.

$1\underline{1}21$

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- For f_1 only look at $(1,2)$, $(2,1)$ in $\Gamma(J)$.
- Concatenate first letters to make a word.
- Ignore underlined letters.
- Cancel 21 pairs *like before*.

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- Cancel 21 pairs *like before*.
- Consider rightmost 1 *like before*.

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Crystal operators (cont.)

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- Add corresponding position to J, and remove from J the position corresponding to underlined 1 to its right (if any).

$1\underline{1}21$

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Crystal operators (cont.)

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$J = \{1, 4, 7, 8\}$. $f_1(J) = \{1, 2, 7, 8\}$.

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Crystal operators (cont.)

Note: f_i is similarly defined based on $(i, i+1)$ in $\Gamma(J)$ for $i \neq 0$ and f_0 is similarly defined based on $(n, 1)$.

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Similar procedure in arbitrary type, using the simple roots α_i for $f_i \neq f_0$ and the longest root θ for $f_i = f_0$.

Main results

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The above conjecture is true in type A and C.

A byproduct is a bijection between $\mathcal{A}(\mu)$ and the filling model for $B^{\otimes\mu}$ in type A and C, which is shown to preserve the corresponding affine crystal structures (cf. Conjecture).

Quantum Lakshmibai-Seshadri paths

Note: Recent work by Lenart, Naito, Sagaki, Schilling and Shimozono: $B^{\otimes\mu}$ is realized in terms of a model which is related to the quantum alcove model, namely the quantum Lakshmibai-Seshadri paths.

Applications of the quantum alcove model

- Computing the energy function on tensor products of KR crystals. (Work of Lenart, Naito, Sagaki, Schilling, Shimozono).

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Thank you