Cumulants of the q-semicircular law, Tutte polynomials, and heaps

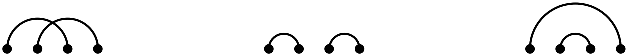
Matthieu Josuat-Vergès

Institut Gaspard Monge, Université de Marne-la-Vallée

FPSAC 2012, Nagoya

Let $m_{2n}(q) = \sum_{\sigma \in \mathcal{M}(2n)} q^{\mathrm{cr}(\sigma)}$ where $\mathcal{M}(2n)$ is the set of matchings on $\{1, \ldots, 2n\}$ and $\mathrm{cr}(\sigma)$ counts the pairs $((i,j),(k,l))$ with $i < k < j < l$.

Example



$$m_2(q) = 1,$$
$$m_4(q) = 2 + q,$$
$$m_6(q) = 5 + 6q + 3q^2 + q^3.$$

$m_{2n}(1) = 1 \times 3 \times 5 \times \cdots \times (2n-1)$, $\quad m_{2n}(0) = C_n$ (Catalan).

$m_n(q)$ is the nth moment of the *q-semicircular* law.
The cumulants $k_n(q)$ are defined by:

$$\sum_{n\geq 1} k_n(q)\frac{z^n}{n!} = \log\left(\sum_{n\geq 0} m_n(q)\frac{z^n}{n!}\right),$$

For example:

$$k_2(q) = 1 \qquad k_4(q) = q-1, \qquad k_6(q) = (q-1)^2(q+5),$$

$$k_8(q) = (q-1)^3(q^3+7q^2+28q+56).$$

We observe that $\frac{k_{2n}(q)}{(q-1)^{n-1}}$ has positive coefficients.

$m_n(q)$ is the nth moment of the q-semicircular probability law $w(x)\mathrm{d}x$, i.e. $m_n(q) = \int x^n w(x)\mathrm{d}x$.
It interpolates between

- the standard gaussian $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}\mathrm{d}x$ at $q = 1$,
- the semicircular ("free Gaussian") law $\frac{1}{2\pi}\sqrt{4 - x^2}\mathrm{d}x$ at $q = 0$.

The "free cumulants" of the q-semicircular are

$$c_{2n}(q) = \sum_{\substack{\sigma \in \mathcal{M}(2n) \\ \sigma \text{ connected}}} q^{\mathrm{cr}(\sigma)}.$$

We will show that the classical cumulants $k_n(q)$ are also related with connected matchings, in a different way involving Tutte polynomials.

Let $G = (V, E)$ a graph.
The Tutte polynomial $T_G(x, y)$ is defined by:

$$T_G(x,y) = \begin{cases} xT_{G/e}(x,y) & \text{if } e \text{ is a bridge,} \\ yT_{G\backslash e}(x,y) & \text{if } e \text{ is a loop,} \\ T_{G/e}(x,y) + T_{G\backslash e}(x,y) & \text{otherwise,} \end{cases}$$

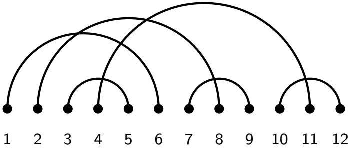
when e is an edge, and $T_G(x, y) = 1$ if G has no edge.

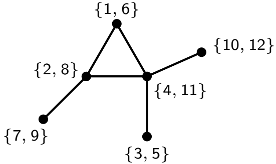
Example

$T_G(x, y) = x^{n-1}$ if G is a tree with n vertices.
$T_G(x, y) = x^{n-1} + \cdots + x^2 + x + y$ if G is a cycle with n vertices.

Definition

Let σ be a matching, its crossing graph $G(\sigma)$ is as follows: vertices are the pairs of σ, edges are the crossings.





Definition

Let $\mathcal{M}^{conn}(2n) \subset \mathcal{M}(2n)$ *be the set of connected matchings, i.e. such that the graph* $G(\sigma)$ *is connected.*

Theorem

$$\frac{k_{2n}(q)}{(q-1)^{n-1}} = \sum_{\sigma \in \mathcal{M}^{conn}(2n)} T_{G(\sigma)}(1, q).$$

Remark

The "free cumulants" $c_{2n}(q)$ of the q-semicircular law are:

$$c_{2n}(q) = \sum_{\sigma \in \mathcal{M}^{conn}(2n)} q^{\mathrm{cr}(\sigma)}.$$

Proof

Let $\mathcal{P}(n)$ the lattice of set partitions on $\{1, \dots, n\}$ ordered by refinement, and μ its Möbius function.

Lemma

We have:

$$k_n(q) = \sum_{\pi \in \mathcal{P}(n)} \mu(\pi, \hat{1}) \prod_{b \in \pi} m_{|b|}(q).$$

Proof.

From $\sum m_{2n}(q) \frac{z^{2n}}{(2n)!} = \exp(\sum k_{2n}(q) \frac{z^{2n}}{(2n)!})$ we have:

$$m_n(q) = \sum_{\pi \in \mathcal{P}(n)} \prod_{b \in \pi} k_{|b|}(q),$$

Then we can use Möbius inversion. □

Lemma

Let $\sigma \in \mathcal{M}(2n)$ *and* $\pi \in \mathcal{P}(2n)$ *with* $\sigma \leq \pi$*, let* $\mathrm{cr}(\sigma,\pi)$ *the number of crossings* $((i,j),(k,l))$ *of* σ *with* $\{i,j,k,l\} \subset b$ *for some* $b \in \pi$*. Then:*

$$\prod_{b\in\pi} m_{|b|}(q) = \sum_{\substack{\sigma\in\mathcal{M}(2n)\\ \sigma\leq\pi}} q^{\mathrm{cr}(\sigma,\pi)}.$$

Proof.

To choose a matching σ finer than a set partition π, we can choose a matching σ_b of b for each block b of π, and take $\sigma = \cup\sigma_b$. This means there is a bijection

$$\{\sigma \in \mathcal{M}(2n) \ : \ \sigma \leq \pi\} \to \prod_{b\in\pi} \mathcal{M}(b)$$

so that $\sum_{\substack{\sigma\in\mathcal{M}(2n)\\ \sigma\leq\pi}} q^{\mathrm{cr}(\sigma,\pi)}$ can be factorized. □

With the previous two lemmas, we have

$$k_{2n}(q) = \sum_{\pi \in \mathcal{P}(2n)} \mu(\pi, \hat{1}) \prod_{b \in \pi} m_{|b|}(q) = \sum_{\pi \in \mathcal{P}(2n)} \mu(\pi, \hat{1}) \sum_{\substack{\sigma \in \mathcal{M}(2n) \\ \sigma \leq \pi}} q^{\mathrm{cr}(\sigma, \pi)}$$

$$= \sum_{\sigma \in \mathcal{M}(2n)} \sum_{\substack{\pi \in \mathcal{P}(2n) \\ \pi \geq \sigma}} \mu(\pi, \hat{1}) q^{\mathrm{cr}(\sigma, \pi)} = \sum_{\sigma \in \mathcal{M}(2n)} W(\sigma),$$

where we denote

$$W(\sigma) = \sum_{\substack{\pi \in \mathcal{P}(2n) \\ \pi \geq \sigma}} \mu(\pi, \hat{1}) q^{\mathrm{cr}(\sigma, \pi)}.$$

$W(\sigma)$ only depends on the crossing graph $G(\sigma)$.
If $G(\sigma) = (V, E)$ we have:

$$W(\sigma) = \sum_{\pi \in \mathcal{P}(V)} q^{i(E,\pi)} \mu(\pi, \hat{1}).$$

where $i(E, \pi)$ counts edges in E such that both endpoints are in the same block of π.

Lemma

Let $G = (V, E)$ be a graph. If $\pi \in \mathcal{P}(V)$, let $i(E, \pi)$ the number of edges in G such that both endpoints are in a same block of π. Let

$$U_G = \frac{1}{(q-1)^{n-1}} \sum_{\pi \in \mathcal{P}(V)} q^{i(E,\pi)} \mu(\pi, \hat{1})$$

Then we have

$$U_G = \begin{cases} \delta_{n1} & \text{if } \#V = n \text{ and } E = \emptyset, \\ qU_{G \setminus e} & \text{if } e \in E \text{ is a loop,} \\ U_{G/e} + U_{G \setminus e} & \text{if } e \in E \text{ is not a loop.} \end{cases}$$

Corollary

$U_G = T_G(1, q)$ *if G connected,* 0 *otherwise.*

Hence

$$\frac{1}{(q-1)^{n-1}} W(\sigma) = \begin{cases} T_{G(\sigma)}(1,q) & \text{if } \sigma \text{ connected,} \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\frac{1}{(q-1)^{n-1}} k_{2n}(q) = \sum_{\sigma \in \mathcal{M}^{conn}(2n)} T_{G(\sigma)}(1,q).$$

The case $q = 2$

In this case:

$$\sum_{n\geq 1} k_n(2)\frac{z^n}{n!} = \log\left(\sum_{n\geq 0} m_n(2)\frac{z^n}{n!}\right).$$

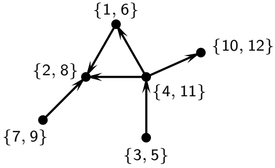
$k_{2n}(2)$ is a positive integer.

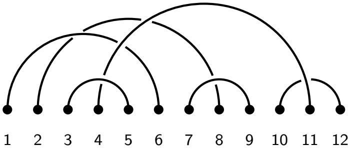
$k_{2n}(2) = \sum_{\sigma\in\mathcal{M}^{conn}(2n)} T_{G(\sigma)}(1,2)$ can be proved with the exponential formula using:

Proposition (Gioan, 2010)

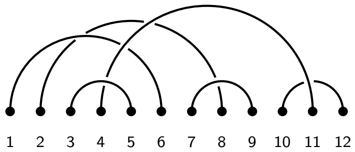
Let G be connected graph with a root r, then $T_G(1,2)$ counts the orientations such that for any vertex v, there is an oriented path from r to v (i.e. root-accessible orientations).

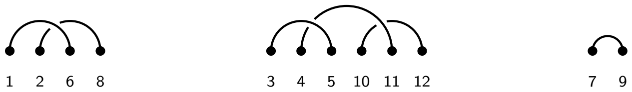
$m_{2n}(2)$ counts pairs (σ, r) where $\sigma \in \mathcal{M}(2n)$ and r is an orientation of $G(\sigma)$.





The block decomposition is as follows: take the leftmost arch, "push" it, it takes others arches with it, and this defines the first block. (Then do the same thing with what remains.)





This defines a decomposition $(\sigma, r) \mapsto ((\sigma_1, r_1), \ldots, (\sigma_k, r_k))$. For each σ_i, the leftmost arch is considered as the root of the crossing graph $G(\sigma_i)$. Then r_i is an orientation such that each vertex is accessible from the root.

The case $q = 0$ (details omitted)

In the case $q = 0$, letting C_n denote the Catalan numbers, we have:

$$-\log\left(\sum_{n\geq 0}(-1)^n C_n \frac{z^{2n}}{(2n)!}\right) = \sum_{n\geq 1}(-1)^{n-1} k_{2n}(0)\frac{z^{2n}}{(2n)!}$$

The integers $(-1)^{n-1}k_{2n}(0)$ form an increasing sequence of positive numbers [Lassalle, 2010].

$(-1)^{n-1}k_{2n}(0) = \sum T_{G(\sigma)}(1,0)$ can be proved via Viennot's theory of "heaps of pieces", using:

Proposition (Greene-Zaslavsky)

If G is connected and has a root r, $T_G(1,0)$ counts acyclic orientations such that for each vertex v there is a directed path from r to v.

The case $q = 0$ can be generalized

Let m_n be any sequence of moments, k_n (resp. c_n) the corresponding cumulants (resp. free cumulants). The relations $m_n \leftrightarrow k_n$ (resp. $m_n \leftrightarrow c_n$) are ruled by Möbius inversion on the lattice of set partitions (resp. noncrossing set partitions).

What about the relations $k_n \leftrightarrow c_n$?

Theorem (Lehner)

$$c_n = \sum_{\substack{\pi \in \mathcal{P}(n) \\ \pi \text{ connected}}} \prod_{b \in \pi} k_{|b|}.$$

This is invertible, but we cannot use the Möbius inversion here. Our method show that:

$$k_n = \sum_{\substack{\pi \in \mathcal{P}(n) \\ \pi \text{ connected}}} (-1)^{1+|\pi|} T_{G(\pi)}(1,0) \prod_{b \in \pi} c_{|b|}.$$

Two (hopefully related) questions:

- Is there a generalization for something that interpolates between the cumulants and free cumulants ?
- Is there a generalization involving $T_{G(\sigma)}(p, q)$ and not just $T_{G(\sigma)}(1, q)$?

thanks for your attention