RIFFLE SHUFFLES WITH BIASED CUTS

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Theorem. (Bayer–Diaconis) On $\mathfrak{S}_n$, we have $P_{\frac{1}{a}} * P_{\frac{1}{b}} = P_{\frac{1}{ab}}$.

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k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
SEP	1.00	1.00	1.00	1.00	1.00	1.00	1.00	.996	.931	.732	.479	.278	.150	.078	.040	.020

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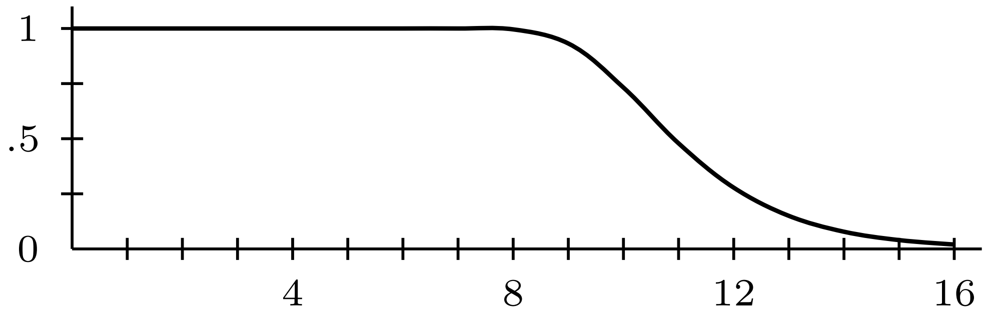

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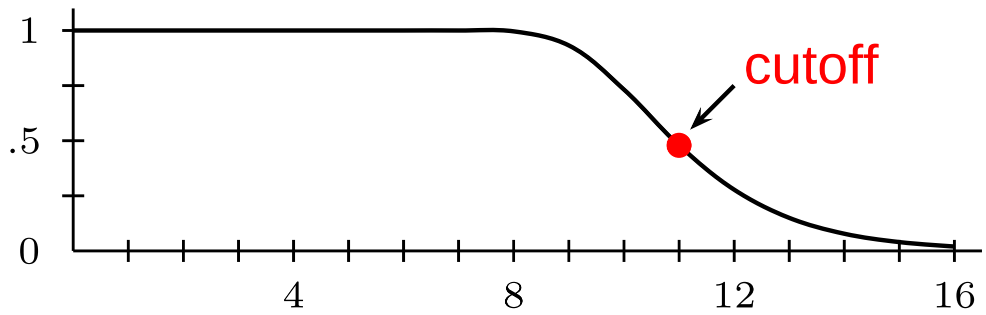

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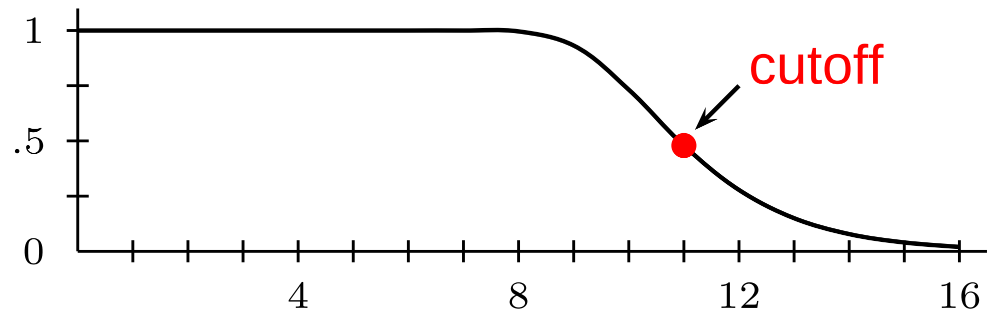

Precisely, for $a_n, b_n \to \infty$ with $b_n/a_n \to 0$, the chains P_n, π_n satisfy an a_n, b_n cutoff if for all real fixed θ with $k_n = \lfloor a_n + \theta b_n \rfloor$

$$\|P_n^{k_n} - \pi_n\| \longrightarrow c(\theta) \quad \text{where} \begin{cases} c(\theta) \to 0 & \text{as } \theta \to \infty \\ c(\theta) \to 1 & \text{as } \theta \to -\infty \end{cases}$$

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Repeated θ-shuffles convolve: $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_a)$ and $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_b)$,

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Theorem. (Diaconis–Fill–Pitman) On $\mathfrak{S}_n$, we have $P_{\boldsymbol{\theta}} * P_{\boldsymbol{\eta}} = P_{\boldsymbol{\theta} * \boldsymbol{\eta}}$.

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Gessel's fundamental quasisymmetric function basis is (subsets)

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The fundamental basis is related to the monomial basis by

$$Q_{D(\beta)}(X) = \sum_{\alpha \text{ refines } \beta} M_\alpha(X)$$

Biased distribution

Theorem. (Fulman,Stanley) For $\sigma \in \mathfrak{S}_n$ and $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_a)$,

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Corollary. On $\mathfrak{S}_n$, we have $\mathrm{SEP}(P_{\boldsymbol{\theta}}) = 1 - n!Q_{[n-1]}(\boldsymbol{\theta})$.

Biased separation

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Theorem. (A-D-S) On $\mathfrak{S}_n$, the separation distance for $P_{\boldsymbol{\theta}}$ is

$$\mathrm{SEP}(k) = 1 - n! \left(\sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} z_\lambda^{-1} \prod_{i=1}^{n} p_i(\boldsymbol{\theta})^{k n_i(\lambda)} \right)$$

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This gives a tight upper bound on separation, establishes the cutoff phenomenon, and shows that unbiased cuts lead to fastest mixing.

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For example, for $n = 3$ we have

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A similarly sharp analysis for total variation remains open for $\theta \neq \frac{1}{2}$.

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Thank you for listening.