From alternating sign matrices to Painlevé VI

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Outline

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Three-coloured chessboards

0	1	2	0
1	2	1	2
2	1	2	1
0	2	1	0

Chessboard of size $(n+1) \times (n+1)$. Paint squares with three colours 0, 1, 2 mod 3.

- Adjacent squares have distinct colour.
- "Domain wall boundary conditions" (DWBC). Read entries mod 3.

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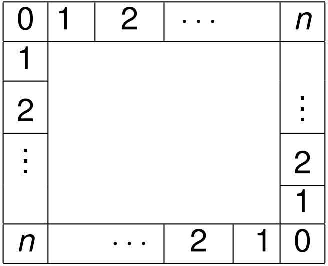

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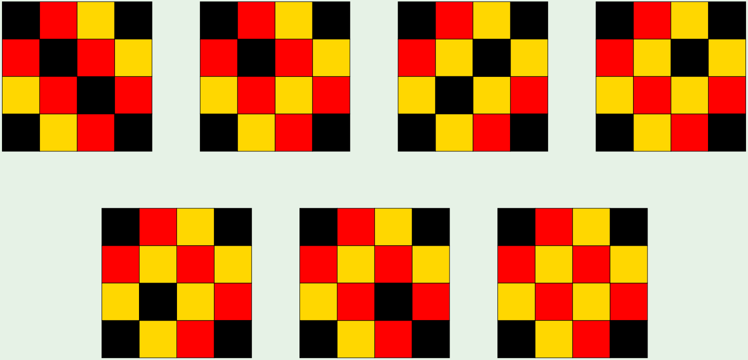
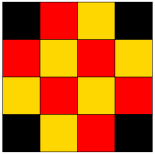
Example

When $n = 3$ there are seven chessboards.
$0 =$ black, $1 =$ red, $2 =$ yellow.



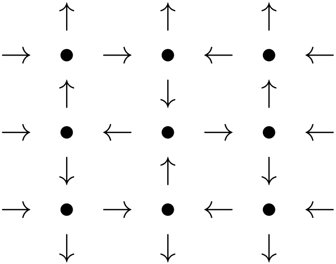
Other descriptions



Chessboard

$$\begin{matrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{matrix}$$

Alternating sign matrix



Ice graph

Bijection to alternating sign matrices

Represent colours (residue classes mod 3) by integers so that neighbours differ by 1.

$$\begin{matrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 3 & 2 & 1 & 0 \end{matrix}$$

Contract each block $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ to $(b+c-a-d)/2 \in \{-1,0,1\}$.

$$\begin{matrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{matrix}$$

Gives bijection to $n \times n$ alternating sign matrices. Non-zero entries in each row and column alternate in sign and add up to 1.

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Bijection to ice graphs (states of six-vertex model)

0	1	2	0
1	2	1	2
2	1	2	1
0	2	1	0

Put arrows between adjacent squares. Larger entry to the right, $0 < 1 < 2 < 0$.

- Each vertex has two incoming and two outgoing edges.
- Domain wall boundary conditions.

Vertex = Oxygen, Incoming arrow = Hydrogen bond.

Bijection to ice graphs (states of six-vertex model)

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1		2		1		2
2		1		2		1
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\begin{array}{ccccccc}
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\rightarrow & & \rightarrow & & \leftarrow & & \leftarrow \\
1 & \uparrow & 2 & \downarrow & 1 & \uparrow & 2 \\
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→	•	←	•	→	•	←
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Example

For $n = 5$, the number of chessboards with exactly k squares of colour 0 and l squares of colour 2 are as follows.

	k=8	9	10	11	12	13	14
l=8							1
9						4	6
10					7	8	15
11				8	12	36	20
12			7	12	36	40	15
13		4	18	36	40	24	6
14	1	6	15	20	15	6	1

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	1	4	6	4	1		
+			1	4	6	4	1
=	1	4	7	8	7	4	1

Generating function (partition function)

$$Z_n(t_0, t_1, t_2) = \sum_{\substack{\text{chessboards} \\ \text{of size } (n+1)\times(n+1)}} t_0^{\#\text{ squares coloured 0}}\, t_1^{\#\text{ squares coloured 1}}\, t_2^{\#\text{ squares coloured 2}}$$

$$Z_5(t_0, t_1, t_2) = t_0^{14} t_1^{14} t_2^{8} + 4 t_0^{13} t_1^{14} t_2^{9} + \cdots .$$

Partition function for three-colour model with DWBC.
Studied by Baxter (1970) for *periodic* boundary conditions.

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Three questions

- How many states are there (for fixed n)?
  What is $Z_n(1,1,1)$?

- How common are the various colors?
  What is

$$\frac{\partial Z_n}{\partial t_j}(1,1,1) = \sum_{\text{chessboards}} \#\text{squares of colour } j?$$

- What is the joint distribution of the three colours?
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Question 1: Enumeration

Alternating sign matrix theorem:

$$\#\text{chessboards} = \frac{1!\,4!\,7!\cdots(3n-2)!}{n!(n+1)!(n+2)!\cdots(2n-1)!}.$$

Conjectured by Mills–Robbins–Rumsey (1983).
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Much simpler proof by Kuperberg (1996),
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We generalize Kuperberg's work using
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8VSOS model (no details)

Introduced by Baxter 1973.

Generalizes both three-colour model and six-vertex model. Same states, but more general weight function.

It gives a "nice" way to put $2n$ extra parameters into three-colour model (or 2 extra parameters into six-vertex model).

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Trigonometric case

The 8VSOS model involves elliptic functions. Using the trigonometric limit case, we can prove that, with $\omega = e^{2\pi i/3}$,

$$Z_n\left(\frac{1}{(1-\lambda)^3}, \frac{1}{(1-\lambda\omega)^3}, \frac{1}{(1-\lambda\omega^2)^3}\right) = \frac{(1-\lambda\omega^2)^2(1-\lambda\omega^{n+1})^2(A_n(1+\omega^n\lambda^2)+(-1)^n C_n\omega^{2n}\lambda)}{(1-\lambda^3)^{n^2+2n+3}},$$

where A_n are alternating sign matrix numbers and

$$C_n = \prod_{j=1}^{n} \frac{(3j-1)(3j-3)!}{(n+j-1)!}$$

count cyclically symmetric plane partitions.

Consequence

The case $\lambda = 0$ is the ASM Theorem: $Z_n(1,1,1) = A_n$.

Applying $\partial/\partial\lambda \Big|_{\lambda=0}$ gives expressions for the first moments

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Answer to Question 2: How common are the colours?

Suppose $n \equiv 0 \mod 6$ and consider the colour 0 (all other cases are similar).

Probability that random square from random chess-board (chosen uniformly) has colour 0 is

$$\frac{1}{3} + \frac{2}{9(n+1)^2}\frac{2\cdot 5\cdots(3n-1)}{1\cdot 4\cdots(3n-2)} + \frac{4}{9(n+1)^2}$$

$$= \frac{1}{3} + \frac{2}{9}\frac{\Gamma(1/3)}{\Gamma(2/3)}n^{-5/3} + \frac{4}{9}n^{-2} + O(n^{-3}), \quad n \to \infty.$$

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Question 3: What is Z_n in general?

Z_n can be split as a sum of two parts.

Each part is a specialized affine Lie algebra character, and a tau function of Painlevé VI.
Moreover, each part satisfies a Toda-type recursion.

Before giving some details,
we mention one more thing.

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Free energy

Supppose t_0, t_1, t_2 are positive. **Conjecture:**

$$\lim_{n\to\infty} \frac{\log Z_n(t_0,t_1,t_2)}{n^2} = \frac{1}{3}\log(t_0t_1t_2) + \log\left(\frac{(\zeta+2)^{\frac{3}{4}}(2\zeta+1)^{\frac{3}{4}}}{2^{\frac{2}{3}}\zeta^{\frac{1}{12}}(\zeta+1)^{\frac{4}{3}}}\right),$$

where ζ is determined by

$$\frac{(t_0t_1+t_0t_2+t_1t_2)^3}{(t_0t_1t_2)^2} = \frac{2(\zeta^2+4\zeta+1)^3}{\zeta(\zeta+1)^4}, \qquad \zeta \geq 1.$$

Compare Baxter's formula for *periodic* boundary conditions:

$$\frac{1}{3}\log(t_0t_1t_2) + \log\left(\frac{2^{\frac{5}{3}}\zeta^{\frac{1}{3}}(\zeta+1)^{\frac{4}{3}}}{(2\zeta+1)^{\frac{3}{2}}}\right).$$

Free energy

Supppose t_0, t_1, t_2 are positive. **Conjecture:**

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Outline

Symmetric polynomials

Let

$$S_n(x_1,\dots,x_n,y_1,\dots,y_n,z) = \frac{\prod_{i,j=1}^{n} G(x_i,y_j)}{\prod_{1\le i<j\le n}(x_j-x_i)(y_j-y_i)} \det_{1\le i,j\le n}\left(\frac{F(x_i,y_j,z)}{G(x_i,y_j)}\right),$$

$$F(x,y,z) = (\zeta+2)xyz - \zeta(xy+yz+xz+x+y+z) + \zeta(2\zeta+1),$$

$$G(x,y) = (\zeta+2)xy(x+y) - \zeta(x^2+y^2) - 2(\zeta^2+3\zeta+1)xy + \zeta(2\zeta+1)(x+y).$$

This is a symmetric (!) polynomial in all $2n+1$ variables, depending on parameter ζ.

It can be identified with a character of $A_{4n-3}^{(2)}$ affine Lie algebra.

"Cauchy-type": all minors have the same form. Very useful!

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It can be identified with a character of $A_{4n-3}^{(2)}$ affine Lie algebra.

"Cauchy-type": all minors have the same form. Very useful!

Symmetric polynomials

Let

$$S_n(x_1,\dots,x_n,y_1,\dots,y_n,z) = \frac{\prod_{i,j=1}^n G(x_i,y_j)}{\prod_{1\le i<j\le n}(x_j-x_i)(y_j-y_i)} \det_{1\le i,j\le n}\left(\frac{F(x_i,y_j,z)}{G(x_i,y_j)}\right),$$

$$F(x,y,z)=(\zeta+2)xyz-\zeta(xy+yz+xz+x+y+z)+\zeta(2\zeta+1),$$

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Relation to three-colour model

Let

$$p_n(\zeta) = \text{elementary factor}$$
$$\times\, S_n\Big(\underbrace{2\zeta+1,\dots,2\zeta+1}_{n+1},\underbrace{\frac{\zeta}{\zeta+2},\dots,\frac{\zeta}{\zeta+2}}_{n}\Big).$$

This is a polynomial in ζ of degree $n(n+1)/2$.

Result: $Z_n(t_0,t_1,t_2)$ is a linear combination (with elementary coefficients) of $p_{n-1}(\zeta)$ and $p_{n-1}(1/\zeta)$, where

$$\frac{(t_0t_1+t_0t_2+t_1t_2)^3}{(t_0t_1t_2)^2} = \frac{2(\zeta^2+4\zeta+1)^3}{\zeta(\zeta+1)^4}.$$

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More precisely . . .

Suppose $n \equiv 0 \mod 6$ and

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$$Z_n(t_0, t_1, t_2) = (t_0 t_1 t_2)^{\frac{n(n+2)}{3}} \left(\frac{2}{\zeta(\zeta + 1)^4} \right)^{\frac{n^2}{12}} \times \left(t_0 \frac{p_{n-1}(\zeta) - \zeta^{\frac{n^2}{2}+1} p_{n-1}(1/\zeta)}{1 - \zeta} - \frac{t_0 t_1 t_2(\zeta^2 + 4\zeta + 1)}{t_0 t_1 + t_0 t_2 + t_1 t_2} \frac{p_{n-1}(\zeta) - \zeta^{\frac{n^2}{2}} p_{n-1}(1/\zeta)}{1 - \zeta^2} \right)$$

Note that only asymmetry comes from t_0.

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Table of p_n

n	$p_n(\zeta)$
-1	1
0	1
1	$3\zeta + 1$
2	$5\zeta^3 + 15\zeta^2 + 7\zeta + 1$
3	$\frac{1}{2}(35\zeta^6 + 231\zeta^5 + 504\zeta^4 + 398\zeta^3 + 147\zeta^2 + 27\zeta + 2)$
4	$\frac{1}{2}(63\zeta^{10} + 798\zeta^9 + 4122\zeta^8 + 11052\zeta^7 + 16310\zeta^6 + 13464\zeta^5 + 6636\zeta^4 + 2036\zeta^3 + 387\zeta^2 + 42\zeta + 2)$

Conjecture: The polynomials p_n have positive coefficients.
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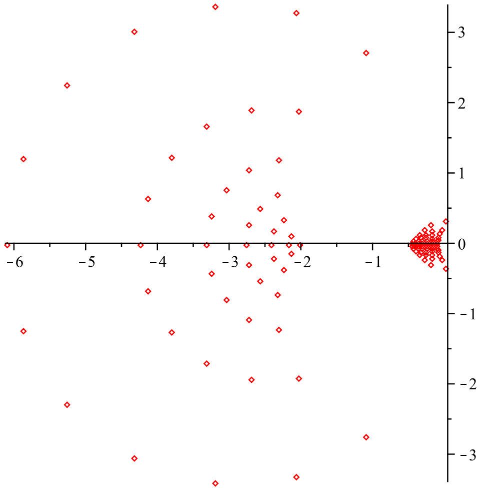
Plot of the 105 complex zeroes of p_{14}.



Outline

Painlevé VI

PVI is the nonlinear ODE
(Painlevé, Fuchs, Gambier 1900–1910)

$$
\begin{aligned}
\frac{d^2y}{dt^2} &= \frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 \\
&\quad - \left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{y-t}\right)\frac{dy}{dt} \\
&\quad + \frac{y(y-1)(y-t)}{2t^2(t-1)^2}\left(\alpha+\beta\frac{t}{y^2}+\gamma\frac{t-1}{(y-1)^2}+\delta\frac{t(t-1)}{(y-t)^2}\right).
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Most general second order ODE such that all movable singularities are poles.

Special functions of the 21st Century?

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Bäcklund transformations

If $y = y(t)$ solves PVI, then so does t/y, with parameters $\alpha \leftrightarrow -\beta$, $\gamma \leftrightarrow \frac{1}{2} - \delta$.

Such Bäcklund transformations $y \mapsto F(t, y, y')$ generate group (extended affine Weyl group) containing $\mathbb{Z}^4$.

Given a "seed" solution $y = y_{0000}$, acting with $\mathbb{Z}^4$ gives new solutions $y_{k_1 k_2 k_3 k_4}$.

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Picard's seed solution

When $\alpha = \beta = \gamma = 0$, $\delta = 1/2$, PVI can be solved in terms of elliptic functions (Picard, 1889).

Picard's solutions include the algebraic solution

$$y^4 - 4ty^3 + 6ty^2 - 4ty + t^2 = 0,$$

which is parametrized by

$$y = \frac{\zeta(\zeta + 2)}{2\zeta + 1}, \qquad t = \frac{\zeta(\zeta + 2)^3}{(2\zeta + 1)^3}.$$

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Example

$$y_{1,2,-3,1} = \frac{\zeta(\zeta+2)(\zeta^3+3\zeta^2+3\zeta+5)(5\zeta^3+15\zeta^2+7\zeta+1)}{(2\zeta+1)(\zeta^3+7\zeta^2+15\zeta+5)(5\zeta^3+3\zeta^2+3\zeta+1)}$$

The non-trivial factors are called tau functions.

Note that

$$5\zeta^3+15\zeta^2+7\zeta+1 = p_2(\zeta).$$

Result: There is a 4-dim cone in $\mathbb{Z}^4$ where tau functions are given by specializations of S_n.

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Tau functions in a cone

Let

$$S_n^{(k_0,k_1,k_2,k_3)}(\zeta) = S_n\Big(\underbrace{2\zeta+1}_{k_0}, \underbrace{\frac{\zeta}{\zeta+2}}_{k_1}, \underbrace{\frac{\zeta(2\zeta+1)}{\zeta+2}}_{k_2}, \underbrace{1}_{k_3}\Big).$$

Then, if k_i are non-negative integers with $k_0 + k_1 + k_2 + k_3 = 2n - 1$,

$$y_{k_0+k_2-n+1,k_1+1,n-k_0-k_1-1,n-k_1-k_2-1}$$
$$= \text{elementary factor} \cdot \frac{S_n^{(k_0+1,k_1,k_2,k_3+1)}(\zeta)\, S_n^{(k_0,k_1+1,k_2+1,k_3)}(\zeta)}{S_n^{(k_0+1,k_1,k_2+1,k_3)}(\zeta)\, S_n^{(k_0,k_1+1,k_2,k_3+1)}(\zeta)},$$

where as before

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Consequence: Recursions

Tau functions satisfy bilinear recursions.
For instance, it follows that

$$p_{n+1}(\zeta)p_{n-1}(\zeta) = A_n(\zeta)p_n(\zeta)^2 + B_n(\zeta)p_n(\zeta)p_n'(\zeta) + C_n(\zeta)p_n'(\zeta)^2 + D_n(\zeta)p_n(\zeta)p_n''(\zeta).$$

with explicit coefficients.

Conjectured by Bazhanov and Mangazeev 2010.

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Questions

- Symmetry classes of three-coloured chessboards.
  For the six-vertex model, various classical Lie algebra characters appear (Kuperberg, Okada,. . .).
  For the three-colour model, we expect various affine Lie algebras.
- Macroscopic boundary effects.
  Arctic curves.
- New phenomenon: Some tau functions for Painlevé VI are specialized affine Lie algebra characters, given by Cauchy-type determinant formulas.
  Does this happen for other solutions to Painlevé equations?

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