The representation of the symmetric group on m-Tamari intervals

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Outline

1. Motivations and results: Toward a combinatorial description of diagonal coinvariant spaces of the symmetric group

2. Proof: solving a differential-catalytic equation

3. Open combinatorial questions

The diagonal coinvariant spaces of the symmetric group

$$X := \begin{pmatrix} x_1 & x_2 & x_3 & \dots & x_n \\ y_1 & y_2 & y_3 & \dots & y_n \\ \vdots & & & & \vdots \\ z_1 & z_2 & z_3 & \dots & z_n \end{pmatrix}$$

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The symmetric group $\mathfrak{S}_n$ acts on X by permuting the columns:

$$\sigma(X) := \begin{pmatrix} x_{\sigma(1)} & x_{\sigma(2)} & \dots & x_{\sigma(n)} \\ y_{\sigma(1)} & y_{\sigma(2)} & \dots & y_{\sigma(n)} \\ \vdots & & & \vdots \\ z_{\sigma(1)} & z_{\sigma(2)} & \dots & z_{\sigma(n)} \end{pmatrix}.$$

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$\mathcal{J}$ is the ideal generated by constant free polynomials such that

$$\sigma \cdot f(X) := f(\sigma(X)), \quad \forall \sigma \in \mathfrak{S}_n.$$

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The diagonal coinvariant space of the symmetric group:

$$\mathcal{DR}_{k,n} := \mathbb{C}[X] \big/ \mathcal{J}.$$

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${\mathcal{DR}_{k,n}}^{\varepsilon}$ is the sign component (the alternants).

'Higher' diagonal coinvariant spaces

A polynomial $f(X)$ is alternant if

$$\sigma \cdot f(X) = (-1)^{\mathrm{sign}(\sigma)} f(X).$$

$\mathcal{A}$ is the ideal generated by alternants.

The 'higher' diagonal coinvariant spaces of the symmetric group:

$$\mathcal{DR}_{k,n}^{m} := \varepsilon^{m-1} \otimes \left(\mathcal{A}^{m-1} \Big/ \mathcal{J}\mathcal{A}^{m-1} \right).$$

${\mathcal{DR}_{k,n}^{m}}^{\varepsilon}$ is the sign component.

Coinvariant spaces ($k = 1$)

Theorem (Artin ∼1950's)

$$\dim(\mathcal{DR}^{m\ \varepsilon}_{1,n}) = 1$$

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$$\dim(\mathcal{DR}^{m}_{1,n}) = n!$$

Bivariate diagonal coinvariant spaces ($k = 2$)

Theorem (Haiman 2002)

$$\dim(\mathcal{DR}_{2,n}^{m}{}^{\varepsilon}) = \frac{1}{(m+1)n+1}\binom{(m+1)n+1}{n}.$$

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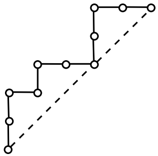
$$\dim(\mathcal{DR}_{2,n}^{m}) = (mn+1)^{n-1}.$$

Combinatorial interpretations of these numbers ($k = 2$)

Theorem (classic)

The number of m-Dyck paths of height n is

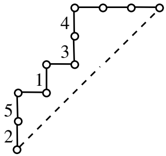
$$\frac{1}{(m+1)n+1}\binom{(m+1)n+1}{n}.$$



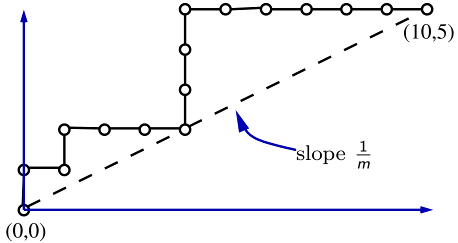
Theorem (classic)

The number of m-parking functions of height n is

$$(mn+1)^{n-1}.$$

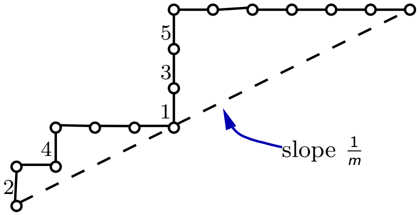


m-Dyck paths ($m = 2$)



- paths consisting of north and east steps.
- starting at (0,0) and finishing at (mn, n).
- never going below the line of equation $my = x$.

m-parking functions ($m = 2$)



- m-Dyck paths.
- n north steps are labelled with the values $\{1, 2, \ldots n\}$.
- labels increase along consecutive north steps.

Trivariate diagonal coinvariant spaces ($k = 3$)

Conjecture (m=1: Haiman 1994; $m > 1$: F. Bergeron 2009)

$$\dim(\mathcal{DR}_{3,n}^{m\ \varepsilon}) = \frac{m+1}{n(mn+1)}\binom{(m+1)^2 n + m}{n-1}.$$

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Combinatorial questions: what do these numbers count?

Combinatorial interpretations of these numbers ($k = 3$)

Theorem ($m = 1$: Chapoton 2006; $m > 1$: MBM, Fusy, LFPR 2011) (Conj. $m > 1$: Bergeron 2009)

The number of intervals in the m-Tamari lattice defined on m-Dyck paths of height n is

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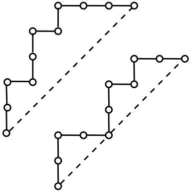
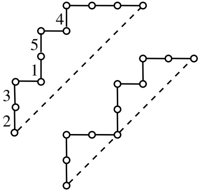


Theorem (MBM, GC, LFPR 2011) (Conj. Bergeron 2008-2009)

The number of labelled *intervals in the m-Tamari lattice defined on m-Dyck paths of height n is*

$$(m+1)^n (mn+1)^{n-2}.$$



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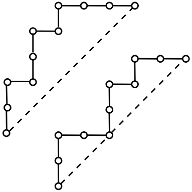

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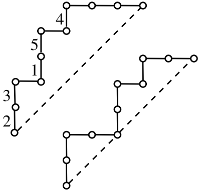
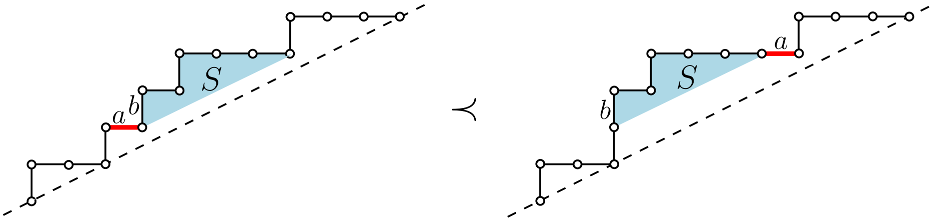


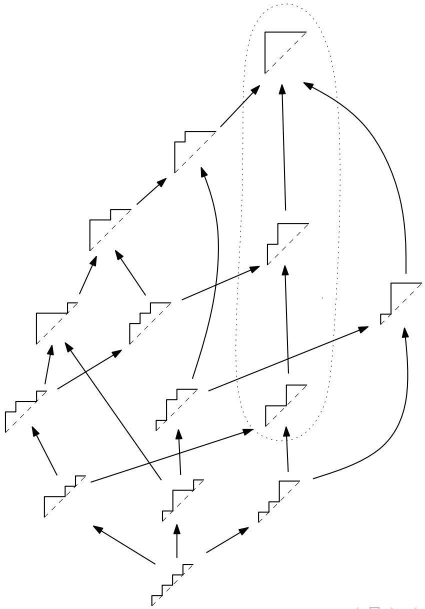
Proof.

Ideas of the proofs will be given in the next section. □

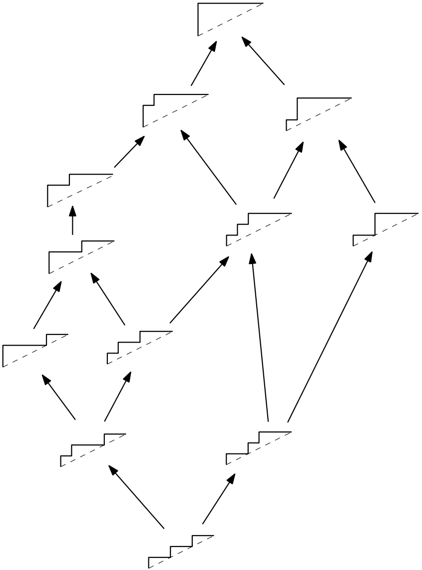
The cover relation in the m-Tamari lattice (m=2)



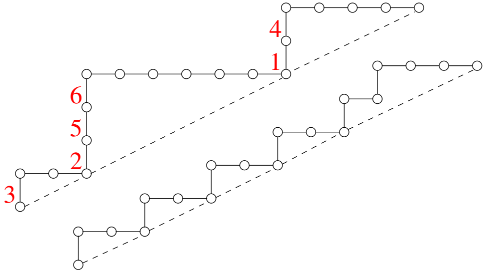
Tamari lattice on Dyck paths of height 4 ($m = 1$)



2-Tamari lattice on 2-Dyck paths of height 3

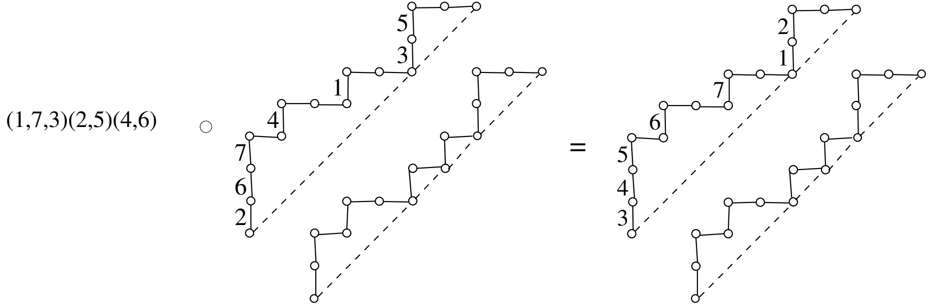


The labelled intervals in the m-Tamari lattice ($m = 2$)



- an interval in the m-Tamari lattice
- the top path is an m-parking function

A representation on labelled intervals in the m-Tamari lattice



We denote this combinatorial representation by $\mathrm{Tam}_m(n)$.

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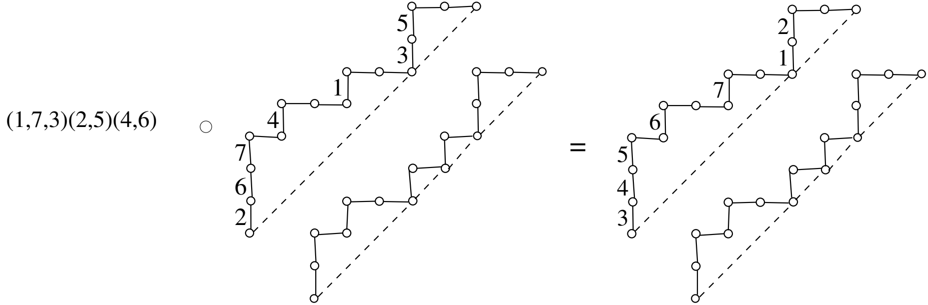

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Conjecture (F. Bergeron, LFPR 2010)

$$\mathcal{DR}^m_{3,n} \;\cong\; \varepsilon \otimes \mathrm{Tam}_m(n).$$

The character of the m-Tamari representation

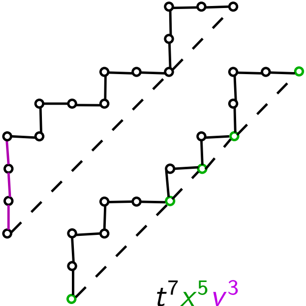
Theorem (MBM, GC, LFPR 2012) (Conj. Bergeron, LFPR 2010)

In the representation $\mathrm{Tam}_m(n)$, *the number of labelled intervals in the* m*-Tamari lattice fixed under a permutation of cycle type* $\lambda = (\lambda_1, .., \lambda_\ell)$ *is given by*

$$(mn+1)^{\ell-2} \prod_{1 \le i \le \ell} \binom{(m+1)\lambda_i}{\lambda_i}.$$

Outline

Parameters for m-Tamari intervals



$t^7 x^5 y^3$

t : height of the paths

x : contacts of the lower path

y : first rise of the top path

$$F^{(m)}(t; x, y) := \sum_{n \geq 0} \sum_{\substack{\alpha \leq \beta \\ \alpha, \beta \in \mathrm{Dyck}_m(n)}} t^n \, x^{contacts(\alpha)} \, y^{rise(\beta)}$$

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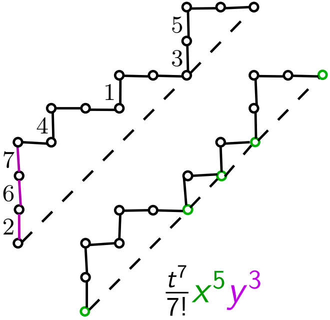

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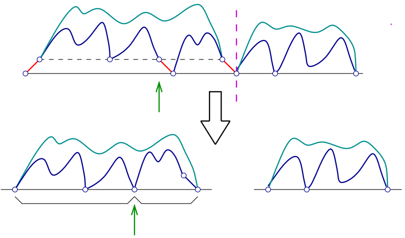
$$G^{(m)}(t; x, y) := \sum_{n \geq 0} \sum_{\substack{\alpha \leq \beta \\ \alpha, \beta \in \mathrm{Dyck}_m(n)}} \sum_{P \in \mathrm{Park}_m(\beta)} \frac{t^n}{n!} x^{contacts(\alpha)} y^{rise(\beta)}$$

Recurrence on unlabelled m-Tamari intervals ($m = 1$)



$$F(t; x, y)$$

$$= x + txy \, \frac{F(t; x, y) - F(t; 1, y)}{x - 1} \, F(t; x, 1).$$

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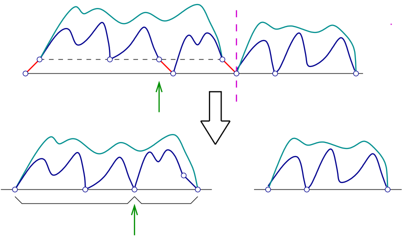

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With $\Delta(R(t; x)) := \frac{R(t;x)-R(t;1)}{x-1}$, this reads:

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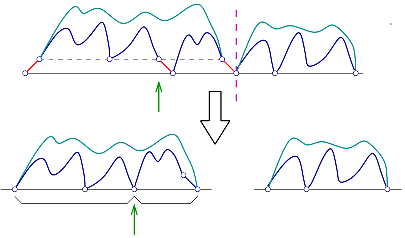

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For general m:

$$F^{(m)}(t;x,y) = x + txy(F^{(m)}(t;x,y)\Delta)^m\,(F^{(m)}(t;x,y)).$$

Polynomial equations with a catalytic variable ($y = 1$)

$$F(t;x) = x + tx\,\frac{F(t;x) - F(t;1)}{x-1}\,F(t;x).$$

Divided difference: x is a catalytic variable (Zeilberger??).

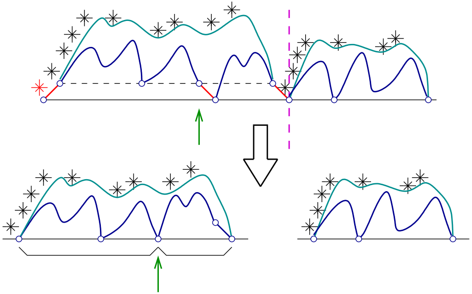
Such equations are ubiquitous in map enumeration.

The solution is always an algebraic series (Brown-Tutte 1960's, Bousquet-Mélou-Jehanne 2006).

i.e. there exist a polynomial Q such that:

$$Q(F(t;x), t, x) = 0.$$

Recurrence on labelled m-Tamari intervals ($m = 1$)



With $\Delta(R(t;x)) := \frac{R(t;x)-R(t;1)}{x-1}$, for general m:

$$G^{(m)}(t;x,y) = x + tx \int \left(G^{(m)}(t;x,1)\Delta\right)^m \left(G^{(m)}(t;x,y)\right) dy.$$

Alternative forms of the functional equations

$$F^{(m)}(t;x,y) = x + txy(F^{(m)}(t;x,1)\Delta)^m \, (F^{(m)}(t;x,y))$$

is equivalent to

$$F^{(m)}(t;x,y) = \frac{1}{1 - t\,x\,y(F^{(m)}(t;x,1)\Delta)^m}\,(x).$$

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is equivalent to

$$G^{(m)}(t;x,y) = e^{txy\,(G^{(m)}(t;x,1)\Delta)^m} \, (x).$$

Guessing a solution for $F^{(m)}(t;x,y)$ and $G^{(m)}(t;x,y)$

$$\Delta(R(t;x)) := \frac{R(t;x) - R(t;1)}{x-1}$$

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algebraic $\Rightarrow$ use Gfun (Maple)

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$$t = z(1-z)^{m^2+2m}$$

$$x = \frac{1+u}{(1+zu)^{m+1}}$$

$$F^{(m)}(t;x,1) = \frac{1+u}{(1-z)^{m+2}}\left(\frac{1+u-(1+zu)^{m+1}}{(1+zu)^{m+1}u}\right)$$

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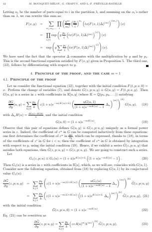
$$t = z(1-z)^{m^2+2m} \qquad t = ze^{-m(m+1)z}$$

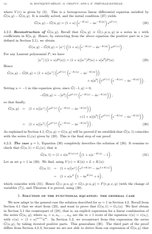
$$x = \frac{1+u}{(1+zu)^{m+1}} \qquad x = \frac{1+u}{e^{mzu}}$$

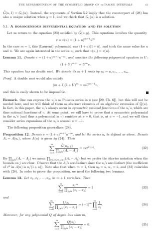
$$F^{(m)}(t;x,1) = \frac{1+u}{(1-z)^{m+2}}\left(\frac{1+u-(1+zu)^{m+1}}{(1+zu)^{m+1}u}\right) \qquad G^{(m)}(t;x,1) = \frac{1+u}{e^{-(m+1)z}}\left(\frac{1+u-e^{mzu}}{e^{(m-1)zu}u}\right)$$

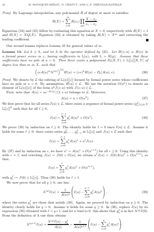
algebraic $\Rightarrow$ use Gfun (Maple) ????

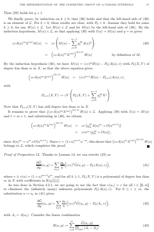
Proof of checking: le début . . .











Outline

Open combinatorial questions

- Simplify the checking ($m > 1$).
- Prove the formulas without guessing.
- Bijective proofs? Connections with certain maps? Why is F(t;x,y) symmetric in x and y?
- Combinatorial problems involving statistics on these objects (the statistics correspond to the grading of these spaces).