# Graded tensor product multiplicities from quantum cluster algebras

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# Outline

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1. Graded tensor products
2. Cluster algebras and quantum cluster algebras
3. Grading from quantization

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# The idea of grading

Tensor products of the 2-dimensional representation $V(\omega) \simeq \mathbb{C}^2$ of $\mathfrak{sl}_2$:

$$V(\omega) \otimes V(\omega) \quad \simeq \quad V(0) \oplus V(2\omega)$$

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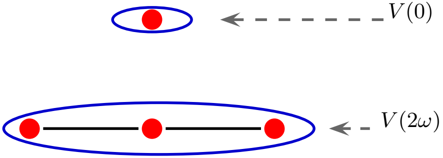

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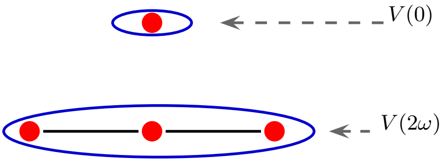


$$\mathrm{ch}_t(V(\omega) \otimes V(\omega)) = \mathrm{ch}(V(2\omega) + t\ \mathrm{ch}(V(0)))$$

# The idea of graded tensor products

- Grading on tensor products of $\mathfrak{g}$ (simple Lie algebra) or $U_q(\mathfrak{g})$ (quantum algebra) modules.

$$\mathrm{ch}\,(V_1 \otimes V_2 \otimes \cdots \otimes V_n) = \sum_{V:\mathrm{irred}} M_{\{V_i\},V}\ \mathrm{ch}V$$

$$\Downarrow \qquad \text{Introduce grading}$$

$$\mathrm{ch}_t\,(V_1 \star V_2 \star \cdots \star V_n) = \sum_{V:\mathrm{irred}} M_{\{V_i\},V}(t)\ \mathrm{ch}V$$

- $M_{\{V_i\},V}(t)$: "graded multiplicity" of the irreducible component $V$ in the graded product.
- What is a good definition of the grading?

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# Algebraic source of grading

| Finite-dimensional algebra | $\subset$ | Infinite-dimensional algebra: |
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| simple Lie algebra $\mathfrak{g}$ | $\subset$ | $\widehat{\mathfrak{g}}$, $Y(\mathfrak{g})$ affine algebra, Yangian |
| quantum algebra $U_q(\mathfrak{g})$ | $\subset$ | $U_q(\widehat{\mathfrak{g}})$ quantum affine algebra |

- The infinite-dimensional algebra is graded: induce a grading on modules $W$.
- Restrict the action to finite-dim subalgebra:

$$M_{W,V} = \dim \mathrm{Hom}_{\mathfrak{g}}(W,V),$$

$W$: finite-dim. $\widehat{\mathfrak{g}}$-mod
$V$: irreducible $\mathfrak{g}$-mod.

- **Hilbert polynomial:** $\mathfrak{g}$ acts on the graded components $W[n]$

$$M_{W,V}(t) := \sum_{n\geq 0} t^n \dim \mathrm{Hom}_{\mathfrak{g}}(W[n],V), \qquad M_{W,V}(1) = M_{W,V}.$$

- **Graded characters:**

$$\mathrm{ch}_t(W) = \sum_{V;\mathrm{irred}} M_{W,V}(t)\mathrm{ch}(V)$$

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- Choose a set of finite-dimensional modules $\{V_1, ..., V_n\}$ of infinite-dim alg:

$$W \simeq V_1 \otimes \cdots \otimes V_N.$$

- Grading on $W$ can be defined, for example:
  1. Combinatorially From Bethe ansatz of generalized Heisenberg model (Yangian). [Kerov, Kirillov, Reshetikhin, '86; Kuniba, Nakanishi, Okado '93]; Physical interpretation from conformal field theory [K., McCoy '91].
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  3. Natural grading of $\widehat{\mathfrak{g}}$ = central extension of $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ by degree in $t$. [Feigin-Loktev "fusion product", '99].

**Theorem**

*If the modules $V_i$ are of sufficiently simple (KR-type) the three ways of defining gradings on the tensor products give the same Hilbert polynomials.*

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- FL defined a (commutative) graded tensor product on $\widehat{\mathfrak{g}}$-modules

$$\mathcal{F}_{\{V_i\}} = V_1 \star V_2 \star \cdots \star V_N$$

- The Hilbert polynomial $M_{\{V_i\},\lambda}(t) := \sum_{n\geq 0} t^n \dim \operatorname{Hom}_{\mathfrak{g}} \left(\mathcal{F}_{\{V_i\}}[n], V(\lambda)\right)$

  $V(\lambda)$=Irreducible $\mathfrak{g}$-module.
- Example 1: If $\mathfrak{g} = \mathfrak{sl}_n$ and $V_i$ are symmetric power representations, $M_{\{V_i\},\lambda}(t)$ is a Kostka polynomial (transition function between Hall-Littlewood polynomials and Schur polynomials).
- Example 2: If $\mathfrak{g} = \mathfrak{sl}_n$ and $V_i = V(m\omega_j)$ (Kirillov-Reshetikhin modules), $M_{\{V_i\},\lambda}(t)$ is a generalized Kostka polynomial [Lascoux, Leclerc, Thibon].
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- Example 1: If $\mathfrak{g} = \mathfrak{sl}_n$ and $V_i$ are symmetric power representations, $M_{\{V_i\},\lambda}(t)$ is a Kostka polynomial (transition function between Hall-Littlewood polynomials and Schur polynomials).
- Example 2: If $\mathfrak{g} = \mathfrak{sl}_n$ and $V_i = V(m\omega_j)$ (Kirillov-Reshetikhin modules), $M_{\{V_i\},\lambda}(t)$ is a generalized Kostka polynomial [Lascoux, Leclerc, Thibon].
- The Hilbert polynomials give Betti numbers of cohomology of Lagrangian quiver varieties (Nakajima, Lusztig, Kodera-Naoi...)
- Have an interpretation as parabolic Kazhdan-Lusztig polynomials...

# Example of Feigin-Loktev product for $\mathfrak{g} = \mathfrak{sl}_2$

$$\mathfrak{sl}_2 = \langle f, h, e \rangle, \qquad \widetilde{\mathfrak{sl}}_2 = \langle x[m] = xt^m \rangle_{x \in \mathfrak{sl}_2, m \in \mathbb{Z}}, \qquad \mathfrak{sl}_2 \simeq \langle x[0] \rangle \subset \widetilde{\mathfrak{sl}}_2$$

Define Action of $\widetilde{\mathfrak{sl}}_2$ on the tensor product of two representations

$$x[m] v_1 \otimes v_2 = z_1^m (x v_1) \otimes v_2 + z_2^m v_1 \otimes (x v_2), \quad v_1 \otimes v_2 \in V_1 \otimes V_2.$$

Filtration of $\mathcal{F} = U(f[i]_{i \geq 0}) v_1 \otimes v_2$:

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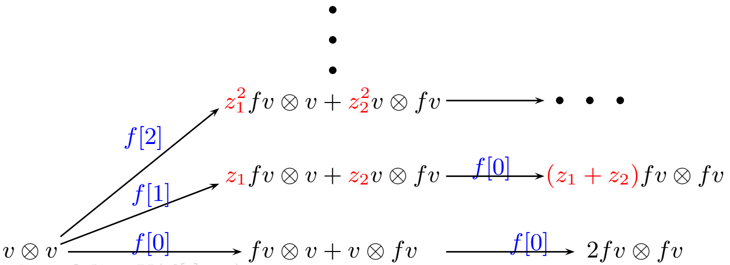
Example: $V_1 = V_2 = V(\omega_1) \simeq \mathrm{Span}\{v, fv\}$ with $f^2 v = 0$:

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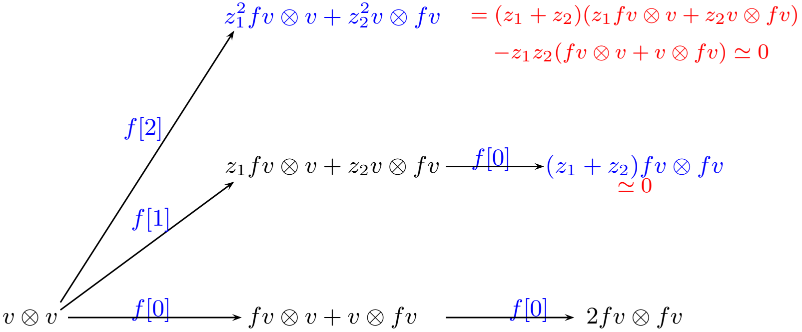
# Example of graded product $V(\omega_1) \star V(\omega_1)$:



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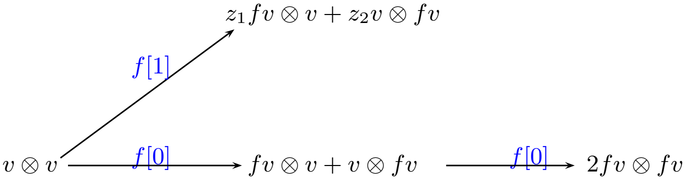

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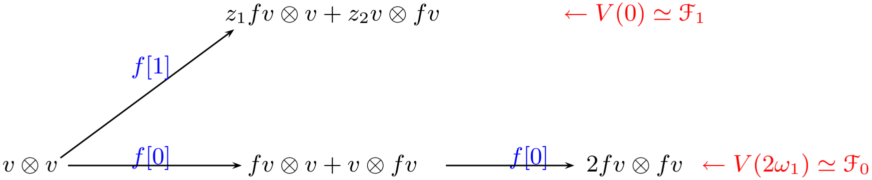

# Example of graded product $V(\omega_1) \star V(\omega_1)$:

$$M_{V^2,0}(t) = t$$

$$M_{V^2,2\omega_1}(t) = 1$$

$$v \otimes v \xrightarrow{f[1]} z_1 fv \otimes v + z_2 v \otimes fv \qquad \leftarrow V(0) \simeq \mathfrak{F}_1$$

$$v \otimes v \xrightarrow{f[0]} fv \otimes v + v \otimes fv \xrightarrow{f[0]} 2fv \otimes fv \quad \leftarrow V(2\omega_1) \simeq \mathfrak{F}_0$$

# Example of graded tensor product for $\mathfrak{sl}_2$

Triple tensor product $V_1 \star V_2 \star V_3$ with $V_1 = V_2 = V_3 = V(\omega_1)$.

$$M_{V^3,\omega_1}(t) = t + t^2; \quad M_{V^3,3\omega_1}(t) = 1$$

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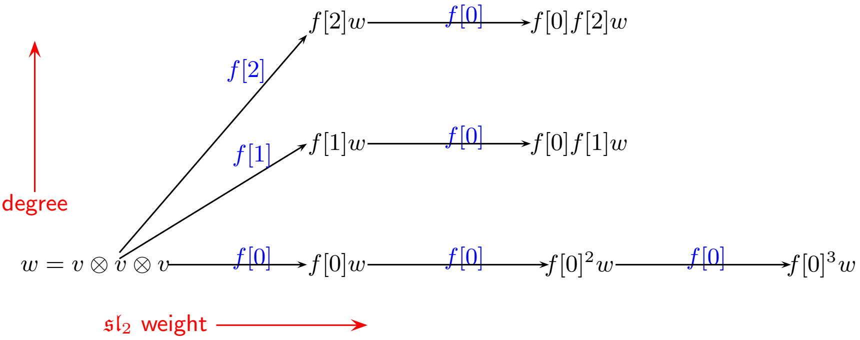

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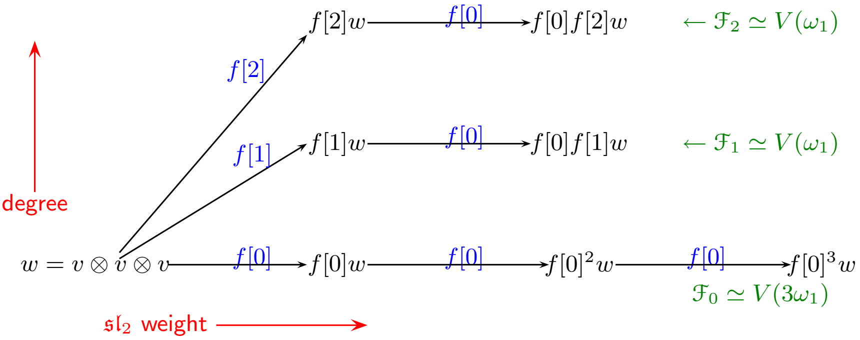

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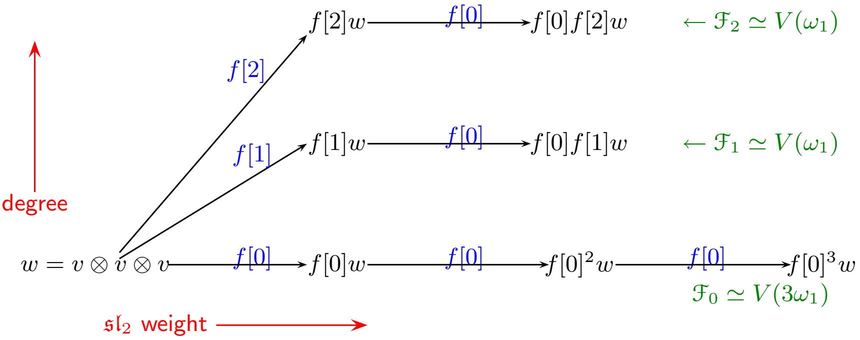

# Explicit formula for graded multiplicities: $\widehat{\mathfrak{sl}}_2$

Choose a collection of irreducible $\mathfrak{sl}_2$-modules:

$$\{V_i\} = \{\underbrace{V(\omega_1), \ldots, V(\omega_1)}_{n_1 \text{ times}}, \underbrace{V(2\omega_1), \ldots, V(2\omega_1)}_{n_2 \text{ times}}, \ldots, \underbrace{V(j\omega_1), \ldots, V(j\omega_1)}_{n_j \text{ times}}, \ldots\}$$

**Theorem:** There is a formula for the multiplicity of irreducible components:

$$\dim \mathrm{Hom}_{\mathfrak{g}}(V_1 \otimes \cdots \otimes V_n, V(\lambda)) = M_{\{V_i\},\lambda} = \sum_{m_1, m_2, \ldots \in \mathbb{Z}_+} \prod_{i \geq 1} \binom{p_i + m_i}{m_i}$$

- Sum $\sum$ is restricted: $\sum_j j(n_j - 2m_j)\omega_1 = \lambda$
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"Fermionic formula"

**Theorem:** Hilbert polynomials of the FL product are Kostka polynomials

$$M_{\{V_i\},\lambda}(t) = \sum_{m_i} t^{\mathbf{m}^t A \mathbf{m}} \prod_i \begin{bmatrix} p_i + m_i \\ m_i \end{bmatrix}_t, \quad [A]_{i,j} = \min(i,j).$$

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For $\mathfrak{g}$ simply-laced with Cartan matrix $C$, choose $\{V_i\}$: Collection of KR-modules: $n_{a,j}$ modules with highest weight $j\omega_a$.

**Theorem:** The FL graded tensor product multiplicities are

$$M_{\mathbf{n},\lambda}(t) = \sum_{\{m_{a,j}\}} t^{\frac{1}{2}\mathbf{m}^t(C\otimes A)\mathbf{m}} \prod_{a,j} \begin{bmatrix} p_{a,j}+m_{a,j} \\ m_{a,j} \end{bmatrix}_t$$

$$\mathbf{p} = (I\otimes A)\mathbf{n} - (C\otimes A)\mathbf{m}, \quad A_{ij} = \min(i,j)$$

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The restrictions on the sum are:

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The restrictions on the sum are:

(a) positive integers $p_{i,j} \geq 0$

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# Explicit formula generalizes to other $\mathfrak{g}$

For $\mathfrak{g}$ simply-laced with Cartan matrix $C$, choose $\{V_i\}$: Collection of KR-modules: $n_{a,j}$ modules with highest weight $j\omega_a$.

**Theorem:** The FL graded tensor product multiplicities are

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1. *For any set of Kirillov-Reshetikhin modules* $\{V_i\}$ *of any simple Lie algebra* $\widehat{\mathfrak{g}}$,

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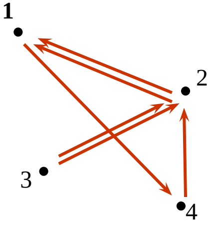
# Cluster algebras: quiver dynamics

Start with a quiver $\Gamma$ with no one-cycles or two cycles:




Corresponds to a skew-symmetric matrix $B$
rows and columns labeled by vertices
The $(i,j)$ entry $=$ number of arrows from $i$ to $j$.

$$B = \begin{pmatrix} 0 & -2 & 0 & 1 \\ 2 & 0 & -2 & -1 \\ 0 & 2 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix}$$

**Dynamics of quiver:** For each vertex label $v$, mutation $\mu_v$ acts on the quiver:
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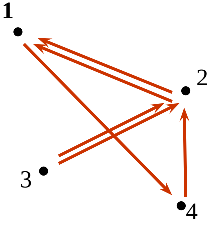


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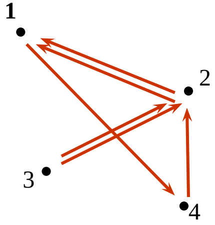


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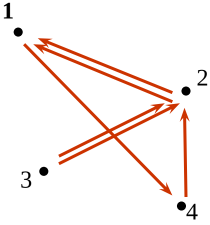


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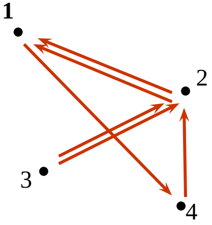


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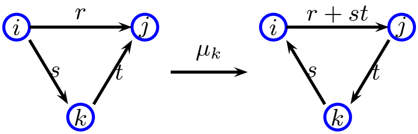
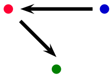



-Reverse incident arrows on node $k$
-create a shortcut for path of length 2 through $k$
-cancel 2-cycles.

# Example of quiver mutations



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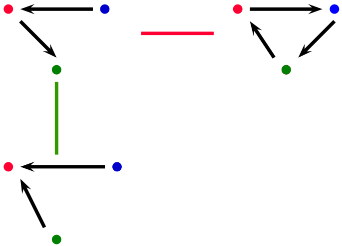

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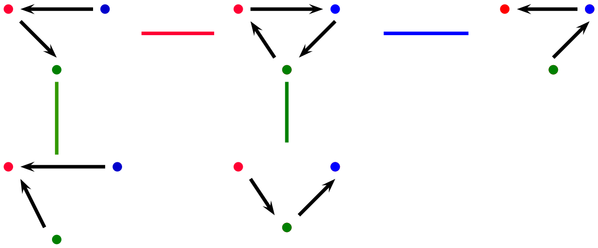

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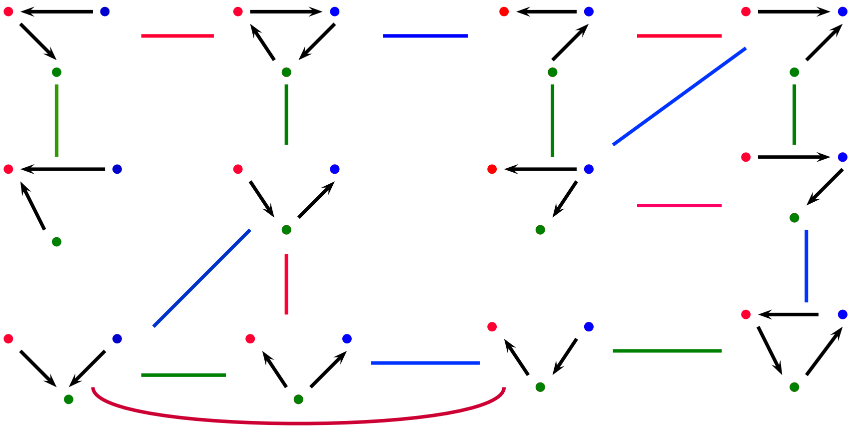

# Cluster variable mutation

- To each node $v$ in $\Gamma$ associate a variable $x_v$. Collection $\mathbf{x} = \{x_v : v \in \Gamma\}$.
- Mutation $\mu_v$ acts on $x_v$ according to the number of incoming and outgoing arrows from vertex $v$:

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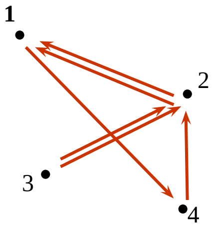


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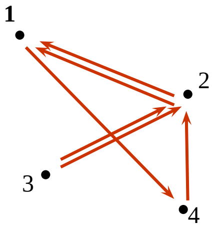


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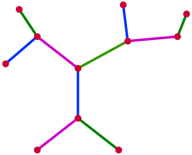
# Evolution tree

- A quiver $\Gamma$ with $n$ nodes $\Longrightarrow$ a complete $n$-tree $\mathbb{T}_n$. Labeled edges.



At each node • of the tree:
data $(\mathbf{x} = \{x_v\}_{v\in\Gamma}, \Gamma)_\bullet = (n$ cluster variables, quiver).

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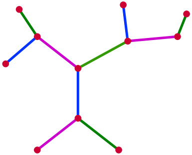

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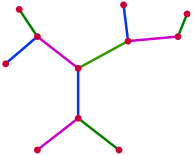

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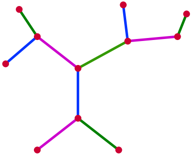

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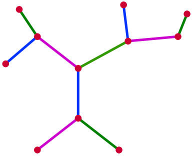

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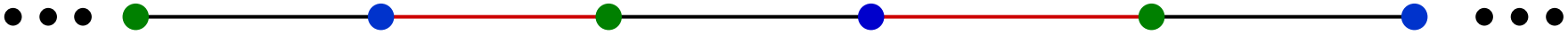

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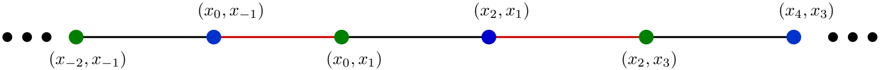

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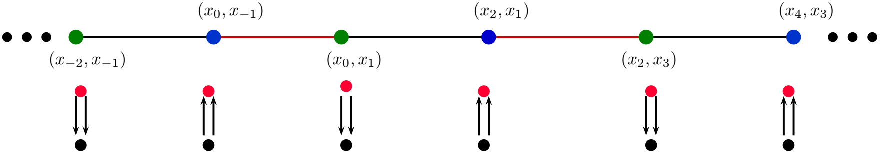

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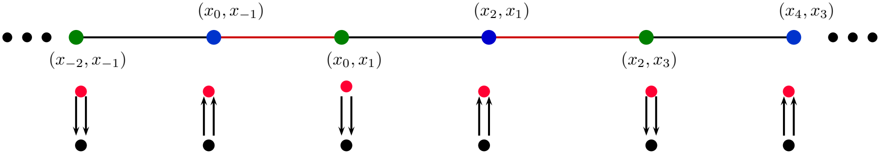


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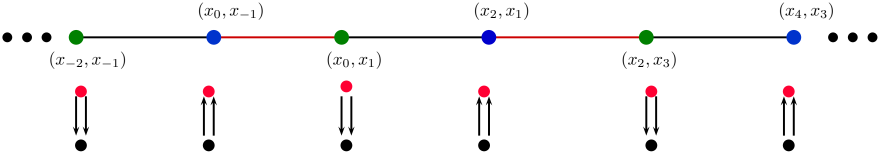


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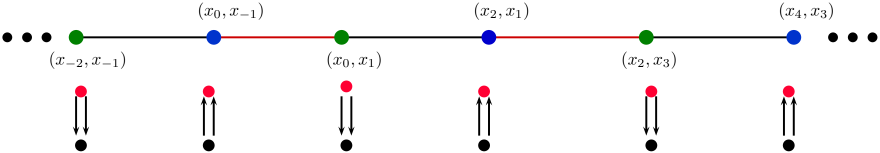


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Laurent polynomials with coefficients in $\mathbb{Z}_+$ in initial data.

# Some facts about cluster algebras

- **Finite** cluster algebras classified by finite simple Lie algebra Dynkin diagrams. [Fomin-Zelevinsky].
- **Quiver-finite** cluster algebras classified by Felikson, Shapiro, Tumarkin.
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- **Finite** cluster algebras classified by finite simple Lie algebra Dynkin diagrams. [Fomin-Zelevinsky].
- **Quiver-finite** cluster algebras classified by Felikson, Shapiro, Tumarkin.
- **Laurent property Theorem:** In terms of any choice of initial data, cluster variables are Laurent polynomials (not just rational functions!).
- **Positivity conjecture:** with coefficients in $\mathbb{Z}_+$. Proof in special cases.

# Quantum cluster algebras

- **Compatible Poisson structure** on cluster algebras [Gekhtman, Shapiro, Vainshtein] [Fock and Goncharov] $\Longrightarrow$ quantum deformation [Berenstein, Zelevinsky].
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  - Generators $\{X_1, \dots, X_n\}$: variables associated to node of $\mathbb{T}_n$;
  - Data $\Gamma \sim$ skew symmetric matrix $B$ (same as in commutative case).
  - New matrix: $\Lambda \propto B^{-1}$ an integer matrix.
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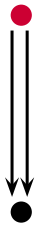
# Example of quantum evolution: Rank 2

Choose $\Gamma =$  Initial data $(X_0, X_1)$

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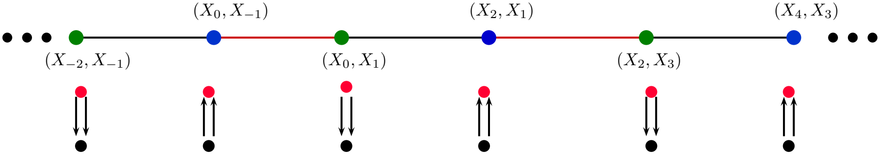

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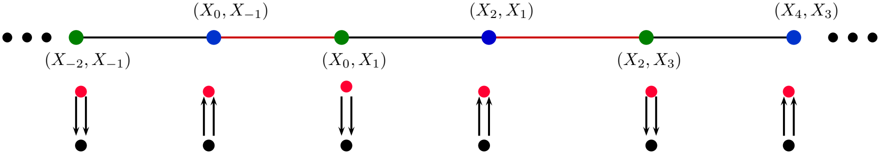


$$\begin{aligned}
X_2 &= (qX_1^2 + 1)X_0^{-1} \\
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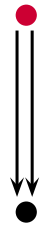
Choose $\Gamma =$  Initial data $(X_0, X_1)$

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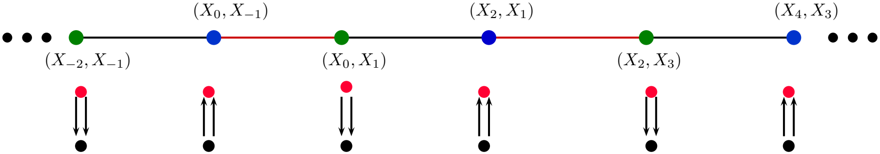


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Laurent polynomials with coefficients in $\mathbb{Z}_+[q, q^{-1}]$ in initial data.

## Next: Grading from Quantization

- There is a cluster algebra associated with the explicit formulas for tensor product multiplicities
- The graded tensor product multiplicities are associated with the quantization of this cluster algebra.

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# Tensor product multiplicities

$$M_{\mathbf{n},\ell}(1) = \sum_{\substack{m_i \geq 0 \\ 2\sum i m_i = \sum_i n_i - \ell \\ p_i \geq 0}} \prod_{i\geq 1} \binom{p_i + m_i}{m_i}$$

Multiplicity formula for $\mathfrak{sl}_2$ tensors
$p_i = \sum \min(i,j)(n_j - 2m_j)$

$\Downarrow$

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Relax restrictions on the sum:
This is not a manifestly non-negative sum!

$\Downarrow$

$$Z_{\mathbf{n},\ell}(y) = \sum_{\{m_i\}} y^p \prod_{i\geq 1} \binom{p_i - p + m_i}{m_i}$$

No restrictions on the sum.
$p \stackrel{\text{def}}{=} \sum i(n_i - 2m_i) - \ell$

The sum $N_{\mathbf{n},\ell}(1)$ is the **constant term** of $Z_{\mathbf{n},\ell}(y)$.

(Repeat for the other Lie algebras $\mathfrak{g}$.)

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# Theorem for generating functions

## Theorem (Di Francesco, K.)

1. *The generating function factorizes:*

$$Z_{\mathbf{n},\ell}(y) = \chi_1 \prod_{i \geq 0} \chi_i^{n_i} \left( \frac{\chi_k}{\chi_{k+1}} \right)^{\ell+1}, \qquad k \gg 0$$

*where $\chi_i$ are solutions of* $\chi_{i+1} = \dfrac{\chi_i^2 - 1}{\chi_{i-1}}, \qquad \chi_0 = 1, \chi_1 = y.$

2. *The modified sum $N_{\mathbf{n},\ell}(1)$ is equal to the multiplicity $M_{\mathbf{n},\ell}(1)$ because the solutions of the recursion $\chi_i$ are polynomials in the initial data $\chi_1$.*

This recursion relation is known as the $Q$-system. (Solutions are Chebyshev polynomials of second type).

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$$Z_{\mathbf{n},\ell}(y) = \chi_1 \prod_{i\geq 0} \chi_i^{n_i} \left(\frac{\chi_k}{\chi_{k+1}}\right)^{\ell+1}, \qquad k \gg 0$$

*where* $\chi_i$ *are solutions of* $\chi_{i+1} = \frac{\chi_i^2 - 1}{\chi_{i-1}}, \qquad \chi_0 = 1, \chi_1 = y.$

2. *The modified sum* $N_{\mathbf{n},\ell}(1)$ *is equal to the multiplicity* $M_{\mathbf{n},\ell}(1)$ *because the solutions of the recursion* $\chi_i$ *are polynomials in the initial data* $\chi_1$.

This recursion relation is known as the $Q$-system. (Solutions are Chebyshev polynomials of second type).

In general, the fact that solutions to $Q$-systems are polynomials follows from two facts:

- The equations are mutations in a cluster algebra.
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# Theorem for generating functions

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# Cluster algebra for the $Q$-system

The $Q$-system for $A_1$

$$\chi_{n+1}\chi_{n-1} = \chi_n^2 - 1$$

For other simply-laced Lie algebras with Cartan matrix $C$:

$$Q_{\alpha,n+1}Q_{\alpha,n-1} = Q_{\alpha,n}^2 + \prod_{\beta\sim\alpha} Q_{\beta,n}$$

Associated with the quiver $\Gamma$ ∼exchange matrix $B$:

$$B = \begin{pmatrix} 0 & -C \\ C & 0 \end{pmatrix}$$

Theorem (K.)

*Each of the $Q$-system relations is a mutation of cluster variables in the mutation tree with initial data $((Q_{\alpha,0}; Q_{\alpha,1})_{1\leq\alpha\leq r}, B)$.*

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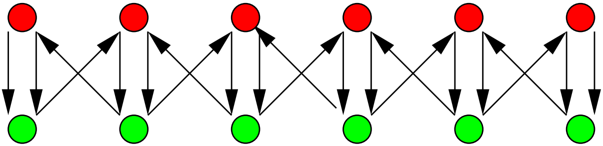
# Example of quiver mutations for $A_6$ $Q$-system



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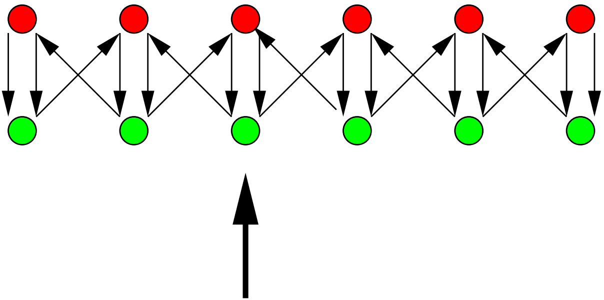

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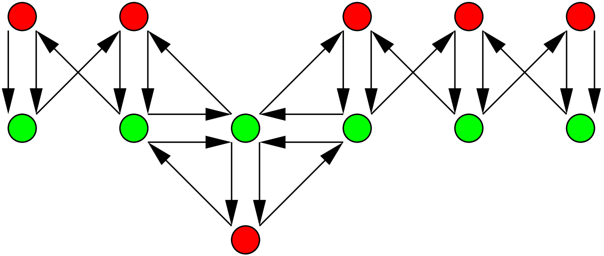

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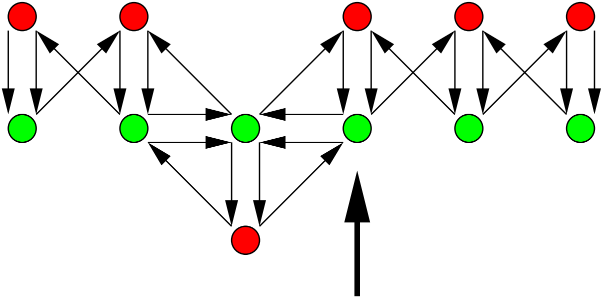

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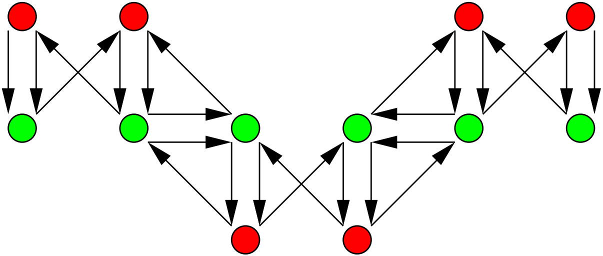

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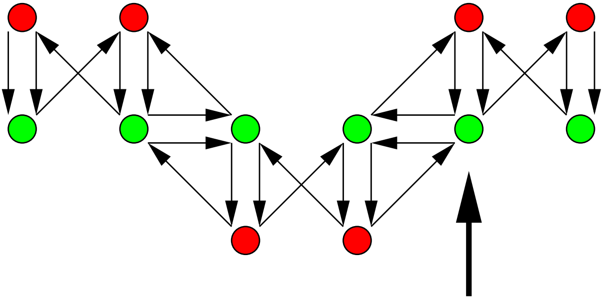

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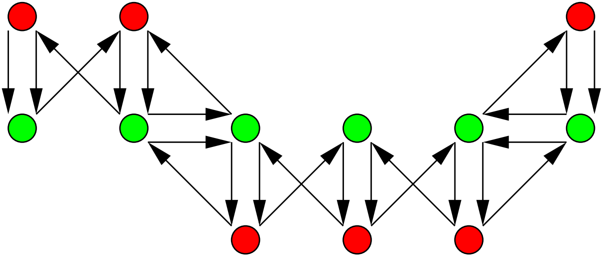

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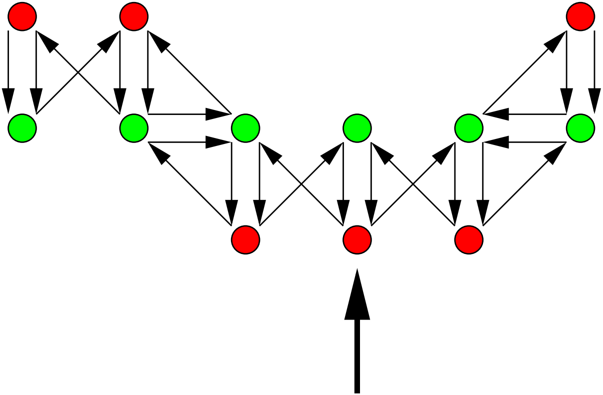

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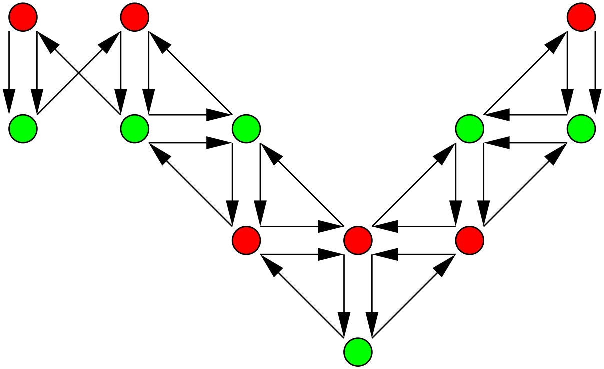

# Graded multiplicities from quantum cluster algebra

**Quantum $Q$-system:** The quantum deformation of the $Q$-system cluster algebra is

$$q^{\lambda_{a,a}} Q_{a,j+1} Q_{a,j-1} = Q_{a,j}^2 + \prod_{a\sim b} Q_{b,j}, \qquad \lambda = \mathrm{Det} C\; C^{-1},$$

**Commutation relations:** $Q_{a,j} Q_{b,j+1} = q^{\lambda_{ab}} Q_{b,j+1} Q_{a,j}$.

**Theorem: The Polynomiality property for the quantum $Q$-system**

Write the ordered expression $\chi_{a,j} = \sum_{m_b, n_b} a_{\mathbf{m},\mathbf{n}} \prod_b \chi_{b,1}^{m_b} \chi_{b,0}^{n_b}$. Then

$$\chi_{a,j}(\chi_{b,0} = 1) \in \mathbb{Z}[q, q^{-1}][\chi_{1,1}, \ldots, \chi_{r,1}] \qquad \text{a polynomial!}$$

follows from Laurent polynomiality for quantum cluster algebras.

Warning: Apply only to normal-ordered expressions!

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# Graded tensor product multiplicities

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Recall: For the ungraded multiplicities:

$$M_{\mathbf{n},\ell}(1) = \sum_{\substack{\{m_i\} \\ 2\sum i m_i = \sum_i n_i - \ell \\ p_i \geq 0}} \prod_{i\geq 1} \binom{p_i + m_i}{m_i}$$

Multiplicity formula for $\mathfrak{sl}_2$ tensors

$$\Downarrow$$

$$N_{\mathbf{n},\ell}(1) = \sum_{\substack{\{m_i\} \\ 2\sum i m_i = \sum_i n_i - \ell}} \prod_{i\geq 1} \binom{p_i + m_i}{m_i}$$

Relax restrictions on the sum

$$\Downarrow$$

$$Z_{\mathbf{n},\ell}(y) = \sum_{\{m_i\}} y^p \prod_{i\geq 1} \binom{p_i - p + m_i}{m_i}$$

No restrictions on the sum.
$p \stackrel{\text{def}}{=} \sum i(n_i - 2m_i) - \ell$

The sum $N_{\mathbf{n},\ell}(1)$ is the **constant term** of $Z_{\mathbf{n},\ell}(y)$.

# Graded tensor product multiplicities

**For the graded multiplicities:**

$$M_{\mathbf{n},\ell}(t) = \sum_{\substack{\{m_i\} \\ 2\sum i m_i = \sum_i n_i - \ell \\ p_i \geq 0}} t^{\mathbf{m}^t A \mathbf{m}} \prod_{i \geq 1} \begin{bmatrix} p_i + m_i \\ m_i \end{bmatrix}_t, \quad A_{ij} = \min(i,j)$$

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$Z_{\mathbf{n},\ell}(t;X,Y)$ is constructed so that the $p=0$ term gives $N_{\mathbf{n},\ell}(t)$ when$X=1$.

# Constant term formula

**Theorem**

- *When $\chi_j$ satisfy the quantum $Q$-system with $q^2 = t$ the generating function*

$$Z_{\mathbf{n},\ell}(t;\chi_0,\chi_1) = t^{f(\mathbf{n})}\chi_1\chi_0^{-1}\left(\overrightarrow{\prod}\chi_j^{n_j}\right)(\chi_k\chi_{k+1}^{-1})^{\ell+1} = \sum a_{i,j}(t)\chi_1^i\chi_0^j$$

*gives* $N_{\mathbf{n},\ell}(t) = \sum_j a_{0,j}(t)$.

- *The polynomiality property for the quantum $Q$-system cluster algebra implies* $M_{\mathbf{n},\ell}(t) = N_{\mathbf{n},\ell}(t)$.

- **Remark:** This is a good thing: $M$-sum is a subtraction-free expression for a multiplicity.
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- Remark: The same quantum $Q$-system is related to the problem of finding canonical bases. The sums $M_{W,V}(t)$ appear as Betti numbers in the cohomology of quiver varieties.

Thank you!

# Summary

**We have established the connections:**

| | | | | |
|---|---|---|---|---|
| Fermionic multiplicity formulas | $\longleftrightarrow$ | $Q$-systems | $\longrightarrow$ | Cluster algebras |
| $\downarrow$ | | | | $\downarrow$ |
| Graded multiplicities | $\longleftarrow$ | Quantum $Q$-system | $\longleftarrow$ | Quantum cluster algebras |

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Thank you!