

Discrete Isoperimetry: Problems, Results, Applications and Methods

Gil Kalai
Hebrew University of Jerusalem

FPSAC' 12, Nagoya, July 2012.

The Isoperimetric Inequality

The Isoperimetric Inequality: The solid of volume V that minimizes the surface area in Euclidean 3-space is the ball.



A sleeping, curled-up sphynx cat called Uthello. Author : Sunny Ripert. Idea: Soul physics blog.

The Isoperimetric Inequality

The Isoperimetric Inequality: The solid of volume V that minimizes the surface area in Euclidean 3-space is the ball.



Combinatorial Isoperimetric Relations - some examples

- ▶ (Erdős-Hanani:) A simple graph with m edges that minimizes the number of vertices is obtained from a complete graph by adding a vertex and some edges.
- ▶ The Kruskal-Katona Theorem.
- ▶ Macaulay's theorem.
- ▶ Harper's theorem.
- ▶ A theorem of Descartes: A simple planar graph with m edges has at least $m/3 + 2$ vertices.

Prelude: Baby Harper and canonical paths

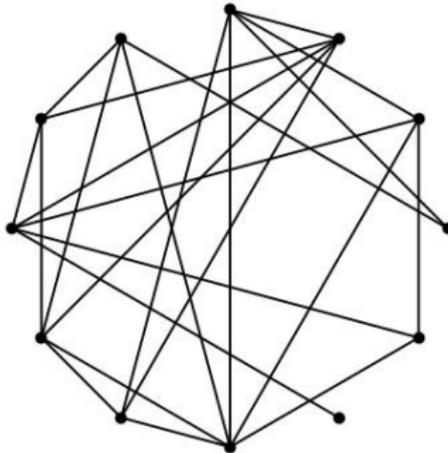
Theorem: For a subset A of the discrete n -cube with $|A| \leq 2^{n-1}$ there are at least $|A|$ edges from A to its complement.

Proof (hint): Consider a canonical path between two 0-1 vectors x and y by flipping the disagreeing-coordinates from left to right. Then look at all canonical paths from all $x \in A$ and $y \notin A$. There are $|A|(2^n - |A|)$ such paths and each of them must contain an edge from A to its complement.

On the other hand, every edge is contained in at most 2^{n-1} canonical paths.

Part I: Isoperimetry, harmonic analysis, and probability

The first part of this lecture is about harmonic analysis applied to discrete isoperimetry. We have several application and potential applications in mind mainly to problems in probability. I will start by mentioning one potential application. It deals with the theory of random graphs initiated by Erdős and Rényi, and the model



$G(n, p)$.
In the picture we see a
random graph with $n = 12$ and $p = 1/3$.

Threshold and Expectation threshold

Consider a random graph G in $G(n, p)$ and the graph property: G contains a copy of a specific graph H . (Note: H depends on n ; a motivating example: H is a Hamiltonian cycle.) Let q be the minimal value for which the expected number of copies of H' in G is at least $1/2$ for every subgraph H' of H . Let p be the value for which the probability that G contains a copy of H is $1/2$.

Conjecture: [Kahn, K. 2006]

$$p/q = O(\log n).$$

The conjecture can be vastly extended to general Boolean functions, and we will hint on possible connection with harmonic analysis and discrete isoperimetry. (Sneak preview: it will require a far-reaching extension of results by Friedgut, Bourgain and Hatami.)

The discrete n -dimensional cube and Boolean functions

The **discrete n -dimensional cube** Ω_n is the set of 0-1 vectors of length n .

A **Boolean function** f is a map from Ω_n to $\{0, 1\}$.

A boolean function f is **monotone** if f cannot decrease when you switch a coordinate from 0 to 1.

The Bernoulli measure

Let $p, 0 < p < 1$, be a real number. The probability measure μ_p is the product probability distribution whose marginals are given by $\mu_p(x_k = 1) = p$. Let $f : \Omega_n \rightarrow \{0, 1\}$ be a Boolean function.

$$\mu_p(f) = \sum_{x \in \Omega_n} \mu_p(x) f(x) = \mu_p\{x : f(x) = 1\}.$$

The total influence

Two vectors in Ω_n are **neighbors** if they differ in one coordinate.

For $x \in \Omega_n$ let $h(x)$ be the number of neighbors y of x such that $f(y) \neq f(x)$.

The **total influence** of f is defined by

$$I^p(f) = \sum_{x \in \Omega_n} \mu_p(x) h(x).$$

If $p = 1/2$ we will omit p as a subscript or superscript.

Russo's lemma

Russo's lemma: For a monotone Boolean function f ,

$$d\mu_p(f)/dp = I^P(f).$$

Very useful in percolation theory and other areas.

The **threshold interval** for a monotone Boolean function f is those values of p so that $\mu_p(f)$ is bounded away from 0 and 1. (Say $0.01 \leq \mu_p(f) \leq 0.99$.)

A typical application of Russo's lemma: If for every value p in the threshold interval $I^P(f)$ is large, then the threshold interval itself is short. This is called a **sharp threshold** phenomenon.

(A version of) Harper's theorem

Harper's theorem: If $\mu_p(f) = t$ then

$$I^p(f) \geq 2t \cdot \log_p t.$$

There is a 3 line proof by induction.

Harmonic analysis proof: without the $\log(1/t)$ factor it follows from Parseval.

Influence of variables on Boolean functions

Let

$$\sigma_k(x_1, \dots, x_{k-1}, x_k, x_{k+1}, \dots, x_n) = (x_1, \dots, x_{k-1}, 1-x_k, x_{k+1}, \dots, x_n).$$

The influence of the k th variable on a Boolean function f is defined by:

$$I_k^p(f) = \mu_p(x \in \Omega_n, f(x) \neq f(\sigma_k(x))).$$

KKL's theorem

Theorem (Kahn, K, Linal, 1988; Bourgain Katznelson KKL 1992; Talagrand 1994 Friedgut K. 1996) There exist a variable k such that

$$I_k^p(f) \geq C \mu_p(f)(1 - \mu_p(f)) \log n/n.$$

A sharp version (due to Talagrand)

$$\sum I_k^p(f) / \log(e + I_k^p(f)) \geq C(p) \mu_p(f)(1 - \mu_p(f)).$$

Discrete Fourier analysis

We assume now $p = 1/2$. Let $f : \Omega_n \rightarrow \mathbf{R}$ be a real function.

Let $f = \sum \hat{f}(S)W_S$ be the **Fourier-Walsh** expansion of f .

Here

$$W_S(x_1, x_2, \dots, x_n) = (-1)^{\sum\{x_i:i \in S\}}.$$

Hypercontractivity and Harper's theorem:

We assume now $p = 1/2$. $f = \sum \hat{f}(S)W_S$ is the **Fourier-Walsh** expansion of f . Key ideas:

- 0 Parseval gives $I(f) = 4 \sum \hat{f}^2(S)|S|$.
- 1 Bonami-Gross-Beckner hypercontractive inequality.

$$\left\| \sum \hat{f}(S)(1/2)^{|S|} \right\|_2 \leq \|f\|_{5/4}.$$

- 2 For Boolean functions the q th power of the q norm is the measure of the support and does not depend on q . If the support is small this means that the q -norm is very different from the r -norm if $r \neq q$.

(See also : Ledoux' book on concentration of measure phenomena)

Part II: Harper's theorem: Stability and Symmetry

Low Influence and Juntas

A **dictatorship** is a Boolean function depending on one variable. A **K -junta** is a Boolean function depending on K variables.

Theorem: (Friedgut, follows easily from KKL) If p is bounded away from 0 and 1 and $IP(f) < C$ then f is close to a $K(C)$ -Junta.

This works if $\log p / \log n = o(1)$ the most interesting applications would be when p is a power of n . There the theorem is not true.

Early stability results for Harper's theorem were obtained in the 70s by Peter Frankl.

The works of Friedgut and Bourgain (1999)

Suppose that f is a Boolean function and

$$I^P(f) < pC,$$

then

Friedgut's theorem (1999): If f represent a monotone graph property then f is close to a “locally defined” function g .

Bourgain's theorem (1999): Unconditionally, f has a substantial “locally defined” ingredient.

Hatami's theorem: Pseudo-juntas

Suppose that for every subset of variables S , we have a function $J_S : \{0, 1\}^S \rightarrow \{0, 1\}$ which can be viewed as a constraint over the variables with indices in S . Now there are two conditions:

A Boolean function is a K -**pseudo-junta** if

(1) the expected number of variables in satisfied constraints is bounded by a constant K .

(2) $f(x) = f(y)$ if the variables in satisfied constraints and also their values are the same for x and y .

Hatami's theorem: For every C there is $K(C)$, such that if

$$I^P(f) < pC,$$

then f is close to a $K(C)$ -pseudo-junta.

A conjectural extension of Hatami's theorem

Conjecture: Suppose that $\mu_p(f) = t$ and

$$I(f) \leq C \log(1/t)t$$

then f is close to a $O(\log(1/t))$ -pseudo-junta.

Stability versions of Harper's theorems

	t bounded away from 0	t small
p bounded away from 0	<p>Friedgut's easy theorem (based on KKL)</p> <p>Many applications to PCP and other areas</p> <p>Some applications to percolation</p>	
p small	<p>Friedgut's hard THM</p> <p>Bourgain's THM</p> <p>Hatami's THM</p> <p>Many applications for proving sharp threshold behavior: k-SAT, connectivity, Ramsey</p>	<p>Super Hatami Conjecture.</p> <p>Potential applications to finding the location of the threshold, and to other things.</p>

Stability versions for Harper's theorems

Symmetry: invariance under transitive group

Theorem: If a monotone Boolean function f with n variables is invariant under a transitive group of permutation of the variables, then

$$I^p(f) \geq C \mu_p(f)(1 - \mu_p(f)) \log n.$$

Proof: Follows from KKL's theorem since all individual influences are the same.

Total influence under symmetry of primitive groups

For a transitive group of permutations $\Gamma \subset S_n$, let $I(\Gamma)$ be the minimum influence for a Γ -invariant function Boolean function with n variables.

Theorem: [Bourgain and K. 1998] If Γ is primitive then one of the following possibilities hold.



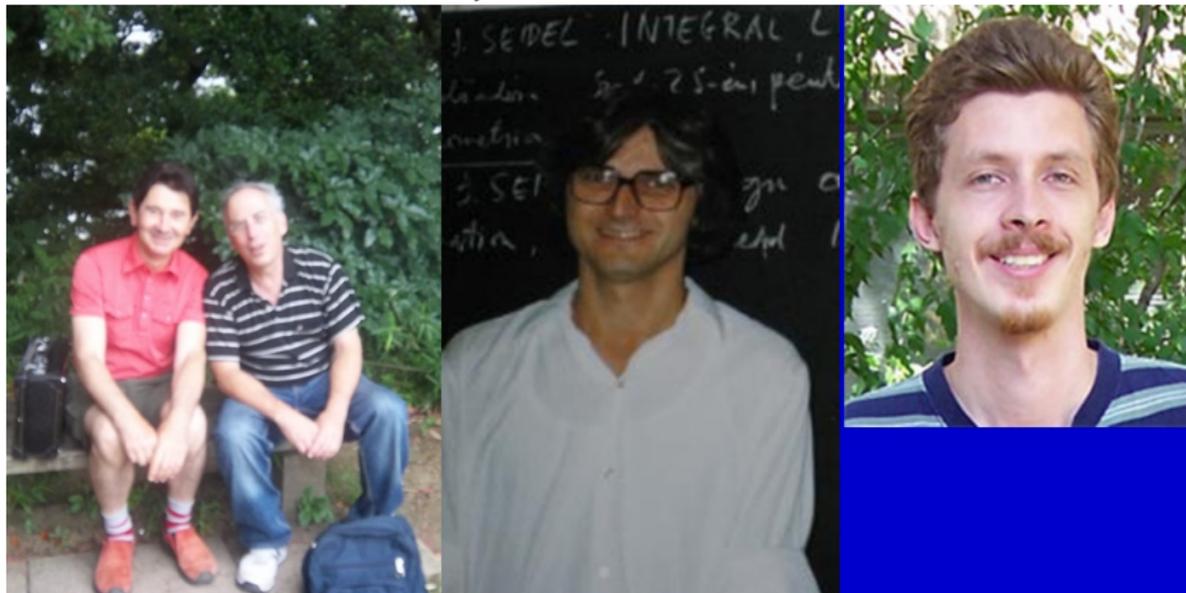
$$I(\Gamma) = \theta(\sqrt{n}),$$



$$(\log n)^{(k+1)/k - o(1)} \leq I(\Gamma) \leq C(\log n)^{(k+1)/k},$$

- ▶ $I(\Gamma)$ behaves like $(\log n)\mu(n)$, where $\mu(n) \leq \log \log n$ is growing in an arbitrary way.

Short Interlude: Kruskal-Katona for colorful (completely balanced) and flag complexes



Frankl-Füredi-K.(88); Frohmader (2008)

An important consequence of Frankl-Furedi-K. Theorem

An important consequence of Frankl-Furedi-K. Theorem

I have Frankl number 1!



Part III: Kruskal-Katona for embedded complexes

Conjecture [high-dimensional crossing conjecture] (Karnabir Sarkaria and K. late 80s) Let K be a d -dimensional simplicial complexes that can be embedded into $2d$ -dimensional space. Then

$$f_d(K) \leq C(d)f_{d-1}(K).$$

Euler's theorem asserts that $C(1) = 3$ it is conjectured that $C(d) = d + 2$.

The high-dimensional crossing conjecture - a known case

Theorem (K. early 90) Let K be a d -dimensional simplicial complexes that can be embedded as a subcomplex into the boundary complex of a $2d + 1$ -dimensional polytope. Then

$$f_d(K) \leq (d + 2)f_{d-1}(K).$$

Shifting

A combinatorial method initiated by Erdős Ko Rado to move from a general set system (or abstract simplicial complex) to a “shifted” one.

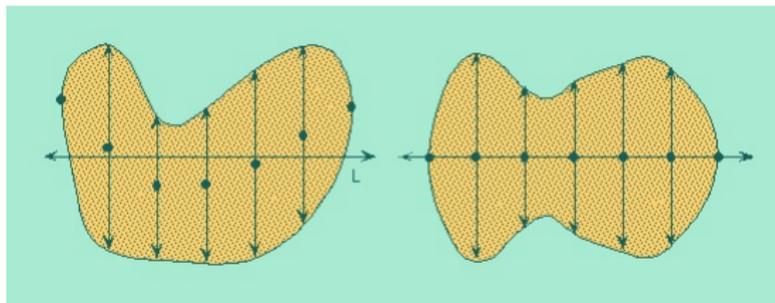
A shifted family of subsets of $\{1, 2, \dots, n\}$ is shifted if whenever you start with a set S in the family and replace an element $j \in S$ with an element $i \notin S$ with $i < j$, then the set R you obtained also belongs to the family.

Example: Give i a real weight w_i . Suppose that $w_1 > w_2 > \dots > w_n$. Let F be the family of all sets where the sum of weights is positive. Then F is shifted.

Shifting is an operation that replace a family of sets by a shifted family of the same cardinality such that various combinatorial properties are preserved.

Symmetrization

The role of shifting in discrete isoperimetry is similar to the role of symmetrization in classical isoperimetry.



Algebraic shifting

Algebraic shifting is an algebraic version of the Erdős-Ko-Rado shifting operation that I developed in my Ph. D. thesis and it was studied by Anders Björner and me in the late 80s and early 90s. It allows to consider homological and algebraic properties of K .

Algebraic shifting maps a general simplicial complex K to a shifted simplicial complex $\Delta(K)$, with the same f -vector.

The high dimensional crossing conjecture - shifting version

Conjecture: Let K be a d dimensional simplicial complex which cannot be embedded into R^{2d} . Then $\Delta(K)$ does not contain the d -skeleton of a $2d + 2$ -dimensional simplex.
For $d = 1$ the conjecture asserts that $\Delta(K)$ does not contain K_5 a complete graph on 5 vertices. This is known.

Recent approach: (Wegner - Nevo)

Theorem (Uli Wagner): The weak version of the high-dimensional crossing conjecture holds for random simplicial complexes (in the Linial-Meshulam model).

Moreover, the result holds if we exclude any subcomplex as a “minor” for a suitable definition of a term “minor”.

Another related recent approach: Balanced complexes and balanced shifting

This is an attack on the problem by restriction our attention to completely balanced complexes: d -complexes whose vertices can be colored by $(d+1)$ colors so that all simplices are rainbow simplices. (Nevo, Novik, K., Babson, Wegner..)



Conclusion

Isoperimetric relations are important in several areas of combinatorics, extremal; probabilistic, geometric and algebraic, and are relevant to connections between combinatorics and other areas.

Thank You!