Multi-cluster complexes

FPSAC 2012

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A few motivations

- Is there a polytopal realization of the multi-associahedron? (Still open)
- How do cluster complexes of finite types are related to subword complexes?
- Do multi-triangulations have a generalization to finite Coxeter groups?

Plan of the talk

Triangulations & multi-triangulations

Subword complexes and cluster complexes

Multi-cluster complexes

Triangulations

Definition

Given a convex m-gon, the dual associahedron, Δ_m : the simplicial complex for which

$$\begin{array}{rcc} \textit{vertices} & \longleftrightarrow & \textit{diagonals of the convex m-gon} \\ r\textit{-faces} & \longleftrightarrow & r\textit{-subsets of non-crossing diagonals} \\ \textit{facets} & \longleftrightarrow & \textit{triangulations of the convex m-gon} \end{array}$$

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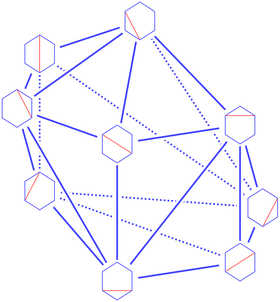

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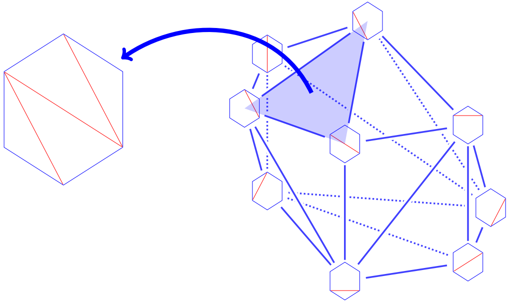

Multi-triangulations

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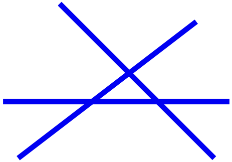

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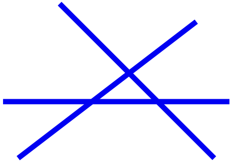
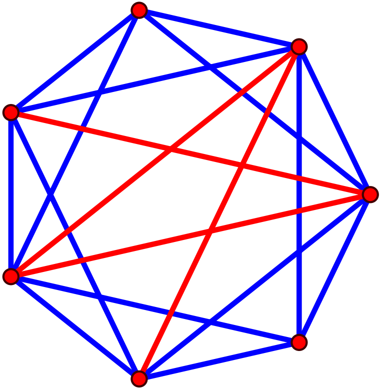


Definition

Multi-triangulation (or k-triangulation): Maximal set of diagonals not containing a $(k+1)$-crossing

Multi-triangulations - An example

A 2-triangulation of the heptagon:



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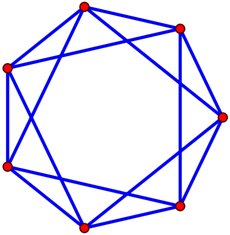

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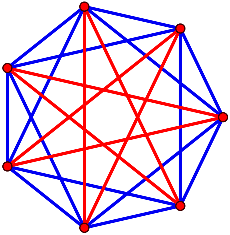

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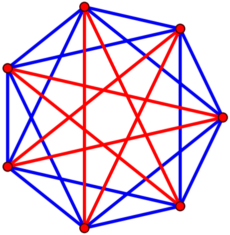

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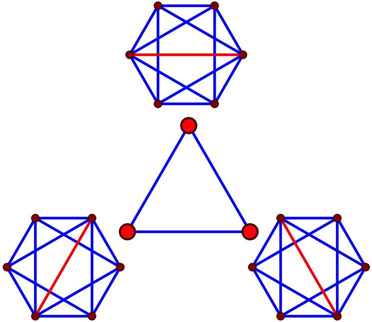
Definition

$\Delta_{m,k}$ *: the simplicial complex of* $(k+1)$*-crossing free sets of k-relevant diagonals:*

$$\text{faces} \quad \longleftrightarrow \quad (k+1)\text{-crossing free sets of } k\text{-relevant diagonals}$$

Simplicial complex $\Delta_{m,k}$ - Examples

Let $m = 6$ and $k = 2$



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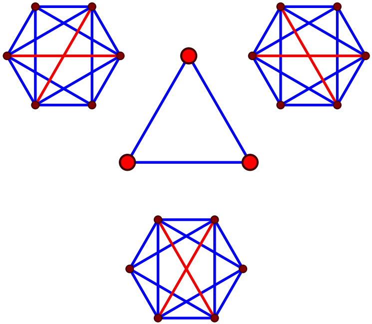

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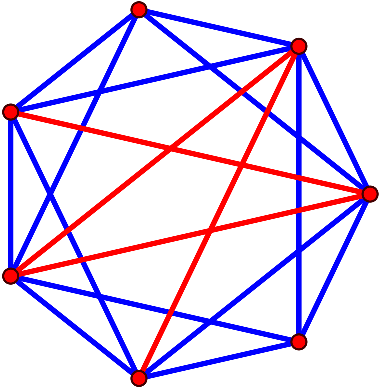

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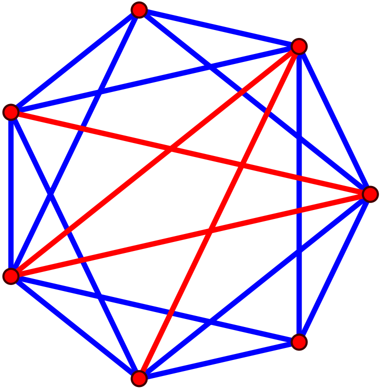

When $m = 2k + 3$, $\Delta_{m,k}$ is a 2k-dimensional cyclic polytope on $2k + 3$ vertices.

Some properties of $\Delta_{m,k}$

- pure, vertex-decomposable simplicial complex (Dress-Koolen-Moulton 2002, Jonsson 2003, Stump 2011)
- facets are in bijection with k-fans of Dyck paths and with plane partitions of height k (Stump-Serrano 2012)
- Its Stanley-Reisner ring is an initial ideal for Pfaffians (Jonsson-Welker 2007)

In this talk

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$\updownarrow$
Type A clusters

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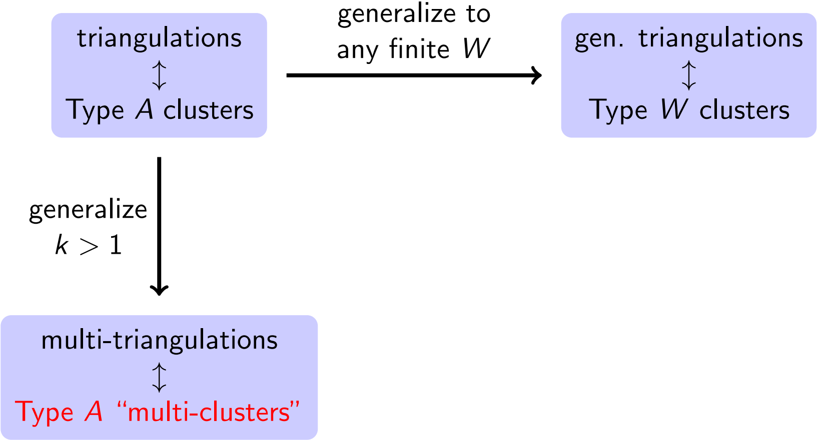
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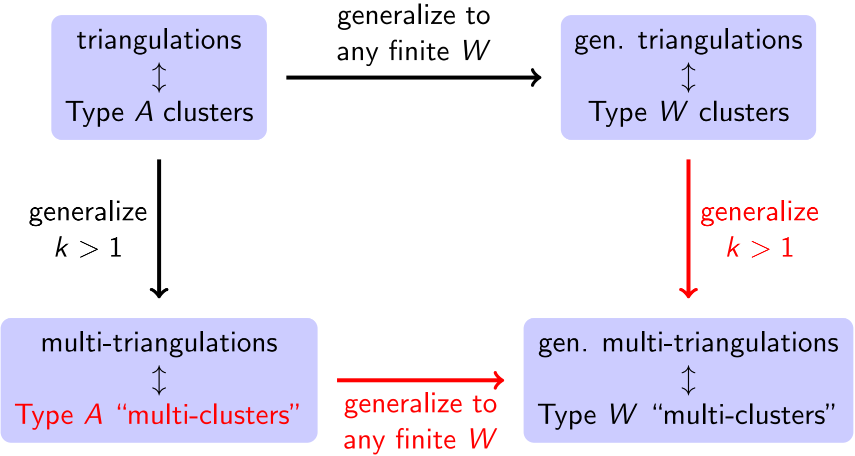
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(W, S) finite Coxeter system of rank n

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Definition (Knutson-Miller, 2004)

The subword complex $\Delta(Q, \pi)$ *is the simplicial complex for which*

$$\textit{faces} \quad \longleftrightarrow \quad \begin{array}{l}\textit{subwords } P \textit{ of } Q \textit{ such that } Q \setminus P \\ \textit{contains a reduced expression of } \pi\end{array}$$

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Subword complexes are topological spheres or balls.

Subword complex - Example 1

Let $W = A_2 = \mathbb{S}_3$, $S = \{s_1, s_2\} = \{(1\ 2), (2\ 3)\}$,

$Q = \begin{matrix} (\ s_1, s_2, s_1, s_2, s_1\ ) \\ q_1, q_2, q_3, q_4, q_5 \end{matrix}$ and $\pi = w_\circ = s_1 s_2 s_1 = s_2 s_1 s_2 = [3\ 2\ 1]$.

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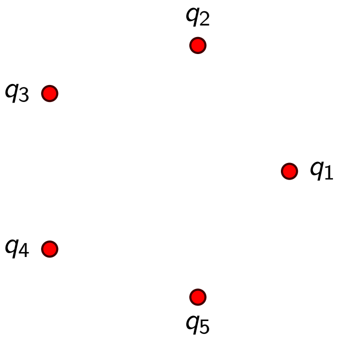

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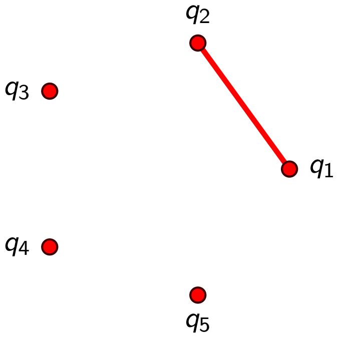

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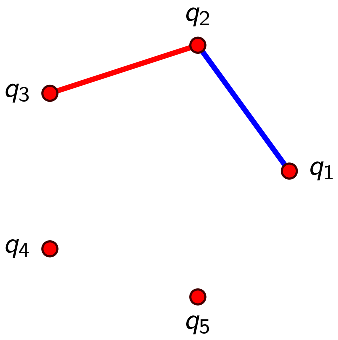

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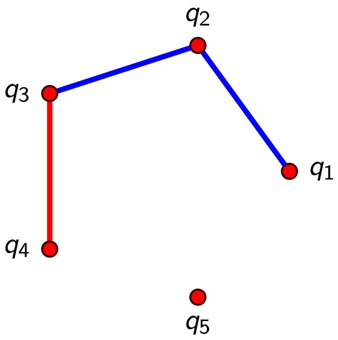

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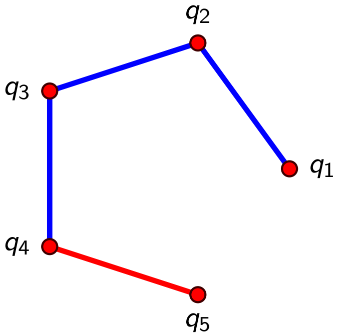

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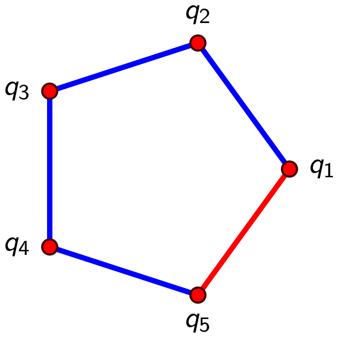

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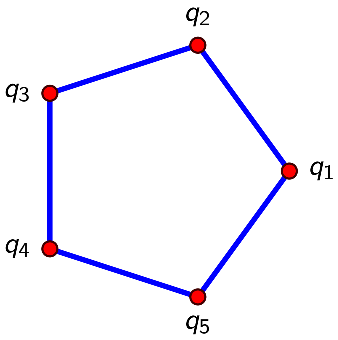

Subword complex - Example 2

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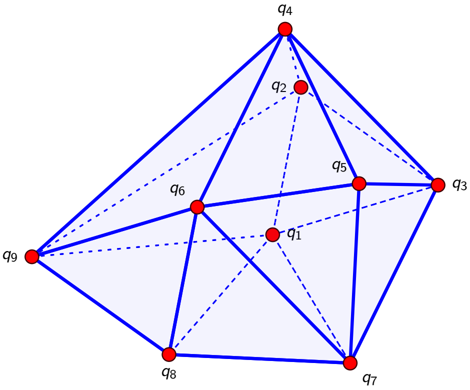

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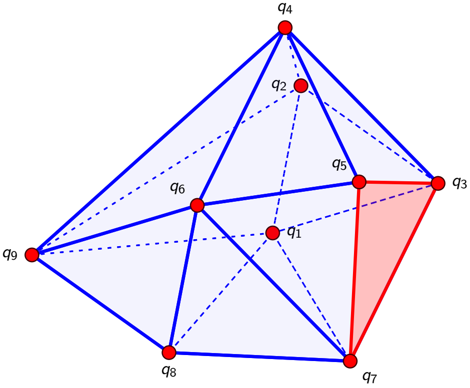

Finite cluster complexes - necessary objects

V

The group W acts on a vector space V of dimension n.

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Φ root system

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$\Phi_{\geq -1}$	almost positive roots: $\Phi^+ \cup -\Delta$

For $s \in S$, the involution $\sigma_s : \Phi_{\geq -1} \longrightarrow \Phi_{\geq -1}$ is given by

$$\sigma_s(\beta) = \begin{cases} \beta & \text{if } -\beta \in \Delta \setminus \{\alpha_s\}, \\ s(\beta) & \text{otherwise.} \end{cases}$$

Finite cluster complexes - Compatibility relations

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Definition (Fomin-Zelevinsky 2003, Reading 2007)

There exists a family $\|_c$ of c-compatibility relations on $\Phi_{\geq -1}$ satisfying the following two properties:

(i) *for $s \in S$ and $\beta \in \Phi_{\geq -1}$,*

$$-\alpha_s \|_c \beta \Leftrightarrow \beta \in \left(\Phi_{\langle s \rangle}\right)_{\geq -1},$$

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$$\beta_1 \, \|_c \, \beta_2 \Leftrightarrow \sigma_s(\beta_1) \, \|_{scs} \, \sigma_s(\beta_2).$$

A maximal subset of pairwise c-compatible almost positive roots is called c-cluster.

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- All c-cluster complexes for the various Coxeter elements are isomorphic (Reading, 2007)
- In crystallographic types, they are isomorphic to the cluster complex as defined by Fomin-Zelevinsky.

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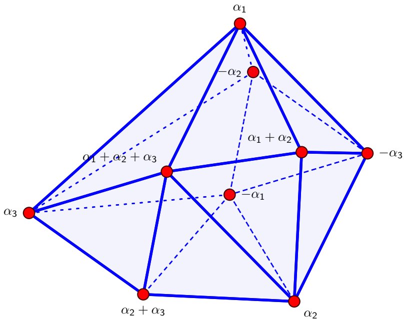

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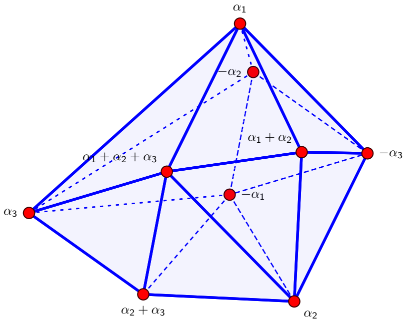

Aim

Obtain cluster complexes of finite types as subword complexes.

Cluster complexes as subword complexes

$\mathbf{w}_\circ(\mathbf{c})$: the lexicographically first (as a sequence of positions) subword of

$$\mathbf{c}^\infty = \mathbf{c}\mathbf{c}\mathbf{c}\dots$$

which is a reduced word for $w_\circ$.

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Theorem (CLS, 2011)

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A subset C of $\Phi_{\geq -1}$ is a c-cluster if and only if the complement of the corresponding subword in $\mathbf{c}\mathbf{w}_\circ(\mathbf{c}) = (c_1, \ldots, c_n, w_1, \ldots, w_N)$ represents a reduced expression for $w_\circ$.

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- A similar result for crystallographic types is due to Igusa & Schiffler (2010)

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Theorem (CLS, 2011)

All multi-cluster complexes $\Delta_c^k(W)$ *for the various Coxeter elements are isomorphic.*

Characterization of sorting words $\mathbf{w}_\circ(\mathbf{c})$

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Given a word $\mathbf{w}$ in S, let $|\mathbf{w}|_s$ denote the number of occurrences of the letter s in $\mathbf{w}$.

Let $\psi : S \to S$ be the involution $\psi(s) = w_\circ^{-1} s w_\circ$.

Theorem (CLS, 2011)

Let $\mathbf{w}_\circ(\mathbf{c})$ *be the c-sorting word of* $w_\circ$ *and let* s, t *be neighbors in the Coxeter graph such that* s *comes before* t *in* $\mathbf{c}$*. Then*

$$|\mathbf{w}_\circ(\mathbf{c})|_s = \begin{cases} |\mathbf{w}_\circ(\mathbf{c})|_t & \text{if } \psi(s) \text{ comes before } \psi(t) \text{ in } c, \\ |\mathbf{w}_\circ(\mathbf{c})|_t + 1 & \text{if } \psi(s) \text{ comes after } \psi(t) \text{ in } c. \end{cases}$$

Multi-cluster complexes of type A and B

Theorem (Pilaud-Pocchiola 2012, Stump 2011)

The multi-cluster complex $\Delta^k_c(A_n)$ $\cong$ *simplicial complex of k-triangulations of a convex m-gon*

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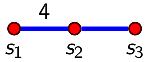
Corollary

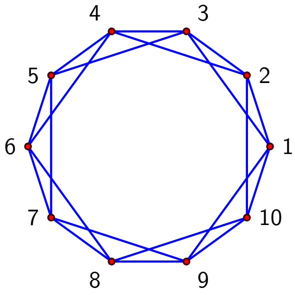
$\Delta_{m,k}^{sym}$ *is a vertex-decomposable simplicial sphere.*

Multi-cluster complex of type B - Example

Let $m = 5$ and $k = 2$ and B_3 :



Example of centrally symmetric 2-triangulation of a 10-gon



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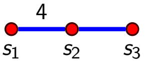

If $c = s_1 s_2 s_3$, then $\mathbf{w}_\circ(\mathbf{c}) = (s_1 s_2 s_3)^3$

$$(s_1 , s_2 , s_3 \mid s_1 , s_2 , s_3 \mid s_1 , s_2 , s_3 , s_1 , s_2 , s_3 , s_1 , s_2 , s_3)$$

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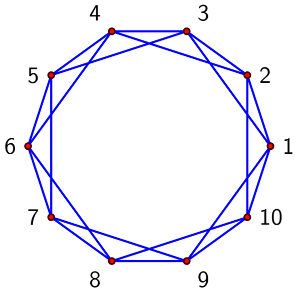

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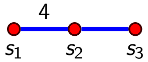

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$\updownarrow$ Bijection

[6,1] ,[6,2],[6,3] | [7,2],[7,3],[7,4] | [8,3],[8,4], [8,5] ,[9,4], [9,5] ,[9,6],[10,5],[10,6],[10,7]

[1,6] ,[1,7],[1,8] | [2,7],[2,8],[2,9] | [3,8],[3,9],[3,10],[4,9],[4,10],[4,1],[5,10], [5,1] , [5,2]

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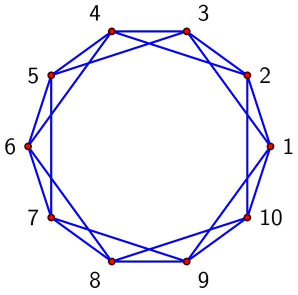

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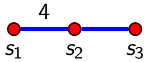
Let $m = 5$ and $k = 2$ and B_3 : 

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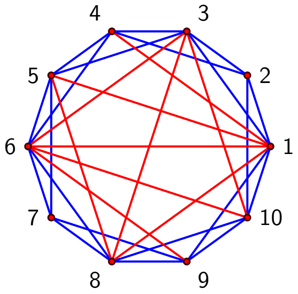

Multi-cluster complexes of type $I_2(m)$

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The multi-cluster complex $\Delta_c^k(I_2(m))$ $\cong$ boundary complex of a 2k-dimensional cyclic polytope on $2k + m$ vertices

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Corollary

The multi-associahedron of type $I_2(m)$ is the simple polytope given by the dual of a $2k$-dimensional cyclic polytope on $2k+m$ vertices.

Universality and polytopality of $\Delta_c^k(W)$

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Charaterize all simplicial spheres that can be realized as a subword complex.

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A simplicial sphere is realized as a subword complex $\iff$ *it is the link of a face of a multi-cluster complex $\Delta_c^k(W)$.*

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Corollary

The following two statements are equivalent.

(i) *Every spherical subword complex is polytopal.*

(ii) *Every multi-cluster complex is polytopal.*

Open problems and Conjectures

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- *True for $k = 1$: Hohlweg-Lange-Thomas (2011), Pilaud-Stump (2012);*
- *True for $I_2(m)$, $k \geq 1$: cyclic polytope;*
- *True for A_3, $k = 2$: Bokowski-Pilaud (2009).*

Ceballos, L. & Stump, *Subword complexes, cluster complexes, and generalized multi-associahedra*, arXiv:1108.1776.

Merci! Thank you! Grazie! Danke! Gracias!
ありがとう！

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