Projective invariants of vector configurations

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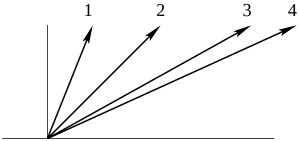
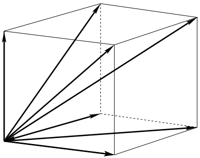
Outline of the talk

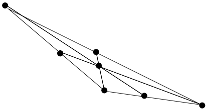
- Vector configurations & orbits thereof.
- Example: 4 vectors in $\mathbb{C}^2$.
- Matroids.
- Equations for $\overline{[v]}$.
- Algebraic invariants.

Vector configurations

A **configuration** is a list $v = (v_1, \ldots, v_n)$, where $v_i \in V \approx \mathbb{C}^r$.







The space of configurations is $V \times \cdots \times V \approx \mathbb{C}^{r \times n}$.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 5 \end{bmatrix} \qquad \begin{bmatrix} 1 & & & 1 & 1 & 1 & 0 \\ & 1 & & 1 & 1 & 0 & 1 \\ & & 1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

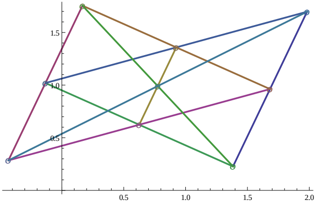
Projective equivalence

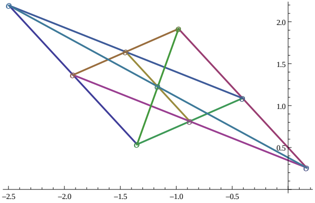
The space of r-by-n matrices comes with an action of

$$GL_r \times \underbrace{\prod^n \mathbb{C}^\times}_{=: T^n}$$

(GL_r on the left, T^n on the right.)

Here are two *projectively equivalent* Pappus configurations:





Projective equivalence ctd.

The space of r-by-n matrices comes with an action of

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Definition

The projective equivalence class of v is $[v] \subseteq \mathbb{C}^{r\times n}$:

$$[v] := \text{ the orbit of } v \text{ under } GL_r \times T^n$$

This set is smooth and locally closed; its closure $\overline{[v]}$ is an irreducible affine variety.

Example. $v \in (\mathbb{C}^{2\times 4})^o$

Let $v = 4$ non-parallel vectors in $\mathbb{C}^2$. There is a unique μ such that

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & \mu \end{bmatrix} \in [v].$$

This μ is the cross ratio of v:

$$\frac{\det(v_1 v_4)\det(v_2 v_3)}{\det(v_1 v_3)\det(v_2 v_4)} = \mu.$$

Example. $v \in (\mathbb{C}^{2\times 4})^o$

v is fixed with cross ratio μ

and we may as well choose this v:

$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{bmatrix} \qquad \textit{and} \qquad v = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & \mu \end{bmatrix}$$

are equivalent if

$$\frac{\det(\mathbf{x}_1\mathbf{x}_4)\det(\mathbf{x}_2\mathbf{x}_3)}{\det(\mathbf{x}_1\mathbf{x}_3)\det(\mathbf{x}_2\mathbf{x}_4)} = \mu$$

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$$(x_1y_4 - x_4y_1)(x_2y_3 - x_3y_2) - \mu(x_1y_3 - x_3y_1)(x_2y_4 - x_4y_2) = 0$$

Easy: this single polynomial cuts out $\overline{[v]}$.

(In fact $\overline{[v]}$ has the largest possible dimension.)

Matroids

On the Combinatorics of linear dependency, H. Whitney (1935).

$$\begin{bmatrix} 1 & & & 1 & 1 & 1 & 0 \\ & 1 & & 1 & 1 & 0 & 1 \\ & & 1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

Record which maximal minors are non-zero, as below:

$$M(v) = \{123, 124, \cancel{125}, 126, \ldots, 567\} =: \text{ the matroid of } v$$

Points of $[v]$ share a matroid.

The ideal of $\overline{[v]}$

What is the ideal of polynomials in $\mathbb{C}[x_{11}, \ldots, x_{rn}]$ that vanish on $\overline{[v]}$?

- What are its generators?

The ideal of $\overline{[v]}$

What is the ideal of polynomials in $\mathbb{C}[x_{11}, \ldots, x_{rn}]$ that vanish on $\overline{[v]}$?

- What are its generators?
- What is its Hilbert series?

The ring $\mathbb{C}[x_{11}, \ldots, x_{rn}]$ is graded by $\mathbf{N}^r \times \mathbf{N}^n$,

$$\deg x_{ij} = (\mathbf{e}_i, \mathbf{e}_j) \in \mathbf{N}^r \times \mathbf{N}^n$$

and the prime ideal I_v of $\overline{[v]}$ is homogeneous.

Definition.

The **multigraded Hilbert series** $\mathrm{Hilb}(v)$ of $\overline{[v]}$ is the generating function for the dimensions of the $\mathbf{N}^r \times \mathbf{N}^n$-graded pieces of

$$\mathbb{C}[x_{11}, \ldots, x_{rn}]/I_v$$

It is a generating function in variables $u_1, \ldots, u_r$ and $t_1, \ldots, t_n$.

Why the Hilbert series?

- $\mathrm{Hilb}(v)$ answers many questions of this form:

Take a subvariety $X \subseteq \mathbb{C}^{r \times n}$ and ask

How many configurations in X are proj. equiv. to v?

For example: resume $(r, n) = (2, 4)$.

How many configurations of the form

$$\begin{bmatrix} 8+12s & 6+9s & 9+6s & 11+14s \\ -3s & -3+s & -1-s & -5s+13 \end{bmatrix}$$

are in $[v]$? Answer: 4. This only depends on the matroid (i.e. on the non-parallel condition.)

Why the Hilbert series?

- $\mathrm{Hilb}(v)$ answers many questions about counting $X \cap \overline{[v]}$.
- It yields info on the irred. decomp. of the S_n-representation spanned by

$$\{v_{w(1)} \otimes v_{w(2)} \otimes \cdots \otimes v_{w(n)} : w \in S_n\}$$

  $\Longrightarrow$ partitions into independent sets [Berget]
- It gives the Tutte polynomial of the matroid. [F–Speyer]

Problems.

- What are the generators of the ideal of $\overline{[v]}$?
- What is its Hilbert series?

Can the answers be determined from $M(v)$?

- Murphy's law suggests "no".
- More convincingly, Mnev's universality thm suggests this too.
- The Grassmannian situation suggests "maybe"...

Gale duality

To get at the ideal of $\overline{[v]}$ we need **Gale duality**.

$$v = \begin{bmatrix} 1 & & & 1 & 1 & 1 & 0 \\ & 1 & & 1 & 1 & 0 & 1 \\ & & 1 & 1 & 0 & 1 & 1 \end{bmatrix} \rightsquigarrow v^{\perp} = \begin{bmatrix} -1 & -1 & -1 & 1 & & & \\ -1 & -1 & 0 & & 1 & & \\ -1 & 0 & -1 & & & 1 & \\ 0 & -1 & -1 & & & & 1 \end{bmatrix}$$

$v^{\perp}$ = any matrix whose row space forms a basis for $\ker(v)$.

Theorem (Berget–F, 2011)

$$v = \begin{bmatrix} 1 & & & 1 & 1 & 1 & 0 \\ & 1 & & 1 & 1 & 0 & 1 \\ & & 1 & 1 & 0 & 1 & 1 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_3 & x_6 & x_7 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 \\ z_1 & z_2 & z_3 & z_4 & z_5 & z_6 & z_7 \end{bmatrix}$$

Theorem (by example)

The common vanishing locus of the following polynomials is $\overline{[v]}$:

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Theorem (by example)

The common vanishing locus of the following polynomials is $\overline{[v]}$:

Take the 7-by-7 minors of the 12-by-7 matrix [Kapranov '91]

$$v^{\perp} \otimes \mathbf{x} = \begin{bmatrix} -1\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} & -1\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} & -1\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} & \begin{pmatrix} x_4 \\ y_4 \\ z_4 \end{pmatrix} & & & \\ -1\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} & -1\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} & \mathbf{0} & & \begin{pmatrix} x_5 \\ y_5 \\ z_5 \end{pmatrix} & & \\ -1\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} & \mathbf{0} & -1\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} & & & \begin{pmatrix} x_6 \\ y_6 \\ z_6 \end{pmatrix} & \\ \mathbf{0} & -1\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} & -1\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} & & & & \begin{pmatrix} x_7 \\ y_7 \\ z_7 \end{pmatrix} \end{bmatrix}$$

AND ...

theorem ctd.

AND ... for all subconfigurations $v_S \subseteq v$, e.g.,

$$v^{\perp}_{123567} = \begin{bmatrix} 1 & & & 1 & 1 & 0 \\ & 1 & & 1 & 0 & 1 \\ & & 1 & 0 & 1 & 1 \end{bmatrix}$$

take the $|S|$-by-$|S|$ minors of

$$v_S \otimes \mathbf{x}_S = \begin{bmatrix} -1\binom{x_1}{\substack{y_1\\z_1}} & -1\binom{x_2}{\substack{y_2\\z_2}} & \mathbf{0} & \begin{pmatrix} x_5\\y_5\\z_5 \end{pmatrix} & & \\ -1\begin{pmatrix} x_1\\y_1\\z_1 \end{pmatrix} & \mathbf{0} & -1\begin{pmatrix} x_3\\y_3\\z_3 \end{pmatrix} & & \begin{pmatrix} x_6\\y_6\\z_6 \end{pmatrix} & \\ \mathbf{0} & -1\begin{pmatrix} x_2\\y_2\\z_2 \end{pmatrix} & -1\begin{pmatrix} x_3\\y_3\\z_3 \end{pmatrix} & & & \begin{pmatrix} x_7\\y_7\\z_7 \end{pmatrix} \end{bmatrix}$$

Comments on the theorem.

- **Conjecture (Berget–F)**. The ideal these polynomials generate is prime.

  Theorem (Berget–F, 2011). If $r = 2$ or $n = r + 2$ and v has a connected matroid then the conj is true.
  In this case, the ideals come out determinantal.

- If v is rank 2 configuration of 4 vectors, recover cross ratio.

$$\det(\mathbf{x}_1\mathbf{x}_4)\det(\mathbf{x}_2\mathbf{x}_3) - \mu\det(\mathbf{x}_1\mathbf{x}_3)\det(\mathbf{x}_2\mathbf{x}_4) = 0$$

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$$\det(\mathbf{x}_1\mathbf{x}_4)\det(\mathbf{x}_2\mathbf{x}_3) - \mu\det(\mathbf{x}_1\mathbf{x}_3)\det(\mathbf{x}_2\mathbf{x}_4) = 0$$

- Dependence on the matroid: the size of $v_S^\perp$ reflects rank v_S.

Back to the Hilbert series

Details in progress:

The matroid of v determines $\mathrm{Hilb}(v)$, the multigraded Hilbert series of the quotient ring

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Our approach: Compare the variety $\overline{[v]}$ with a torus orbit on the Grassmannian, where we have [F–Speyer].

With Weyman's geometric technique, the comparison is a cohomology computation on toric vector bundles.

$\Longrightarrow \overline{[v]}$ has rational singularities.

Known cases

- $r = 2$. For the uniform matroid, by our last theorem:

$$\mathrm{Hilb}(v) = 1 - s_{(2,2)}(u)e_4(t) + s_{(3,2)}(u)e_5(t) - (2s_{(4,2)}(u) + s_{(3,3)}(u))e_5(t) + \dots$$

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Parallel extensions suffice for every rank 2 configuration.
In equivariant cohomology:

$$\text{class of } \overline{[v]} = \sum_{k=1}^{n/2} \max\{0, \mu'_1 + \cdots + \mu'_k - 2k\}\, s_{(n-k-1,k)}(u)$$

where μ'_k is the number of parallelism classes of $\geq k$ points.

Known cases ctd.

- $r = 2$.
- The uniform matroid, at least in eqvt. cohomology.
- Certain coefficients in an arbitrary configuration:

$$\mathrm{Hilb}(v) \equiv 1 - \sum_{D \in \mathcal{D}(M)} s_{(|D|-\mathrm{rk}(D),1^{\mathrm{rk}(D)})}(u) \prod_{j \in D} t_j$$
$$\bmod \langle s_\lambda(u) : \lambda \text{ not a hook} \rangle + \langle t_1^2, \ldots, t_n^2 \rangle,$$

where $\mathcal{D}(M)$ denotes the dependent sets of the matroid of v.

Thanks for listening!