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# *The pentagram map and $Y$-patterns*

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**Abstract.** The pentagram map, introduced by R. Schwartz, is defined by the following construction: given a polygon as input, draw all of its "shortest" diagonals, and output the smaller polygon which they cut out. We employ the machinery of cluster algebras to obtain explicit formulas for the iterates of the pentagram map.

**Résumé.** L'application pentagramme de R. Schwartz est définie par la construction suivante: on trace les diagonales "les plus courtes" d'un polygone donné en entrée et on retourne en sortie le plus petit polygone que ces diagonales découpent. Nous employons la machinerie des algèbres "clusters" pour obtenir des formules explicites pour les itérations de l'application pentagramme.

**Keywords:** pentagram map, cluster algebra, $Y$-pattern, alternating sign matrix

## 1 Introduction and main formula

The pentagram map, introduced by Richard Schwartz, is a geometric construction which produces one polygon from another. Figure 1 gives an example of this operation. Schwartz [8] uses a collection of cross ratio coordinates to study various properties of the pentagram map. In this paper, we work with a related set of quantities, which we term the $y$-parameters. A polygon can be reconstructed (up to a projective transformation) from its $y$-parameters together with one additional quantity. The other quantity transforms in a very simple manner under the pentagram map, so we focus on the $y$-parameters. Specifically, we derive a formula for the $y$-parameters of a polygon resulting from repeated applications of the pentagram map.

We show that the transition equations of the $y$-parameters under the pentagram map coincide with mutations in the $Y$-pattern associated to a certain cluster algebra. We exploit this connection to prove our formulas for the iterates of the pentagram map. These formulas depend on the $F$-polynomials of the corresponding cluster algebra, which in general are defined recursively. In this instance, a non-recursive description of these polynomials can be found. Specifically, the $F$-polynomials are generating functions for the order ideals of a certain sequence of partially ordered sets. These posets were originally defined by N. Elkies, G. Kuperberg, M. Larsen, and J. Propp [1]. It is clear from this description of the $F$-polynomials that they have positive coefficients, verifying that the Laurent positivity conjecture of S. Fomin and A. Zelevinsky [2] holds in this case.

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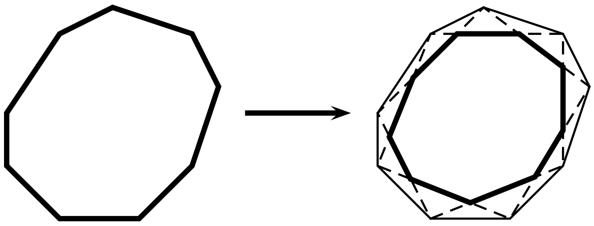

**Fig. 1:** The pentagram map

This paper is organized as follows. In the remainder of this section we state our main result, the formula for the $y$-parameters of the iterated pentagram map. This formula is proven in the subsequent sections. Section 2 gives the transition equations of the $y$-parameters under a single application of the pentagram map. In Section 3, we explain the connection to $Y$-patterns. This connection is used in Section 4 to derive our main formula in terms of the $F$-polynomials. Section 4 also provides an analogous formula expressed in the original coordinate system used by Schwartz. Section 5 contains the proof of the formula for the $F$-polynomials in terms of order ideals. Lastly, Section 6 applies the results of this paper to axis-aligned polygons, expanding on a result of Schwartz. Detailed proofs of all statements given in this paper can be found in the full version [4].

Schwartz [8] studies the pentagram map on a class of objects called twisted polygons. A *twisted polygon* is a sequence $A = (A_i)_{i \in \mathbb{Z}}$ of points in the projective plane that is periodic modulo some projective transformation $\phi$, i.e., $A_{i+n} = \phi(A_i)$ for all $i \in \mathbb{Z}$. Two twisted polygons $A$ and $B$ are said to be *projectively equivalent* if there exists a projective transformation $\psi$ such that $\psi(A_i) = B_i$ for all $i$. Let $\mathcal{P}_n$ denote the space of twisted $n$-gons modulo projective equivalence.

The *pentagram map*, denoted $T$, inputs a twisted polygon $A$ and constructs a new twisted polygon $T(A)$ given by the following sequence of points:

$$\ldots, \overleftrightarrow{A_{-1}A_1} \cap \overleftrightarrow{A_0A_2}, \overleftrightarrow{A_0A_2} \cap \overleftrightarrow{A_1A_3}, \overleftrightarrow{A_1A_3} \cap \overleftrightarrow{A_2A_4}, \ldots$$

(we denote by $\overleftrightarrow{AB}$ the line passing through $A$ and $B$). Note that this operation is only defined for generic twisted polygons. The pentagram map preserves projective equivalence, so it is well defined for generic points of $\mathcal{P}_n$.

If $A \in \mathcal{P}_n$ then the vertices of $B = T(A)$ naturally correspond to edges of $A$. To reflect this, we use $\frac{1}{2} + \mathbb{Z} = \{\ldots, -0.5, 0.5, 1.5, 2.5 \ldots\}$ to label the vertices of $B$. Specifically, we let

$$B_i = \overleftrightarrow{A_{i-\frac{3}{2}}A_{i+\frac{1}{2}}} \cap \overleftrightarrow{A_{i-\frac{1}{2}}A_{i+\frac{3}{2}}}$$

for all $i \in (\frac{1}{2} + \mathbb{Z})$. This indexing scheme is illustrated in Figure 2. Similarly, if $B$ is a sequence of points indexed by $\frac{1}{2} + \mathbb{Z}$ then $T(B)$ is defined in the same way and is indexed by $\mathbb{Z}$. Let $\mathcal{P}_n^*$ denote the space of twisted $n$-gons indexed by $\frac{1}{2} + \mathbb{Z}$, modulo projective equivalence.

The *cross ratio* of 4 real numbers $a, b, c, d$ is defined to be

$$\chi(a, b, c, d) = \frac{(a-b)(c-d)}{(a-c)(b-d)}$$

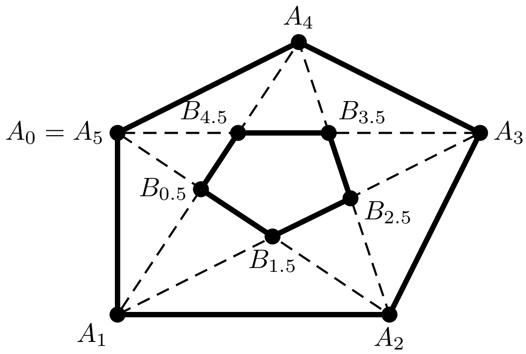


**Fig. 2:** The pentagon $B = T(A)$ is indexed by $\frac{1}{2} + \mathbb{Z}$.

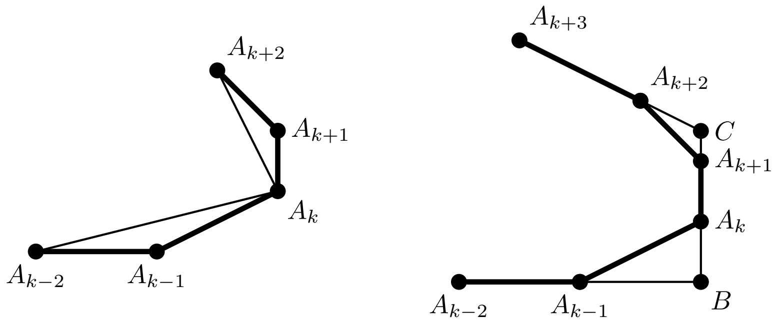


**Fig. 3:** The cross ratios corresponding to the $y$-parameters. On the left, $-(y_{2k}(A))^{-1}$ is the cross ratio of the 4 lines through $A_k$. On the right, $y_{2k+1}(A) = -\chi(B, A_k, A_{k+1}, C)$.

Define similarly the cross ratio of 4 collinear points in the plane, or dually, 4 lines which pass through a common point.

**Definition 1.1** *Let $A$ be a twisted polygon indexed either by $\mathbb{Z}$ or $\frac{1}{2} + \mathbb{Z}$. The $y$-*parameters *of $A$ are the real numbers $y_j(A)$ for $j \in \mathbb{Z}$ defined as follows. For each index $k$ of $A$ let*

$$y_{2k}(A) = -\left(\chi(\overleftrightarrow{A_k A_{k-2}}, \overleftrightarrow{A_k A_{k-1}}, \overleftrightarrow{A_k A_{k+1}}, \overleftrightarrow{A_k A_{k+2}})\right)^{-1} \tag{1.1}$$

$$y_{2k+1}(A) = -\chi(\overleftrightarrow{A_{k-2} A_{k-1}} \cap L, A_k, A_{k+1}, \overleftrightarrow{A_{k+2} A_{k+3}} \cap L) \tag{1.2}$$

*where $L = \overleftrightarrow{A_k A_{k+1}}$.*

Note that the 4 lines in (1.1) all pass through the point $A_k$, and the 4 points in (1.2) all lie on the line $L$. Therefore the cross ratios are defined. These cross ratios are illustrated in Figure 3.

As will be demonstrated, each $y$-parameter of $T(A)$ can be expressed as a rational function of the $y$-parameters of $A$. It follows that each iterate of $T$ corresponds to a rational map of the $y$-parameters. Our formulas for these maps involve the $F$-polynomials of a particular cluster algebra. These can in turn be expressed in terms of certain posets which we define now.

The original definition of the posets, given by Elkies, Kuperberg, Larsen, and Propp [1], involves height functions of domino tilings. For our purposes, the following self-contained definition will suffice. Let $Q_k$ be the set of triples $(r,s,t) \in \mathbb{Z}^3$ such that

$$2|s| - (k-2) \leq t \leq k - 2 - 2|r|$$

and

$$2|s| - (k-2) \equiv t \equiv k - 2 - 2|r| \pmod 4$$

Let $P_k = Q_{k+1} \cup Q_k$. The partial order on $P_k$ is defined by saying that $(r', s', t')$ covers $(r, s, t)$ if and only if $t' = t + 1$ and $|r' - r| + |s' - s| = 1$. We denote by $J(P_k)$ the set of *order ideals* in $P_k$, i.e., subsets $I \subseteq P_k$ such that $x \in I$ and $y < x$ implies $y \in I$. The partial order on $P_k$ restricts to a partial order on $Q_k$. The Hasse diagram for $P_2$ is given in Figure 7(a).

**Theorem 1.2** *Let $A \in \mathcal{P}_n$ and let $y_j = y_j(A)$ for all $j \in \mathbb{Z}$. If $k \geq 1$ then the $y$-parameters of $T^k(A)$ are given by*

$$y_j(T^k(A)) = \begin{cases} \left( \prod_{i=-k}^{k} y_{j+3i} \right) \dfrac{F_{j-1,k} F_{j+1,k}}{F_{j-3,k} F_{j+3,k}}, & j+k \text{ even} \\ \left( \prod_{i=-k+1}^{k-1} y_{j+3i}^{-1} \right) \dfrac{F_{j-3,k-1} F_{j+3,k-1}}{F_{j-1,k-1} F_{j+1,k-1}}, & j+k \text{ odd} \end{cases} \tag{1.3}$$

*where*

$$F_{j,k} = \sum_{I \in J(P_k)} \prod_{(r,s,t) \in I} y_{3r+s+j} \tag{1.4}$$

A sample computation of $F_{j,k}$ using (1.4) is given at the end of Section 5.

Throughout this paper, we adopt the convention that $\prod_{i=a}^{a-1} z_i = 1$ and $\prod_{i=a}^{b} z_i = \prod_{i=b+1}^{a-1} (1/z_i)$ for $b < a - 1$. This will frequently allow a single formula to encompass what otherwise would require several cases. With this convention, the property $\prod_{i=a}^{b} z_i \prod_{i=b+1}^{c} z_i = \prod_{i=a}^{c} z_i$ holds for all $a, b, c \in \mathbb{Z}$.

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## 2 The transition equations

Let $A$ be a twisted $n$-gon. Since the cross ratio is invariant under projective transformations, it follows that $y_{j+2n}(A) = y_j(A)$ for all $j$. In this section, we show that each $y$-parameter of $T(A)$ is a rational function of $y_1(A), \ldots, y_{2n}(A)$. The proof of this fact makes use of the cross ratio coordinates $x_1, \ldots, x_{2n}$ introduced by Schwartz [8].

For each index $k$ of $A$ let

$$x_{2k}(A) = \chi(A_{k-2}, A_{k-1}, \overleftrightarrow{A_k A_{k+1}} \cap \overleftrightarrow{A_{k-2} A_{k-1}}, \overleftrightarrow{A_{k+1} A_{k+2}} \cap \overleftrightarrow{A_{k-2} A_{k-1}})$$
$$x_{2k+1}(A) = \chi(A_{k+2}, A_{k+1}, \overleftrightarrow{A_k A_{k-1}} \cap \overleftrightarrow{A_{k+2} A_{k+1}}, \overleftrightarrow{A_{k-1} A_{k-2}} \cap \overleftrightarrow{A_{k+2} A_{k+1}})$$

This definition makes sense as all 4 points in the first cross ratio lie on the line $\overleftrightarrow{A_{k-2}A_{k-1}}$ and those in the second all lie on the line $\overleftrightarrow{A_{k+2}A_{k+1}}$. As with the $y_j$, we have that the $x_j$ are periodic mod $2n$.

**Proposition 2.1 ([8])** *The functions $x_1, \ldots, x_{2n}$ are (generically) a set of coordinates of the space $\mathcal{P}_n$ and of the space $\mathcal{P}^*_n$.*

As observed by V. Ovsienko, R. Schwartz, and S. Tabachnikov in [5], the products $x_j x_{j+1}$ are themselves cross ratios. In fact, $x_j x_{j+1}$ equals the cross ratios used in (1.1)–(1.2) to define $y_j$. Therefore

$$y_j = -(x_j x_{j+1})^{-1} \tag{2.1}$$

if $j/2$ is an index of $A$ and

$$y_j = -(x_j x_{j+1}) \tag{2.2}$$

otherwise. It follows that $y_1 y_2 y_3 \cdots y_{2n} = 1$ for any twisted polygon, so the $y$-parameters do not give a complete set of coordinates on $\mathcal{P}_n$. However, the $y$-parameters of $T(A)$ can be expressed in terms of the $y$-parameters of $A$ as follows.

**Proposition 2.2** *Let $(y_1, \ldots, y_{2n})$ be the $y$-parameters of $A$. If $A$ is indexed by $\frac{1}{2}+\mathbb{Z}$ then $y_j(T(A)) = y'_j$ where*

$$y'_j = \begin{cases} y_{j-3}y_j y_{j+3}\frac{(1+y_{j-1})(1+y_{j+1})}{(1+y_{j-3})(1+y_{j+3})}, & j \text{ even} \\ y_j^{-1}, & j \text{ odd} \end{cases} \tag{2.3}$$

*If $A$ is indexed by $\mathbb{Z}$ then $y_j(T(A)) = y''_j$ where*

$$y''_j = \begin{cases} y_j^{-1}, & j \text{ even} \\ y_{j-3}y_j y_{j+3}\frac{(1+y_{j-1})(1+y_{j+1})}{(1+y_{j-3})(1+y_{j+3})}, & j \text{ odd} \end{cases} \tag{2.4}$$

Let $\alpha_1$ be the rational map $(y_1, \ldots, y_{2n}) \mapsto (y'_1, \ldots, y'_{2n})$ defined by (2.3). Similarly, let $\alpha_2$ be the rational map $(y_1, \ldots, y_{2n}) \mapsto (y''_1, \ldots, y''_{2n})$ defined by (2.4). Proposition 2.2 implies that the $y$-parameters transform under the map $T^k$ according to the rational map $\ldots \circ \alpha_1 \circ \alpha_2 \circ \alpha_1 \circ \alpha_2$ (the composition of $k$ functions), assuming the initial polygon is indexed by integers.

# 3 The associated $Y$-pattern

The equations (2.3)–(2.4) can be viewed as transition equations of a certain $Y$-pattern. $Y$-patterns represent a part of cluster algebra dynamics; they were introduced by Fomin and Zelevinsky [3]. A simplified (but sufficient for our current purposes) version of the relevant definitions is given below.

**Definition 3.1** *A $Y$-seed is a pair $(\mathbf{y}, B)$ where $\mathbf{y} = (y_1, \ldots, y_n)$ is an $n$-tuple of quantities and $B$ is an $n \times n$ skew-symmetric, integer matrix. The integer $n$ is called the* rank *of the seed. Given a $Y$-seed $(\mathbf{y}, B)$ and some $k = 1, \ldots, n$, the* seed mutation *$\mu_k$ in direction $k$ results in a new $Y$-seed $\mu_k(\mathbf{y}, B) = (\mathbf{y}', B')$ where*

$$y'_j = \begin{cases} y_j^{-1}, & j = k \\ y_j y_k^{[b_{kj}]_+}(1+y_k)^{-b_{kj}}, & j \neq k \end{cases}$$

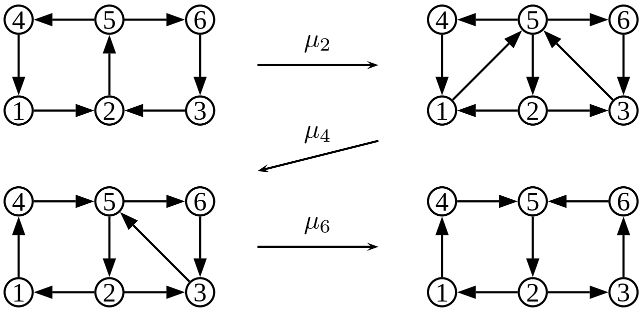

**Fig. 4:** Some quiver mutations

*and $B'$ is the matrix with entries*

$$b'_{ij} = \begin{cases} -b_{ij}, & i = k \text{ or } j = k \\ b_{ij} + \operatorname{sgn}(b_{ik})[b_{ik}b_{kj}]_+, & \text{otherwise} \end{cases}$$

*In these formulas,* $[x]_+$ *is shorthand for* $\max(x, 0)$.

The data of the exchange matrix $B$ can alternately be represented by a *quiver*. This is a directed graph on vertex set $\{1, \dots, n\}$. For each $i$ and $j$, there are $|b_{ij}|$ arcs connecting vertex $i$ and vertex $j$. Each such arc is oriented from $i$ to $j$ if $b_{ij} > 0$ and from $j$ to $i$ if $b_{ij} < 0$. In terms of quivers, the mutation $\mu_k$ consists of the following three steps

1. For every length 2 path $i \to k \to j$, add an arc from $i$ to $j$.
2. Reverse the orientation of all arcs incident to $k$.
3. Remove all oriented 2-cycles.

Figure 4 illustrates a sequence of quiver mutations. Note that in this example the mutated quiver is the same as the initial one except that all of the arrows have been reversed. The is an instance of a more general phenomenon described by the following lemma.

**Lemma 3.2** *Suppose that $(\mathbf{y}, B)$ is a $Y$-seed of rank $2n$ such that $b_{ij} = 0$ whenever $i, j$ have the same parity. Assume also that for all $i$ and $j$ the number of length 2 paths in the quiver from $i$ to $j$ equals the number of length 2 paths from $j$ to $i$. Then the $\mu_i$ for $i$ odd pairwise commute as do the $\mu_i$ for $i$ even. Moreover, $\mu_{2n-1} \circ \cdots \circ \mu_3 \circ \mu_1(\mathbf{y}, B) = (\mathbf{y}', -B)$ and $\mu_{2n} \circ \cdots \circ \mu_4 \circ \mu_2(\mathbf{y}, B) = (\mathbf{y}'', -B)$ where*

$$y'_j = \begin{cases} y_j \prod_k y_k^{[b_{kj}]_+}(1 + y_k)^{-b_{kj}}, & j \text{ even} \\ y_j^{-1}, & j \text{ odd} \end{cases} \tag{3.1}$$

$$y''_j = \begin{cases} y_j^{-1}, & j \text{ even} \\ y_j \prod_k y_k^{[b_{kj}]_+}(1 + y_k)^{-b_{kj}}, & j \text{ odd} \end{cases} \tag{3.2}$$

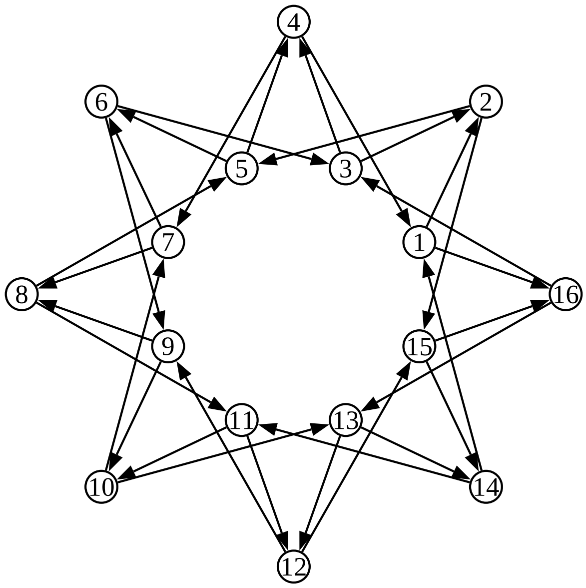


**Fig. 5:** The quiver associated with the exchange matrix $B_0$ for $n = 8$

Let $\mu_{\text{even}}$ be the compound mutation $\mu_{\text{even}} = \mu_{2n} \circ \ldots \circ \mu_4 \circ \mu_2$ and let $\mu_{\text{odd}} = \mu_{2n-1} \circ \ldots \circ \mu_3 \circ \mu_1$. Equations (2.3)–(2.4) and (3.1)–(3.2) suggest that $\alpha_1$ and $\alpha_2$ are instances of $\mu_{\text{odd}}$ and $\mu_{\text{even}}$, respectively. Indeed, let $B_0$ be the matrix with entries

$$b^0_{ij} = \begin{cases} (-1)^j, & i - j \equiv \pm 1 \pmod{2n} \\ (-1)^{j+1}, & i - j \equiv \pm 3 \pmod{2n} \\ 0, & \text{otherwise} \end{cases}$$

The corresponding quiver in the case $n = 8$ is shown in Figure 5.

**Proposition 3.3** $\mu_{\textit{even}}(\mathbf{y}, B_0) = (\alpha_2(\mathbf{y}), -B_0)$ *and* $\mu_{\textit{odd}}(\mathbf{y}, -B_0) = (\alpha_1(\mathbf{y}), B_0)$.

# 4 The formula for an iterate of the pentagram map

Let $A$ be a twisted $n$-gon indexed by $\mathbb{Z}$, and let $\mathbf{y} = (y_1, \ldots, y_{2n})$ be its $y$-parameters. For $k \geq 0$ let $\mathbf{y}_k = (y_{1,k}, \ldots, y_{2n,k})$ be the $y$-parameters of $T^k(A)$. In other words, $\mathbf{y}_0 = \mathbf{y}$, $\mathbf{y}_{2m+1} = \alpha_2(\mathbf{y}_{2m})$, and $\mathbf{y}_{2m} = \alpha_1(\mathbf{y}_{2m-1})$. The results of the previous section show that the $\mathbf{y}_k$ are related by seed mutations:

$$(\mathbf{y}_0, B_0) \xrightarrow{\mu_{\text{even}}} (\mathbf{y}_1, -B_0) \xrightarrow{\mu_{\text{odd}}} (\mathbf{y}_2, B_0) \xrightarrow{\mu_{\text{even}}} (\mathbf{y}_3, -B_0) \xrightarrow{\mu_{\text{odd}}} \cdots$$

Note that each $y_{j,k}$ is a rational function of $y_1, \ldots, y_{2n}$. In the language of cluster algebras, this rational function is denoted $Y_{j,k} \in \mathbb{Q}(y_1, \ldots, y_{2n})$. By (3.1) and (3.2) we have that $Y_{j,k} = 1/Y_{j,k-1}$ for $j + k$

odd, so it suffices to compute the $Y_{j,k}$ for $j+k$ even. Proposition 3.13 of [3], specialized to the present context, says that if $j+k$ is even then $Y_{j,k}$ can be written in the form

$$Y_{j,k} = M_{j,k}\frac{F_{j-1,k}F_{j+1,k}}{F_{j-3,k}F_{j+3,k}} \tag{4.1}$$

Here, $M_{j,k}$ is a Laurent monomial in $y_1,\ldots,y_{2n}$ and the $F_{i,k}$ are certain polynomials over $y_1,\ldots,y_{2n}$. A description of these component pieces follows.

The monomial $M_{j,k}$ is given by the evaluation of the rational expressions $Y_{j,k}$ in the tropical semifield $\mathbb{P} = Trop(y_1,\ldots,y_{2n})$. This is carried out as follows. First of all, $Y_{j,k}$ is expressed in such a manner that no minus signs appear (that this is possible is clear from transition equations of the $Y$-pattern.) Next, each plus sign is replaced by the auxiliary addition $\oplus$ symbol. This is a binary operation on Laurent monomials defined by $\prod_i y_i^{a_i} \oplus \prod_i y_i^{a'_i} = \prod_i y_i^{\min(a_i,a'_i)}$. Finally, this operation together with multiplication and division of monomials is used to compute a result. As an example, by (2.4) we know

$$Y_{3,1} = y_0y_3y_6\frac{(1+y_2)(1+y_4)}{(1+y_0)(1+y_6)}$$

so

$$M_{3,1} = y_0y_3y_6\frac{(1+y_2)(1+y_4)}{(1+y_0)(1+y_6)}\bigg|_{\mathbb{P}} = y_0y_3y_6\frac{(1\oplus y_2)(1\oplus y_4)}{(1\oplus y_0)(1\oplus y_6)} = y_0y_3y_6$$

**Proposition 4.1**

$$M_{j,k} = \prod_{i=-k}^{k} y_{j+3i} \tag{4.2}$$

*for $j+k$ even.*

The $F_{j,k}$ for $j+k$ odd are defined recursively as follows. Put $F_{j,-1}=1$ for $j$ even, $F_{j,0}=1$ for $j$ odd, and

$$F_{j,k+1} = \frac{F_{j-3,k}F_{j+3,k} + M_{j,k}F_{j-1,k}F_{j+1,k}}{(1\oplus M_{j,k})F_{j,k-1}}$$

for $j+k$ even and $k \geq 0$. Recall, $M_{j,k} = \prod_{i=-k}^{k} y_{j+3i}$ so the formula simplifies to

$$F_{j,k+1} = \frac{F_{j-3,k}F_{j+3,k} + (\prod_{i=-k}^{k} y_{j+3i})F_{j-1,k}F_{j+1,k}}{F_{j,k-1}} \tag{4.3}$$

For example, $F_{j,1} = 1 + y_j$ and

$$F_{j,2} = (1+y_{j-3})(1+y_{j+3}) + y_{j-3}y_jy_{j+3}(1+y_{j-1})(1+y_{j+1}) \tag{4.4}$$

Although it is not clear from this definition, the $F_{j,k}$ are indeed polynomials. This is a consequence of general cluster algebra theory.

Equations (4.1)–(4.2) and the fact that $Y_{j,k} = 1/Y_{j,k-1}$ for $j+k$ odd combine to prove that the formula given in Theorem 1.2 is of the right form. What remains is to prove (1.4), which expresses the $F$-polynomials in terms of order ideals. This proof will be outlined in the next section. Before moving on, we point out that Theorem 1.2 can be used to prove an analogous formula for the $x$-coordinates of $T^k(A)$.

**Theorem 4.2** *Let $A \in \mathcal{P}_n$, $x_j = x_j(A)$, and $y_j = y_j(A)$. Then $x_{j,k} = x_j(T^k(A))$ is given by*

$$x_{j,k} = \begin{cases} x_{j-3k} \left( \prod_{i=-k}^{k-1} y_{j+1+3i} \right) \dfrac{F_{j+2,k-1}F_{j-3,k}}{F_{j-2,k-1}F_{j+1,k}}, & j+k \text{ even} \\ x_{j+3k} \left( \prod_{i=-k}^{k-1} y_{j+1+3i} \right) \dfrac{F_{j-3,k-1}F_{j+2,k}}{F_{j+1,k-1}F_{j-2,k}}, & j+k \text{ odd} \end{cases} \tag{4.5}$$

It will be convenient in the following section to define $M_{j,k}$ and $F_{j,k}$ for all $j,k$ (as opposed to just for $j+k$ even or, respectively, odd). This is done by asserting that (4.2) and (4.3) hold for all $j,k$.

# 5 Computation of the $F$-polynomials

This section proves the formula for the $F$-polynomials given in (1.4).

Define Laurent monomials $m_{i,j,k}$ for $k \geq -1$ recursively as follows. Let

$$m_{i,j,0} = \prod_{l=0}^{j-1} \prod_{m=0}^{l} y_{3i+j-4l+6m-1} \tag{5.1}$$

and $m_{i,j,-1} = 1/m_{i,j,0}$ for all $i,j \in \mathbb{Z}$. For $k \geq 1$, put

$$m_{i,j,k} = \frac{m_{i-1,j,k-1}m_{i+1,j,k-1}}{m_{i,j,k-2}} \tag{5.2}$$

Note that in (5.1), if $j \leq 0$ the conventions for products mentioned in the introduction are needed. Applying these conventions and simplifying yields $m_{i,-1,0} = m_{i,0,0} = 1$ and

$$m_{i,j,0} = \prod_{l=j}^{-2} \prod_{m=l+1}^{-1} y_{3i+j-4l+6m-1}$$

for $j \leq -2$. A portion of the array $m_{i,j,0}$ is given in Figure 6.

**Proposition 5.1** *Let $f_{i,j,k} = m_{i,j,k}F_{3i+j,k}$ for all $i,j,k$ with $k \geq -1$. Then*

$$f_{i,j,k-1}f_{i,j,k+1} = f_{i-1,j,k}f_{i+1,j,k} + f_{i,j-1,k}f_{i,j+1,k} \tag{5.3}$$

*for all $i,j,k$ with $k \geq 0$.*

The difference equation (5.3) is known as the octahedron recurrence. Applied recursively, it can be used to express $f_{0,0,k} = F_{0,k}$ as a rational function of the $f_{i,j,0} = m_{i,j,0}$ and the $f_{i,j,-1} = 1/m_{i,j,0}$. D. Robbins and H. Rumsey proved [6] that this rational function is in fact a Laurent polynomial whose terms are indexed by pairs of alternating sign matrices. After reviewing the necessary terminology, we will apply this result to write a formula for $F_{0,k}$.

An *alternating sign matrix* is a square matrix of 1's, 0's, and -1's such that

- the non-zero entries of each row and column alternate in sign and

$$
\begin{array}{rccccc}
 & & \vdots & \vdots & \vdots & \\
(j=2) & \cdots & y_{-6}y_{-2}y_0 & y_{-3}y_1y_3 & y_0y_4y_6 & \cdots \\
(j=1) & \cdots & y_{-3} & y_0 & y_3 & \cdots \\
(j=0) & \cdots & 1 & 1 & 1 & \cdots \\
(j=-1) & \cdots & 1 & 1 & 1 & \cdots \\
(j=-2) & \cdots & y_{-4} & y_{-1} & y_2 & \cdots \\
(j=-3) & \cdots & y_{-7}y_{-5}y_{-1} & y_{-4}y_{-2}y_2 & y_{-1}y_1y_5 & \cdots \\
 & & \vdots & \vdots & \vdots & \\
 & & (i=-1) & (i=0) & (i=1) &
\end{array}
$$

**Fig. 6:** The monomials $m_{i,j,0}$

- the sum of the entries of each row and column is 1.

Let $ASM(k)$ denote the set of $k$ by $k$ alternating sign matrices.

A bijection is given by Elkies, Kuperberg, Larsen, and Propp in [1] between $ASM(k)$ and the set of order ideals of $Q_k$, where $Q_k$ is the poset defined in Section 1. Call order ideals $I \subseteq Q_{k+1}$, $J \subseteq Q_k$ compatible if $I \cup J$ is an order ideal of $P_k = Q_{k+1} \cup Q_k$. Call alternating sign matrices $A \in ASM(k+1)$ and $B \in ASM(k)$ compatible if they correspond under the bijection to compatible order ideals.

The initial data of the recurrence (5.3) can be gathered into matrices of the form

$$X_k = \begin{bmatrix} m_{-k+1,0,0} & m_{-k+2,1,0} & \cdots & m_{0,k-1,0} \\ m_{-k+2,-1,0} & m_{-k+3,0,0} & \cdots & m_{1,k-2,0} \\ \vdots & \vdots & & \vdots \\ m_{0,-k+1,0} & m_{1,-k+2,0} & \cdots & m_{k-1,0,0} \end{bmatrix}$$

In the following, the notation $X^A$, with $X$ and $A$ matrices of the same dimensions, represents the product $\prod\limits_i \prod\limits_j x_{ij}^{a_{ij}}$.

**Proposition 5.2**

$$F_{0,k} = \sum_{A,B} (X_{k+1})^A (X_k)^B \tag{5.4}$$

*where the sum is over all compatible pairs $A \in ASM(k+1)$, $B \in ASM(k)$.*

Alternatively, $F_{0,k}$ can be expressed as a sum over compatible pairs of order ideals of $Q_{k+1}$ and $Q_k$. Such pairs are in turn in bijection with order ideals of $P_k$. The following proposition indicates how to translate the individual terms of (5.4) to the language of order ideals.

**Proposition 5.3** *If $A \in ASM(k)$ and $I \subseteq Q_k$ is the associated order ideal then*

$$X_k^A = \prod_{(r,s,t) \in I} y_{3r+s}$$


(a) (b)

**Fig. 7:** (a) The poset $P_2 = Q_3 \cup Q_2$. Here $Q_3 = \{(-1,0,-1),(1,0,-1),(0,-1,1),(0,1,1)\}$, $Q_2 = \{(0,0,0)\}$. (b) The poset $P_2$ with each element $(r,s,t)$ labeled by $y_{3r+s}$.

**Theorem 5.4**

$$F_{j,k} = \sum_{I \in J(P_k)} \prod_{(r,s,t) \in I} y_{3r+s+j}$$

*where $J(P_k)$ denotes the set of order ideals of $P_k$.*

As an example, let $j = 0$ and $k = 2$. The poset $P_2$ (see Figure 7(b)) has eight order ideals. The four which do not include $y_0$ have weights which sum to $(1+y_{-3})(1+y_3)$. The other four have weights summing to $y_{-3}y_0y_3(1+y_{-1})(1+y_1)$. Adding these yields a formula for $F_{0,2}$ which agrees with (4.4).

# 6 Axis-aligned polygons

The remainder of this paper is devoted to axis-aligned polygons, i.e. polygons whose sides are alternately parallel to the $x$ and $y$ axes.

**Lemma 6.1** *Let $A$ be a twisted polygon indexed either by $\mathbb{Z}$ or $\frac{1}{2}+\mathbb{Z}$. Suppose that no 3 consecutive points of $A$ are collinear. Then for each index $i$ of $A$:*

1. *$A_{i-2}, A_i, A_{i+2}$ are collinear if and only if $y_{2i}(A) = -1$.*
2. *$\overleftrightarrow{A_{i-2}A_{i-1}}, \overleftrightarrow{A_iA_{i+1}}, \overleftrightarrow{A_{i+2}A_{i+3}}$ are concurrent if and only if $y_{2i+1}(A) = -1$.*

Let $A \in \mathcal{P}_{2n}$ be an axis-aligned polygon. Suppose in addition that $A$ is closed, i.e. $A_{i+2n} = A_i$ for all $i \in \mathbb{Z}$. Let $s_{2j+1}$ denote the signed length of the side joining $A_j$ and $A_{j+1}$, where the sign is taken to be positive if and only if $A_{j+1}$ is to the right of or above $A_j$. An example of an axis-aligned octagon is given in Figure 8. It follows from the second statement in Lemma 6.1 that $y_{2j+1}(A) = -1$ for all $j \in \mathbb{Z}$. On the other hand, the even $y$-parameters can be expressed directly in terms of the side lengths.

**Lemma 6.2** *For all $j \in \mathbb{Z}$*

$$y_{2j}(A) = -\frac{s_{2j-1}s_{2j+1}}{s_{2j-3}s_{2j+3}} \tag{6.1}$$

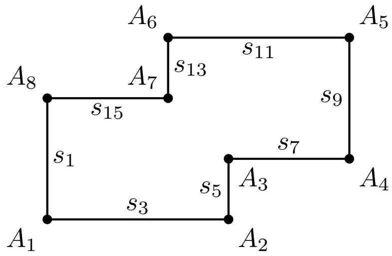


**Fig. 8:** An axis-aligned octagon. The side lengths $s_3, s_5, s_7$, and $s_9$ are positive and the others are negative.

**Theorem 6.3 (Schwartz)** *Let $A$ be a closed, axis-aligned $2n$-gon. Then the odd vertices of $T^{n-2}(A)$ are collinear, as are its even vertices.*

Theorem 6.3 is stated for all $n$ in [7] and proven for $n$ even (i.e. the number of sides of $A$ divisible by 4) in [8]. The results of this paper lead to a new proof which works for all $n$. By the above lemmas, $y_{2j+1}(A) = -1$ and the $y_{2j}(A)$ are given by (6.1). A calculation using Theorem 1.2 shows that half of the $y_j(T^{n-2}(A))$ also equal $-1$ in this case. The first statement of Lemma 6.1 then shows that the vertices of $T^{n-2}(A)$ lie alternately on 2 lines, as claimed.

As an extension of this theorem, suppose that $A$ is not closed but twisted with $A_{i+2n} = \phi(A_i)$. Amazingly, under certain assumptions on $\phi$ a result similar to Theorem 6.3 still holds.

**Theorem 6.4** *Let $A$ be a twisted, axis-aligned $2n$-gon with $A_{i+2n} = \phi(A_i)$ and suppose that $\phi$ fixes every point at infinity. Then the odd vertices of $T^{n-1}(A)$ are collinear, as are its even vertices.*

## References

[1] N. Elkies, G. Kuperberg, M. Larsen, and J. Propp, Alternating-sign matrices and domino tilings, *J. Algebraic Combin.* **1** (1992), 111–132.

[2] S. Fomin and A. Zelevinsky, Cluster algebras I: Foundations, *J. Amer. Math. Soc.* **15** (2002), 497–529.

[3] S. Fomin and A. Zelevinsky, Cluster algebras IV: Coefficients, *Compos. Math.* **143** (2007), 112–164.

[4] M. Glick, The pentagram map and $Y$-patterns, arXiv:1005.0598v2.

[5] V. Ovsienko, R. Schwartz, and S. Tabachnikov, The pentagram map: a discrete integrable system, *Comm. Math. Phys.* **299** (2010), 409-446.

[6] D. Robbins and H. Rumsey, Determinants and alternating sign matrices, *Adv. in Math.* **62** (1986), 169–184.

[7] R. Schwartz, Desargues theorem, dynamics, and hyperplane arrangements, *Geom. Dedicata* **87** (2001), 261–283.

[8] R. Schwartz, Discrete monodromy, pentagrams, and the method of condensation, *J. Fixed Point Theory Appl.* **3** (2008), 379–409.