The Mukhin–Varchenko conjecture for type A

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Atle Selberg (June 14, 1917 – August 6, 2007)

The hypergeometric equation

$$x(1-x)\frac{\mathrm{d}^2F}{\mathrm{d}x^2}+\Big(c-\big(a+b+1\big)x\Big)\frac{\mathrm{d}F}{\mathrm{d}x}-abF=0$$

$$F(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!} \qquad \text{(Gauss)}$$

where

$$(a)_k = a(a+1)\cdots(a+k-1)$$

is a Pochhammer symbol.

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$$F(a,b;c;x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-xt)^a}\, \mathrm{d}t \qquad \text{(Euler)}$$

When $x = 1$ Euler's solution reduces to the beta integral

$$F(a, b; c; 1) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-a-1}\,\mathrm{d}t$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \; B(b, c-a-b)$$

$$= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

Differential equation in x

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$$\Downarrow$$

Solution as integral

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For a special choice of x, the integral may be evaluated in terms of gamma functions

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For this to work we need

- Differential equations based on Lie algebras.
- Solutions as integrals.
- A whole lot more ...

The Knizhnik–Zamolodchikov equation

Notation:

- Simple Lie algebra $\mathfrak{g}$ of rank n.
- Chevalley generators e_i, f_i, h_i, $i \in [n] := \{1, \ldots, n\}$.
- Simple roots α_i, $i \in [n]$.
- Fundamental weights Λ_i, $i \in [n]$.
- The Casimir element $\Omega \in \mathfrak{g} \otimes \mathfrak{g}$.
- Highest weight modules V_λ and V_μ of highest weight λ and μ.

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The Knizhnik–Zamolodchikov (KZ) equation for a function $u(z, w)$ taking values in $V_\lambda \otimes V_\mu$ is the system of partial differential equations

$$\kappa \frac{\partial u}{\partial z} = \frac{\Omega}{z - w} u$$

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Let $(\cdot,\cdot)$ the standard bilinear symmetric form on the dual of the Cartan subalgebra. Then

$$(\alpha_i, \Lambda_j) = \delta_{ij}$$

and

$$\Big(\, (\alpha_i, \alpha_j) \,\Big)_{i,j=1}^{n} = \text{Cartan matrix of } \mathfrak{g}$$

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- Example: $\mathfrak{g} = A_n$

$$\Big((\alpha_i, \alpha_j) \Big)_{i,j=1}^{n} = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & & \ddots & & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}$$

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The Dynkin diagram of A_n.

To the simple root α_i attach a set of k_i integration variables

$$\{t_j\}_{j=k_1+\cdots+k_{i-1}+1}^{k_1+\cdots+k_i}$$

and write

$$\alpha_{t_j} = \alpha_i$$

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Then the master function is defined as

$$\Phi(z, w; \mathbf{t}) = (z-w)^{(\lambda,\mu)} \prod_{i=1}^{k} (t_i - z)^{-(\lambda,\alpha_{t_i})} (t_i - w)^{-(\mu,\alpha_{t_i})} \times \prod_{1\le i<j\le k} (t_i - t_j)^{(\alpha_{t_i},\alpha_{t_j})}$$

where $k = k_1 + \cdots + k_n$.

Schechtman and Varchenko proved that in the subspace of singular vectors of weight $\lambda + \mu - \sum_i k_i\alpha_i$

$$u(z,w) = \sum^* u_{IJ}(z,w)\, f^I v_\lambda \otimes f^J v_\mu$$

with

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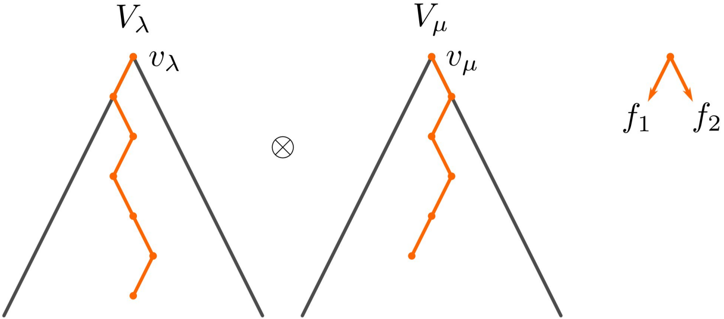
The sum is over multisets I and J with elements taken from $\{1,\dots,n\}$ such that their union contains the number i exactly k_i times, v_λ and v_μ are the highest weight vectors of V_λ and V_μ, and

$$f^I v = \Big(\prod_{i \in I} f_i\Big) v$$



$$f_1 f_2^2 f_1 f_2 f_1 v_\lambda \otimes f_1^2 f_2 f_1 f_2 v_\mu \in V_\lambda \otimes V_\mu$$

- The case $\mathfrak{g} = \mathfrak{sl}_2 = A_1$, $k = 1$

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$$[e, f] = h, \qquad [h, e] = 2e, \qquad [h, f] = -2f$$

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$$[e, f] = h, \qquad [h, e] = 2e, \qquad [h, f] = -2f$$

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Casimir element

$$\Omega = e \otimes f + f \otimes e + \frac{1}{2} h \otimes h$$

Highest weights $\lambda = m_1\Lambda_1$ and $\mu = m_2\Lambda_1$.

$$u(z,w) = u_0(z,w)\, v_1 \otimes f\, v_2 + u_1(z,w)\, f\, v_1 \otimes v_2$$

with

$$u_0(z,w) = (z-w)^{\frac{m_1 m_2}{2\kappa}} \int_\gamma (z-t)^{-m_1/\kappa}(t-w)^{-m_2/\kappa-1}\mathrm{d}t$$

$$u_1(z,w) = (z-w)^{\frac{m_1 m_2}{2\kappa}} \int_\gamma (z-t)^{-m_1/\kappa-1}(t-w)^{-m_2/\kappa}\mathrm{d}t$$

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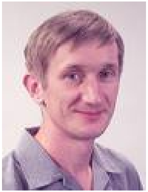
If $w = 0$ and $z = 1$ one can deform γ to

$$\gamma = \{t \in \mathbb{R},\ 0 < t < 1\}$$

Both u_0 and u_1 become Euler beta integrals and can therefore be evaluated in terms of gamma functions.

The Mukhin–Varchenko conjecture



Define the specialised master function (i.e., $w = 0$, $z = 1$)

$$\Phi(\mathbf{t}) = \prod_{i=1}^{k} t_i^{-(\lambda,\alpha_{t_i})}(1-t_i)^{-(\mu,\alpha_{t_i})} \prod_{1\le i<j\le k} (t_i - t_j)^{(\alpha_{t_i},\alpha_{t_j})}$$

Then the example of $\mathfrak{sl}_2$, $k = 1$ "justifies" the following conjecture:

Conjecture. If the space of singular vectors is one-dimensional then there exists a (real) domain of integration D such that

$$\int_D |\Phi(\mathbf{t})|^{1/\kappa} d\mathbf{t}$$

evaluates as a product of gamma functions.

- The case $\mathfrak{g} = \mathfrak{sl}_2 = A_1$, general k

Highest weights $\lambda = m_1\Lambda_1$ and $\mu = m_2\Lambda_1$,

$$(\alpha, \beta, \gamma) := (1 - m_1/\kappa, 1 - m_2/\kappa, 1/\kappa)$$

and

$$D = \{\mathbf{t} \in \mathbb{R}^k,\ 0 \le t_k \le \cdots \le t_1 \le 1\}$$

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$$\int_D \prod_{i=1}^{k} t_i^{\alpha-1}(1-t_i)^{\beta-1} \prod_{1\le i<j\le k} (t_i - t_j)^{2\gamma}\, \mathrm{d}\mathbf{t} = \prod_{i=0}^{k-1} \frac{\Gamma(\alpha+i\gamma)\Gamma(\beta+i\gamma)\Gamma(\gamma+i\gamma)}{\Gamma\big(\alpha+\beta+(k+i-1)\gamma\big)\Gamma(\gamma)}$$

This is the Selberg integral!

The Mukhin–Varchenko (ex-)conjecture for type A

- The case $\mathfrak{g} = \mathfrak{sl}_{n+1} = \mathsf{A}_n$, general $k_1 \le k_2 \le \cdots \le k_n$

Highest weights $\lambda = \lambda_n \Lambda_n$ and $\mu = \mu_1 \Lambda_1 + \cdots + \mu_n \Lambda_n$,

$$(\alpha, \beta_1, \ldots, \beta_n, \gamma) := \left(1 - \frac{\lambda_n}{\kappa}, 1 - \frac{\mu_1}{\kappa}, \ldots, 1 - \frac{\mu_n}{\kappa}, \frac{1}{\kappa}\right)$$

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$$D = \{\mathbf{t} \in \mathbb{R}^k, \text{ chain}\} \quad \text{(in the algebraic topology sense)}$$

$$\int_D |\Phi(\mathbf{t})|^\gamma d\mathbf{t}$$

$$= \prod_{1 \leq s \leq r \leq n} \prod_{i=1}^{k_s - k_{s-1}} \frac{\Gamma(\beta_s + \cdots + \beta_r + (i + s - r - 1)\gamma)}{\Gamma\left(\alpha\delta_{r,n} + \beta_s + \cdots + \beta_r + (i + s - r + k_r - k_{r+1} - 2)\gamma\right)}$$

$$\times \prod_{s=1}^{n} \prod_{i=1}^{k_s} \frac{\Gamma(\alpha\delta_{s,n} + (i - k_{s+1} - 1)\gamma)\Gamma(i\gamma)}{\Gamma(\gamma)}$$

Proof is based on Macdonald polynomial theory.

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To the initiated this should not come as a surprise. In  Macdonald uses his polynomials to prove Kadell's extension of the Selberg integral:

$$\int_D P_\lambda^{(1/\gamma)}(\mathbf{t}) \prod_{i=1}^{k} t_i^{\alpha-1}(1-t_i)^{\beta-1} \prod_{1\le i<j\le k} (t_i - t_j)^{2\gamma}\, \mathrm{d}\mathbf{t}$$
$$= \prod_{1\le i<j\le k} \frac{\Gamma((j-i+1)\gamma+\lambda_i-\lambda_j)}{\Gamma((j-i)\gamma+\lambda_i-\lambda_j)}$$
$$\times \prod_{i=1}^{k} \frac{\Gamma(\alpha+(k-i)\gamma+\lambda_i)\Gamma(\beta+(i-1)\gamma)}{\Gamma(\alpha+\beta+(2k-i-1)\gamma+\lambda_i)}$$

where $P_\lambda^{(\alpha)}(\mathbf{t})$ is the Jack polynomial.

Cheers