Enumerating alternating tree families

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Outline of the talk

1. Alternating trees
2. Enumerating alternating trees
3. Influence of alternating labelling on tree structure

Alternating trees

Alternating trees: Definition

Alternating trees (intransitive trees):

- Unordered trees
- Unrooted trees
- Labelled trees: size n tree labelled by $\{1, 2, \ldots, n\}$
- Labels on each path satisfy:

either $i_1 < i_2 > i_3 < i_4 > \cdots$ or $i_1 > i_2 < i_3 > i_4 < \cdots$

up – down – up – up $\cdots$ down – up – down – up $\cdots$

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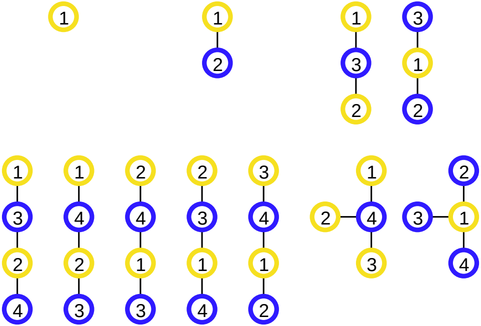
Alternating trees: Example

Example: alternating trees of size $n \leq 4$



F_n: # alternating trees of size n

n	1	2	3	4	5	6
F_n	1	1	2	7	36	246

Alternating trees: Relations to other objects

Relations to other combinatorial objects:

- **Hyperplane arrangements:**

$\mathcal{A}_n$ arrangement of hyperplanes in $\mathbb{R}^n$:

$$x_i - x_j = 1, \qquad 1 \leq i < j \leq n.$$

R_n: number of regions of $\mathcal{A}_n$

Postnikov and Stanley [2000]: $R_n = F_{n+1}$, for $n \geq 1$

- **Hypergeometric systems:**

Gelfand, Graev and Postnikov [1997]:

F_n enumerates admissible bases in certain hypergeometric systems

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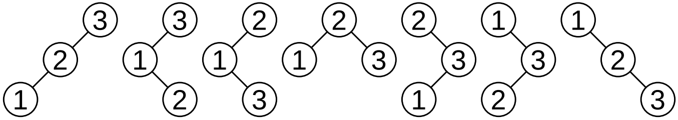
Alternating trees: Relations to other objects

- **Local binary search trees (introduced by I. Gessel):**
  - Labelled binary trees
  - Every left child has smaller label than parent
  - Every right child has larger label than parent



Postnikov [1997]:

\# size-n local binary search trees $= F_{n+1}$

Alternating trees: Known enumeration results

Known enumeration results for alternating trees:

- **Unordered unrooted alternating trees:**

Postnikov [1997]:

$$F_n = \frac{1}{n2^{n-1}} \sum_{k=1}^{n} \binom{n}{k} k^{n-1}$$

- **Unordered rooted up-down alternating trees:**

T_n: number of rooted up-down alternating trees of size n

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Functional equation for generating function $T(z) = \sum_{n\geq 1} T_n \frac{z^n}{n!}$:

$$z = \frac{2T(z)}{1 + e^{T(z)}}$$

- **Ordered rooted up-down alternating trees:**

Chauve, Dulucq and Rechnitzer [2001]:

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Questions:

- **Enumeration results** for alternating labelled trees from other families of trees?
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- Family of rooted trees
  (e.g., binary trees, d-ary trees, Motzkin trees, etc.)

- All up-down alternating labellings:
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T_n: number of up-down alternating trees of size n

- **Explicit results** for T_n ?
- **Asymptotic results** for T_n ?

Generating function: $$T(z) = \sum_{n \geq 1} T_n \frac{z^n}{n!}$$

- **Characterization** of $T(z)$?

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Enumerating alternating trees: Results

Results:

- General generating functions approach for enumerating up-down alternating rooted tree families
- Characterization of $T(z)$ for various labelled tree classes:
  - Ordered trees [Chauve, Dulucq and Rechnitzer 2001]: each node has sequence of children
  - Unordered trees [Postnikov 1997]: each node has set of children
  - d-ary trees (contains, e.g., binary trees): each node has d positions, where either a child is attached or not
  - Strict binary trees: each node has either 0 or 2 children

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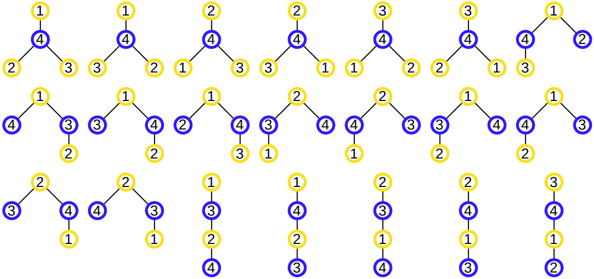
Enumerating alternating trees: Results

- Motzkin trees:
  each node has either 0, 1, or 2 children

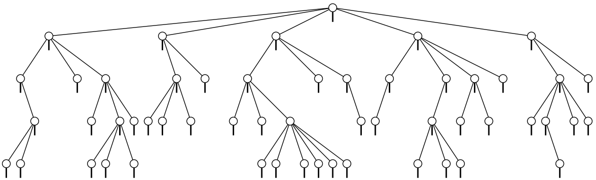
All 21 up-down alternating labelled Motzkin trees of size 4:



Enumerating alternating trees: Results

- *d*-bundled trees:
  each node has *d* positions, where sequence of children is attached

Example of an unlabelled 2-bundled tree:



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- Combinatorial model for "preferential attachement" growth models for trees [Pan & Prodinger 2007]
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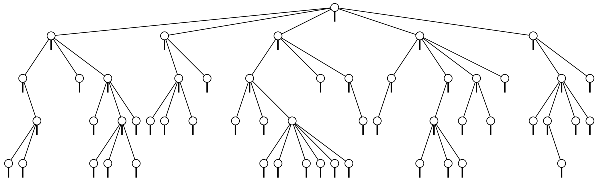

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Exponential generating function $T(z)$ implicitly given as solution of following functional equations:

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$$z = \frac{2}{\left(1 + (1 + T(z))^{d+1}\right)^{\frac{d-1}{d+1}}} \int_0^{T(z)} \frac{dx}{\left(1 + (1 + x)^{d+1}\right)^{\frac{2}{d+1}}}$$

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- Motzkin trees: $$z = \int_0^{T(z)} \frac{4\,dx}{3 + s^2(x)}$$ with

$$s^3(x) + 9s(x) - r(x) = 0$$
$$r(x) = 8(T^3(z) - x^3) + 12(T^2(z) - x^2) + 24(T(z) - x) + 10$$

- Strict binary trees: $$z = \int_0^{T(z)} \frac{dx}{1 + s^2(x)}$$ with

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Enumerating alternating trees: Results

Asymptotic results for T_n: $T_n \sim C\rho^{-n} n^{-\frac{3}{2}} n!$

- Ordered trees: $\rho = \frac{1}{e} \approx 0.367879\ldots$, $C = \frac{1}{\sqrt{2\pi}\, e} \approx 0.146762\ldots$
- Unordered trees: $\rho = -2W(-e^{-1}) \approx 0.556929\ldots$,
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- d-ary trees: $\rho = \frac{2}{(d-1)(1+\tau)^d}$, $C = \sqrt{\frac{1+(1+\tau)^{d+1}}{2d(d-1)(1+\tau)^{d-1}\pi}}$,
  with τ the positive real solution of the equation

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Enumerating alternating trees: Results

Exact formulaæ for T_n:

- Ordered trees: [Chauve, Dulucq and Rechnitzer 2001]

$$T_n = (n-1)^{n-1}$$

- Unordered trees: [Postnikov 1997]

$$T_n = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} k^{n-1}$$

- 3-bundled trees:

$$T_n = \frac{(n-1)!}{2^{n+1}} \sum_{k=0}^{2n} \binom{2n}{k} \binom{\frac{5n-3}{2} - k}{n-1}$$

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Basic idea of proof:

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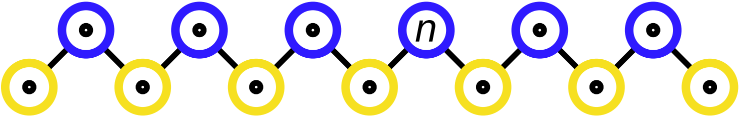
[Andre 1881]: enumeration of odd length alternating permutations



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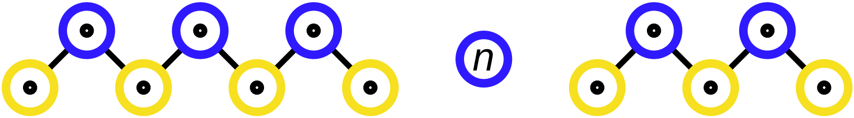

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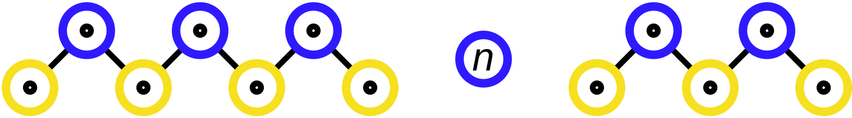

- **DE for generating function $U(z)$:** $U'(z) = 1 + U^2(z)$
- **solution:** $U(z) = \tan z$ $\Rightarrow$ tangent numbers

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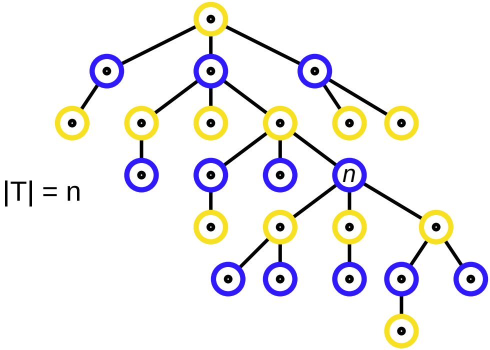
Decomposition of a tree T of family $\mathcal{T}$:



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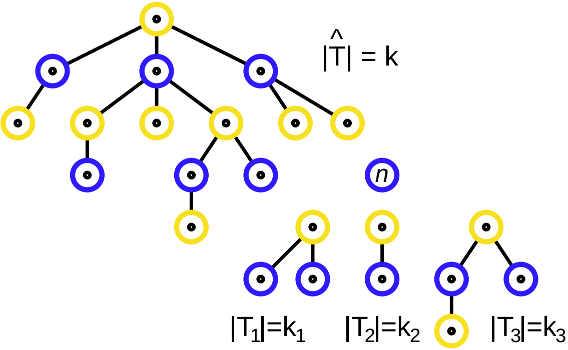


Decomposition: $T \longrightarrow \hat{T}, \textcircled{n}, T_1, T_2, \ldots, T_r$

Enumerating alternating trees: Proof

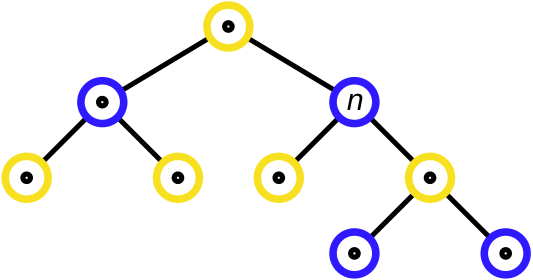
First difficulty:

$\hat{T}$: in general not a member of original tree family $\mathcal{T}$

- Families of ordered, unordered, d-ary, d-bundled, Motzkin trees: $\hat{T} \in \mathcal{T}$
- E.g., family of strict binary trees: $\hat{T} \notin \mathcal{T}$, in general

Enumerating alternating trees: Proof

T



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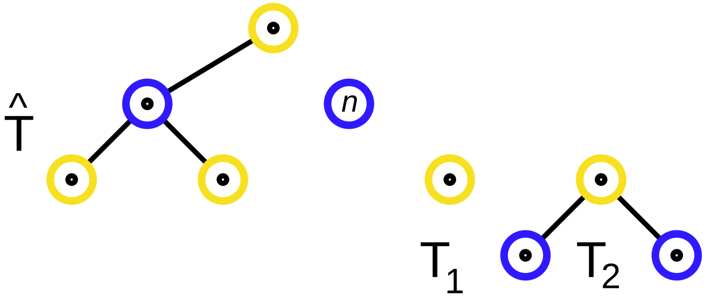

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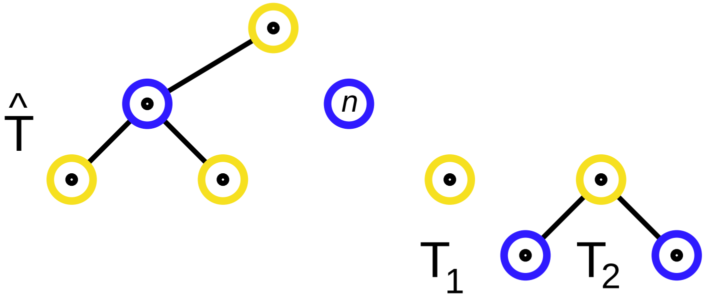

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Enumerating alternating trees: Proof

How many different trees T lead to same sequence $\hat{T}$, ⓝ, T_1, T_2, ..., T_r ?

- Distribution of labels: $\binom{n-1}{k_1,k_2,\ldots,k_r}$
- Possibilities of attaching subtrees $T_1, T_2, \ldots, T_r$ to node n: unordered trees: $\frac{1}{r!}$, d-ary trees: $\binom{d}{r}$, etc.
- Number of possible positions, where node n can be attached to $\hat{T}$, such that up-down alternating labelling is preserved:

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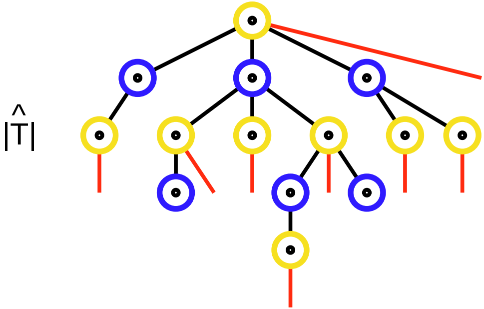

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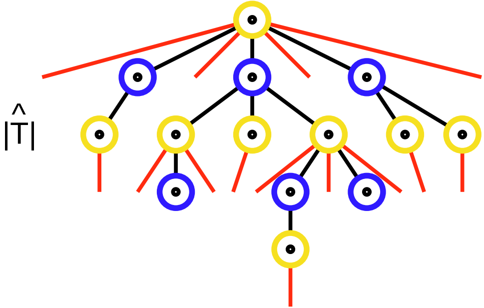
- Ordered trees:

$$\text{factor } w = \sum_{v \in V} (\deg^+(v) + 1) = |V| + |\hat{T} \setminus V| = |\hat{T}|$$



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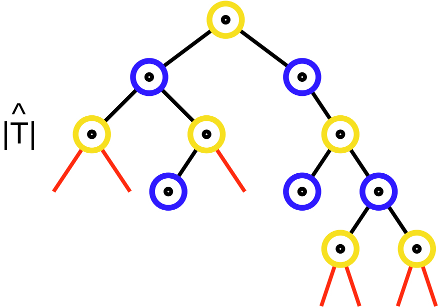
- *d*-ary trees:

$$\text{factor } w = \sum_{v \in V} (d - \deg^{+}(v)) = (d+1)|V| - |\hat{T}|$$



Enumerating alternating trees: Proof

- *d*-bundled trees:

$$\text{factor } w = \sum_{v \in V} (\deg^+(v) + d) = (d-1)|V| + |\hat{T}|$$

- Motzkin trees:

$$\text{factor } w = \sum_{v \in V^{[0]}} 1 + \sum_{v \in V^{[1]}} 2 = |V^{[0]}| + 2|V^{[1]}|$$

Set of "yellow nodes" with 0 children: $V^{[0]}$

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Enumerating alternating trees: Proof

- *d*-bundled trees:

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If tree family is not closed under decomposition ?

Consider larger tree family $\mathcal{S} \supseteq \mathcal{T}$, such that $\mathcal{S}$ is closed under decomposition

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Yellow nodes: 0, 1 left, 1 right, or 2 children

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- **Tree decomposition**

  $\Rightarrow$ Recursive description of T_n

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Enumerating alternating trees: Proof

- Ordered trees: $$T'(z) - 1 = \frac{zT'(z)}{1 - T(z)}$$
- Unordered trees: $$F_z(z,u) - u = ue^{F(z,u)} F_u(z,u)$$
- d-ary trees: $$F_z - u = (1+F)^d \big((d+1)uF_u - zF_z\big)$$
- d-bundled trees: $$F_z - u = \frac{1}{(1-F)^d}\big((d-1)uF_u + zF_z\big)$$
- Motzkin trees: $$F_z - u_0 = \left(1 + F + F^2\right)\left(u_1 F_{u_0} + 2F_{u_1}\right)$$
- Strict binary trees: $$F_z - u_0 = \left(1 + F^2\right)\left(2u_1 F_{u_0} + F_{u_1}\right)$$

Enumerating alternating trees: Proof

- **Solving PDE via Method of Characteristics:**
  - study system of characteristic DE
  - searching for functions, which are constant along any characteristic curve (first integrals)
  - suitable transformation of variables
  - first order ordinary linear DE

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  - Implicit function theorem: locate all dominant singularities
  - Weierstrass preparation theorem: behaviour in complex neighbourhood of singularities

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  Transfer lemmata of Flajolet and Odlyzko [1990]

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Influence of alternating labelling on tree structure

Influence on tree structure: Parameters studied

How much randomness gets lost due to up-down alternating labelling?

Parameters studied for ordered up-down alternating trees:

- Label of root node
- Degree of root node
- Depth of random node

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Influence on tree structure: Label of root

Label of root node:

$T_{n,j}$: number of trees of size n, where root has label j

$$T_{n,j} = (n-j)(n-1)^{j-2} n^{n-j-1}$$

Generating function: $F(z,v) = \sum_{n\geq 1} \sum_{1\leq j\leq n} T_{n,j} \frac{z^{j-1}}{(j-1)!} \frac{v^{n-j}}{(n-j)!}$

$$F(z,v) = e^{ve^{W(z+v)}}$$

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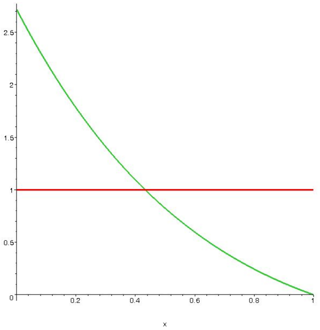
Random variable L_n: $\mathbb{P}\{L_n = j\} = \frac{T_{n,j}}{T_n}$

Limiting distribution result:

$\frac{L_n}{n} \xrightarrow{(d)} L$ with density $f(x) = (1-x)e^{1-x}, \ 0 \le x \le 1$

random labelling

up-down labelling



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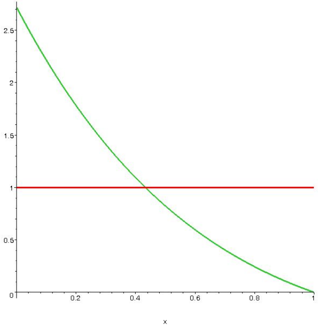

Influence on tree structure: Label of root

Expectation:

$$\mathbb{E}(L_n) = 3n - 1 - \frac{n^n}{(n-1)^{n-1}} \sim (3-e)n \approx (0.281718\ldots) \cdot n$$

"Smaller labels are preferred to become label of root node"

Influence on tree structure: Degree of root

Degree of root node:

$T_{n,m}$: number of trees of size n, where root has degree m

$$T_{n,m} = H_m (n-1)^{n-1} + \sum_{\ell=1}^{m} \binom{m}{\ell} (-1)^{\ell} \frac{\ell+1}{\ell} (n-1-\ell)^{n-1}$$

H_m: harmonic numbers

Generating function: $F(z,v) = \sum_{n \geq 1} \sum_{m \geq 0} T_{n,m} \frac{z^n}{n!} v^m$

$$F(z,v) = \frac{W(z) e^{-W(z)}}{1-v} - \frac{e^{-W(z)}}{1-v} \log\left(\frac{1}{1-v(1-e^{-W(z)})}\right)$$

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Expectation: (randomly labelled: $\mathbb{E}(R_n) \sim 3$)

$$\mathbb{E}(R_n) = \frac{1}{2}\left(\left(\frac{n+1}{n-1}\right)^{n-1} - 1\right) \sim \frac{e^2-1}{2} \approx 3.194528\ldots$$

"On average root of alternating tree has slightly higher degree than root of randomly labelled tree"

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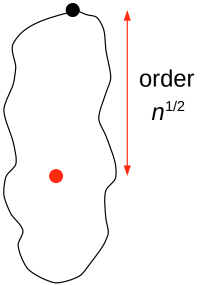
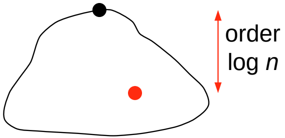
Random labelled ordered trees:



Increasing labelled ordered trees:



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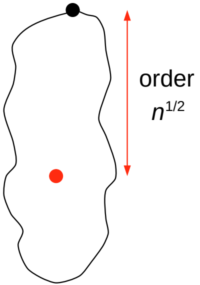

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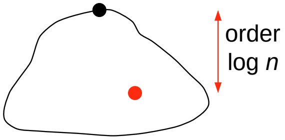

Alternating labelled ordered trees: ?

Influence on tree structure: Depth of random node

Random variable D_n: depth of random node in random size n tree

Limiting distribution result:

Depth is asymptotically Rayleigh distributed:

$$\frac{D_n}{\sqrt{n}} \xrightarrow{(d)} R_{2/3} \quad \text{with density} \quad f(x) = \frac{9x}{4} e^{-\frac{9x^2}{8}}$$

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"Alternating labelled tree is **only slightly shorter** compared to randomly labelled tree"

"On average: depth of random node is about 1/3 smaller than for randomly labelled tree"