Flag enumerations of matroid base polytopes

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Outline

1. Matroid base polytopes
2. Hyperplane splits of a matroid base polytope
3. The **cd**-index of a matroid base polytope
4. Question/Problem

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Matroid base polytopes

Definition

A matroid base polytope $Q(M)$ for a matroid M on $[n]$ is the polytope in $\mathbb{R}^n$ whose vertices are the incidence vectors of the bases of M.

Example

A matroid M on $[4]$ — The matroid base polytope $Q(M)$

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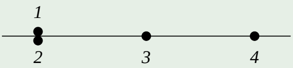

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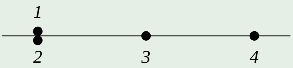

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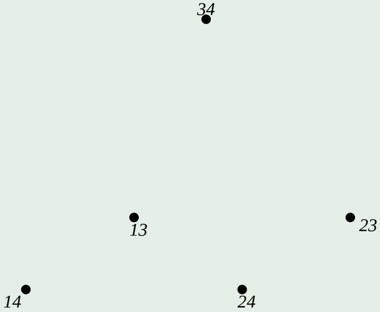

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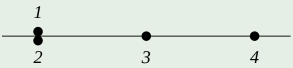

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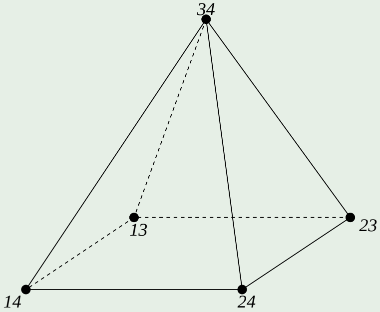

The matroid base polytope $Q(M)$

Faces of a matroid base polytope

Proposition(Ardila and Klivans, 2006)

For $\omega \in \mathbb{R}^n$, let $Q(M)_\omega$ be the face of $Q(M)$ at which $\sum_{i=1}^n \omega_i x_i$ attains its minimum.

- $Q(M)_\omega = Q(M_\omega)$ for some matroid M_ω.
- M_ω depends only on

$$\mathcal{F}(\omega) := \{\emptyset = S_0 \subset S_1 \subset \cdots \subset S_k \subset S_{k+1} = [n]\},$$

where ω is constant on $S_i - S_{i-1}$ and $\omega|_{S_i - S_{i-1}} < \omega|_{S_{i+1} - S_i}$.

- In fact,

$$M_{\mathcal{F}} := M_\omega = \bigoplus_{i=1}^{k+1} (M|_{S_i})/S_{i-1}.$$

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Definition

- $\alpha, \beta \in [n]$ are equivalent if there are bases B and B' of M such that $\alpha \in B$ and $B' = B - \{\alpha\} \cup \{\beta\}$.
- The equivalence classes are called connected components.
- A matroid M is connected if it has only one connected component.

Definition

A flag $\mathcal{F} = \{\emptyset = S_0 \subset S_1 \subset \cdots \subset S_k \subset S_{k+1} = [n]\}$ is factor-connected if $(M|_{S_i})/S_{i-1}$ are connected for $i = 1, \ldots, k+1$.

Definition

Two factor-connected flags $\mathcal{F}$ and $\mathcal{F}'$ of same length are equivalent if they are equal in all but rank j and $(M|_{S_{j+1}})/S_{j-1}$ has two connected components. Then take the transitive closure.

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Proposition(K.)

Two factor-connected flags $\mathcal{F}$ and $\mathcal{F}'$ are equivalent if and only if $M_{\mathcal{F}} = M_{\mathcal{F}'}$.

Example

A matroid M on $[4]$

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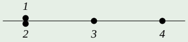

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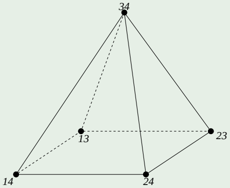

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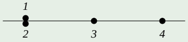

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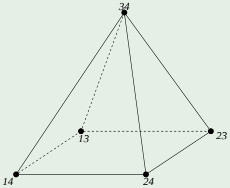

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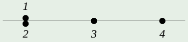

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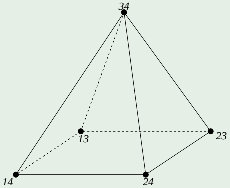

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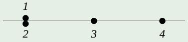

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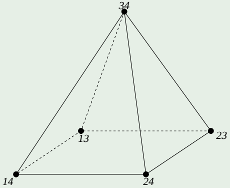

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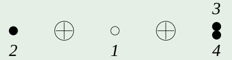

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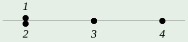

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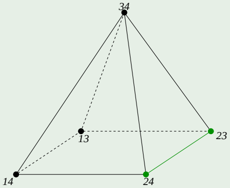

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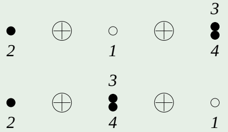

Faces of a matroid base polytope

Theorem (K.)

For a face σ of $Q(M)$,

- $L_\sigma := \bigcup_{\substack{\mathcal{F} \text{ factor connected,} \\ Q(M_\mathcal{F})=\sigma}} \mathcal{F}$ *forms a distributive lattice.*
- $L_\sigma \cong J(P_\sigma)$ *where P_σ is the poset defined by*
  - *The elements of P_σ are the connected components C_i of M_σ*
  - *$C_1 < C_2$ if and only if*

$$\sigma \subset \{x \in \mathbb{R}^n : \sum_{e \in S} x_e = r(S)\} \text{ and } C_2 \subset S \subset [n] \text{ implies } C_1 \subset S$$

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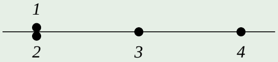

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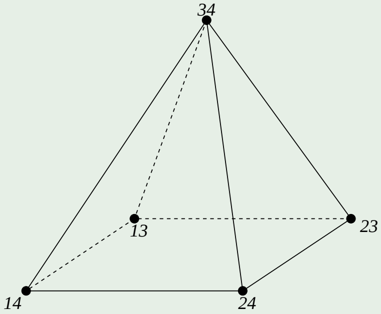

$Q(M)$

The face poset of $Q(M)$

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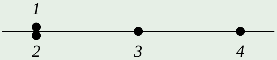

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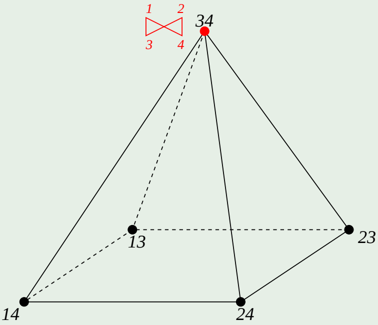

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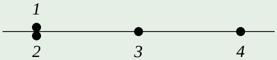

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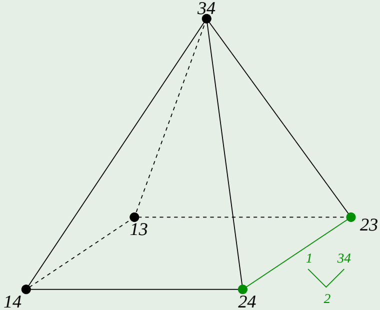

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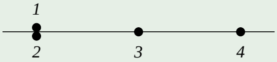

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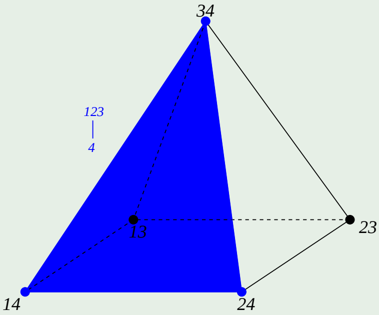

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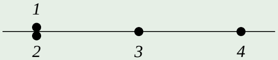

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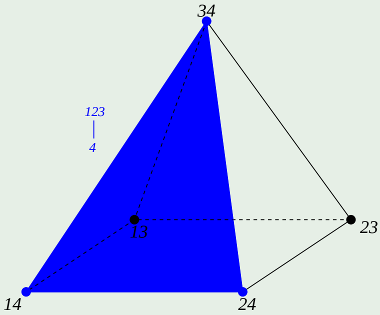

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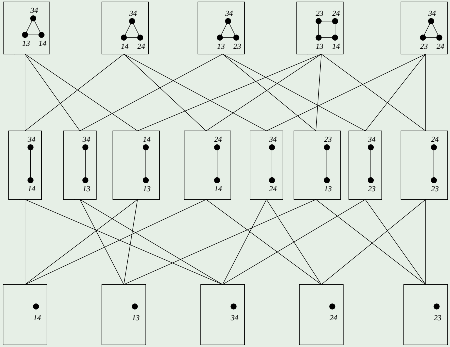

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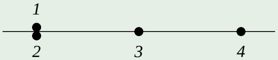

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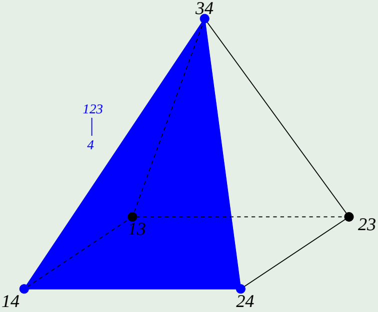

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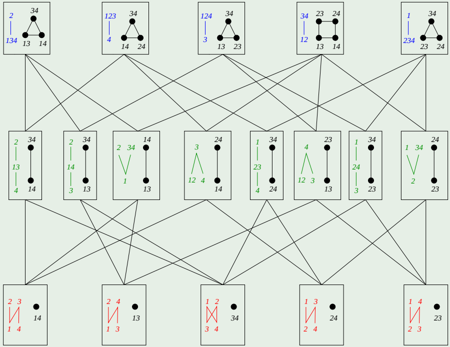

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Hyperplane splits of a matroid base polytope

Definition

A hyperplane split of $Q(M)$ is a decomposition of $Q(M)$ as $Q(M_1) \cup Q(M_2)$ where

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Hyperplane splits of a matroid base polytope

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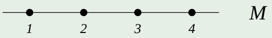

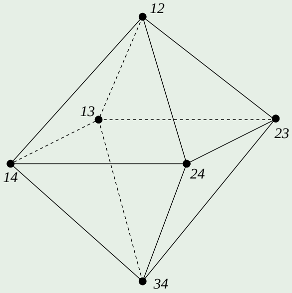

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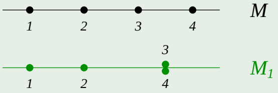

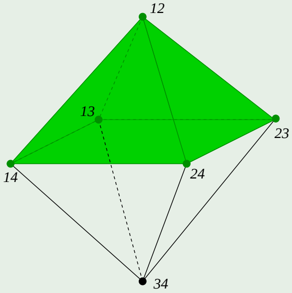

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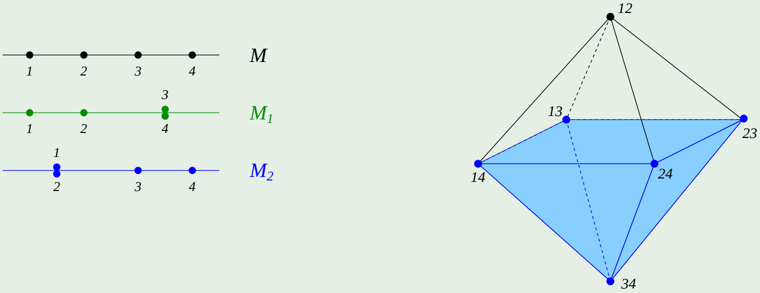

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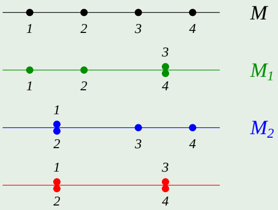

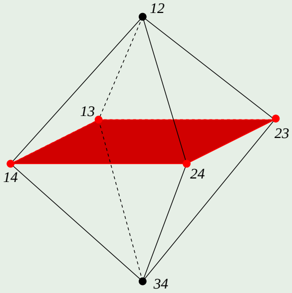

Hyperplane splits of a matroid base polytope

Theorem (K.)

Let M be a rank r matroid on $[n]$ and H be a hyperplane in $\mathbb{R}^n$ defined by $\sum_{e \in S} x_e = k$. Then H gives a hyperplane split of $Q(M)$ if and only if

- $r(S) > k$ *and* $r(S^c) > r - k$,
- *if* I_1 *and* I_2 *are* k*-element independent subset of* S *such that* $(M/I_1)|_{S^c}$ *and* $(M/I_2)|_{S^c}$ *have rank* $r - k$, *then*

$$(M/I_1)|_{S^c} = (M/I_2)|_{S^c}.$$

Example

$H = \{x \in \mathbb{R}^4 | x_1 + x_2 = 1\}$
$S = \{1, 2\}, k = 1$

- $r(S) = 2 > 1, r(S^c) = 2 > 1$
- $I_1 := \{1\} \to B((M/I_1)|_{S^c}) = \{3, 4\},$
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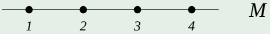

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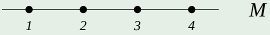

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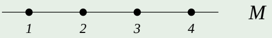

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Hyperplane splits of a matroid base polytope

Corollary

Let M be a rank 2 matroid on $[n]$ and H be a hyperplane in $\mathbb{R}^n$ defined by $\sum_{e \in S} x_e = 1$. Then H gives a hyperplane split of $Q(M)$ if and only if S and S^c are both unions of at least two parallelism classes.

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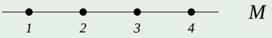

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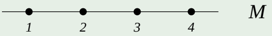

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Outline

The **cd**-index of a polytope

Definition

The **cd**-index $\Psi(Q)$ of a polytope Q, a polynomial in noncommutative variables **c** and **d**, is a very compact encoding of the flag numbers of a polytope.

Example

$$\Psi(P) = \mathbf{c}^3 + 3\mathbf{cd} + 3\mathbf{dc}$$

$$\Downarrow \quad (\mathbf{c} = \mathbf{a} + \mathbf{b}, \mathbf{d} = \mathbf{ab} + \mathbf{ba})$$

$$\mathbf{ab}(P) = \mathbf{a}^3 + 4\mathbf{a}^2\mathbf{b} + 7\mathbf{aba} + 4\mathbf{ab}^2$$

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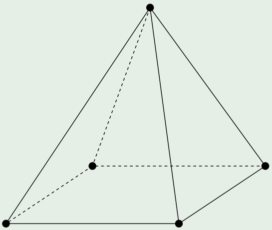

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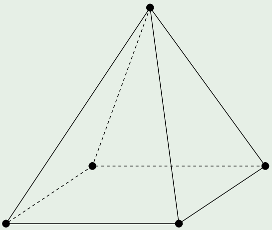

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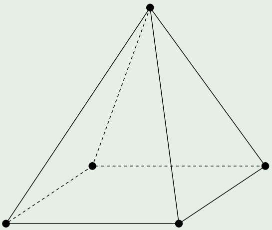

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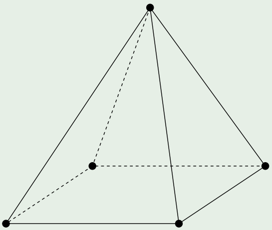

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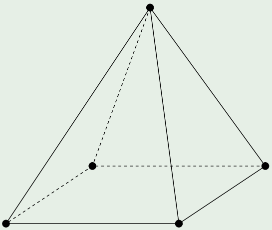

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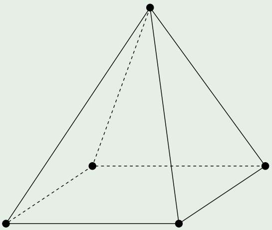

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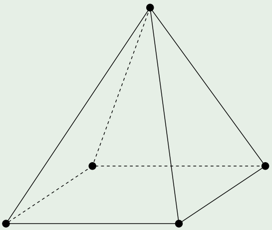

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The **cd**-index of a polytope

Theorem (Ehrenborg and Readdy, 1998)

Let Q be a polytope. Then

$$\Psi(\mathrm{Pyr}(Q)) = \frac{1}{2}\left[\Psi(Q)\cdot \mathbf{c} + \mathbf{c}\cdot\Psi(Q) + \sum_{\sigma}\Psi(\sigma)\cdot\mathbf{d}\cdot\Psi(Q/\sigma)\right],$$

$$\Psi(\mathrm{Prism}(Q)) = \Psi(Q)\cdot\mathbf{c} + \sum_{\sigma}\Psi(\sigma)\cdot\mathbf{d}\cdot\Psi(Q/\sigma),$$

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where the sum is over all proper faces σ of Q.

Theorem (Ehrenborg and Fox, 2003)

For polytopes P and Q, $\Psi(P\times Q)$ can be computed from $\Psi(P)$ and $\Psi(Q)$.

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Theorem (K.)

If Q is a polytope and H a hyperplane in $\mathbb{R}^n$ intersecting relint Q, then

$$\Psi(Q) = \Psi(Q^+) + \Psi(Q^-) - \Psi(\widehat{Q}) \cdot \mathbf{c} - \sum_{\sigma} \Psi(\hat{\sigma}) \cdot \mathbf{d} \cdot \Psi(\widehat{Q}/\hat{\sigma}),$$

where $Q^+ = Q \cap H^+$, $Q^- = Q \cap H^-$, $\widehat{Q} = Q \cap H$, $\hat{\sigma} = \sigma \cap H$, and the sum is over all proper faces σ of Q intersecting both open halfspaces obtained by H nontrivially.

Example

$$\Psi(1234) = \Psi(125) + \Psi(1345) - \Psi(15) \cdot \mathbf{c} - \Psi(5) \cdot \mathbf{d} \cdot \Psi(15/5)$$

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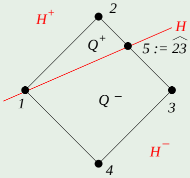

$$\begin{aligned}\Psi(1234) = &\Psi(125) + \Psi(1345)\\ &- \Psi(15) \cdot \mathbf{c}\\ &- \Psi(5) \cdot \mathbf{d} \cdot \Psi(15/5)\end{aligned}$$

The **cd**-index of a polytope

Theorem (K.)

If Q is a polytope and H a hyperplane in $\mathbb{R}^n$ intersecting relint Q, then

$$\Psi(Q) = \Psi(Q^+) + \Psi(Q^-) - \Psi(\widehat{Q}) \cdot \mathbf{c} - \sum_{\sigma} \Psi(\hat{\sigma}) \cdot \mathbf{d} \cdot \Psi(\widehat{Q}/\hat{\sigma}),$$

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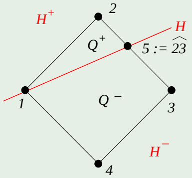

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The **cd**-index of $Q(M)$ for a rank 2 matroid M

Fact

A (loopless) rank 2 matroid on $[n]$ is determined up to isomorphism by the composition $\alpha(M)$ of $[n]$ that gives the sizes α_i of its parallelism classes.

Proposition (K.)

The **cd**-index of $Q(M_\alpha)$ for a rank 2 matroid M_α can be expressed using the **cd**-indices of matroid base polytopes for rank 2 matroids corresponding to compositions with 3 or less parts.

Fact

- If α has only one part, i.e., $\alpha = (\alpha_1)$, then $Q(M_\alpha) = \Delta_{\alpha_1}$.
- If α has two parts, i.e., $\alpha = (\alpha_1, \alpha_2)$, then $Q(M_\alpha) = \Delta_{\alpha_1} \times \Delta_{\alpha_2}$.

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The **cd**-index of $Q(M)$ for a rank 2 matroid M

Proposition (Precise version)

Let M_α be a rank 2 matroid corresponding to a composition α. Then

$$\Psi(Q(M_\alpha)) = \sum_{i=2}^{l(\alpha)-1} \Psi(Q(M_{\lambda(\alpha,i)})) - \left(\sum_{i=2}^{l(\alpha)-2} \Psi(\Delta_{\mu(\alpha,i)_1} \times \Delta_{\mu(\alpha,i)_2}) \right) \cdot \mathbf{c}$$

$$- \sum_{\substack{\beta < \alpha \\ l(\bar{\beta}) \geq 4}} \prod_{j=1}^{l(\alpha)} \binom{\alpha_j}{\beta_j} \left(\sum_{i=2}^{l(\bar{\beta})-2} \Psi(\Delta_{\mu(\bar{\beta},i)_1} \times \Delta_{\mu(\bar{\beta},i)_2}) \right) \cdot \mathbf{d} \cdot \Psi(\Delta_{n-|\bar{\beta}|}).$$

Example

$$\Psi(M_{2,2,1,1}) = \Psi(M_{2,2,2}) + \Psi(M_{4,1,1}) - \Psi(M_{4,2}) \cdot \mathbf{c}$$

$$- 2\Psi(M_{3,2}) \cdot \mathbf{d} \cdot \Psi(\Delta_1) - 2\Psi(M_{3,2}) \cdot \mathbf{d} \cdot \Psi(\Delta_1)$$

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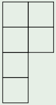

The **cd**-index of $Q(M)$ for a rank 2 matroid M

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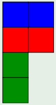

The **cd**-index of $Q(M)$ for a rank 2 matroid M

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Let M_α be a rank 2 matroid corresponding to a composition α. Then

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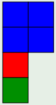

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Let M_α be a rank 2 matroid corresponding to a composition α. Then

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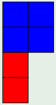

The **cd**-index of $Q(M)$ for a rank 2 matroid M

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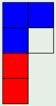

The **cd**-index of $Q(M)$ for a rank 2 matroid M

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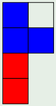

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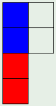

Outline

Question/Problem

- Find a matroidal formula for the **cd**-index of $Q(M)$ for hyperplane splits.
- Find a simple formula for the **cd**-index of $Q(M_\alpha)$ for rank 2 matroids corresponding to compositions α with 3 parts.
- Find an explicit CL-labeling for the face poset of $Q(M)$.

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