Deodhar elements in Kazhdan–Lusztig theory

Brant Jones
brant@math.ucdavis.edu
University of California, Davis

June 25, 2008

Overview of K–L polynomials

Let W be a Coxeter group generated by $S = \{s_1, s_2, \ldots, s_n\}$ with relations $s_i^2 = 1, (s_i s_j)^{m_{ij}} = 1$ for some m_{ij}.

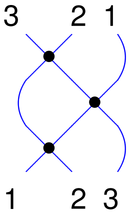
Example

$W = S_{n+1}$ generated by **adjacent transpositions** $s_i = (i, i+1)$.

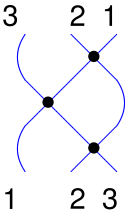
If $w = s_{i_1} s_{i_2} \cdots s_{i_p}$ is an expression for w of minimal length, we say it is *reduced* and $l(w) = p$ is the *length*.

Example

$s_1 s_2 s_1 =$



$= [321] =$



$= s_2 s_1 s_2$

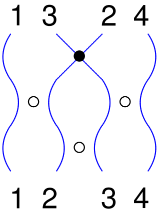
Overview of K–L polynomials

Fix a reduced expression $w = s_{i_1} \cdots s_{i_p}$. We call $\sigma \in \{0,1\}^{l(w)}$ a *mask* for w. Let w^σ be the result of multiplying all of the mask-value 1 entries of w together.

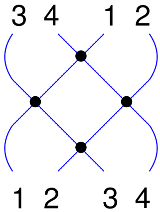
If there exists $\sigma \in \{0,1\}^{l(w)}$ such that $w^\sigma = x$ then we say $x \leq w$ in *Bruhat order*, a poset on the elements of W that is ranked by length.

Example

$s_2 =$



$\leq$



$= s_2 s_1 s_3 s_2$

Overview of K–L polynomials

We say that $w = [w_1 \cdots w_n]$ *contains the permutation pattern* $p = [p_1 \cdots p_k]$ if there exists a subsequence $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that the entries $w_{i_1} w_{i_2} \cdots w_{i_k}$ are in the same relative order as $p_1 p_2 \cdots p_k$. If w does not contain p then we say it *avoids* it.

Example

$w = [\underline{3}4\underline{1}6\underline{5}\underline{7}2]$ contains the permutation pattern $p = [2134]$ but avoids the permutation pattern $p' = [12345]$.

This notion has been generalized by Billey, Postnikov and Braden to other types using root subsystems. Also, Woo and Yong have used *interval pattern avoidance* to study invariants of singularities in Schubert varieties.

Overview of K–L polynomials

The *Hecke algebra* H over $R = \mathbb{Z}[q^{1/2}, q^{-1/2}]$ has two bases.
The *standard basis* $\{T_w : w \in W\}$ satisfies:

- $T_s T_w = T_{sw}$ if $l(sw) > l(w)$, and
- $T_s^2 = (q-1)T_s + q$.

The *Kazhdan–Lusztig basis* $\{C'_w : w \in W\}$ satisfies:

- $\overline{C'_w} = C'_w$ (where $\overline{q} = q^{-1}$ and $\overline{T_w} = T_{w^{-1}}^{-1}$), and
- $C'_w = q^{-\frac{1}{2}l(w)} \sum_{x \leq w} P_{x,w}(q) T_x$.

Here, $P_{x,w} \in \mathbb{Z}[q]$, $P_{w,w} = 1$ and

$$\text{degree } P_{x,w}(q) \leq \frac{1}{2}(l(w) - l(x) - 1).$$

Overview of K–L polynomials

Conjecture

Kazhdan–Lusztig, 1979 *The coefficients of $P_{x,w}(q)$ are nonnegative in the Hecke algebra associated to any Coxeter group.*

When W is the Weyl group of a semisimple algebraic group (like $SL_n(\mathbb{C})$), there exist **Schubert varieties** X_w for $w \in W$.

Theorem

Kazhdan–Lusztig, 1980 *When W is a finite or affine Weyl group, we have*

$$P_{x,w}(q) = \sum_{i \geq 0} \dim IH^{2i}(X_w)_{xB} \;\; q^i.$$

Hence, we have nonnegativity in these cases.

Deodhar's result

Theorem

Deodhar, 1990 *Let W be a Coxeter group such that $P_{x,w}(q)$ has nonnegative coefficients. Fix a reduced expression $w = s_{i_1} s_{i_2} \cdots s_{i_p}$. Then there exists a set $E_w \subset \{0,1\}^{l(w)}$ of* masks *such that*

$$P_{x,w}(q) = \sum_{\sigma \in E_w, w^\sigma = x} q^{d(\sigma)}.$$

Here, w^σ is the result of multiplying the mask-value 1 entries of w together and $d(\sigma)$ is # positions j such that

$$s_{i_1}^{\sigma_1} \cdots s_{i_{j-1}}^{\sigma_{j-1}} s_{i_j} < s_{i_1}^{\sigma_1} \cdots s_{i_{j-1}}^{\sigma_{j-1}},$$

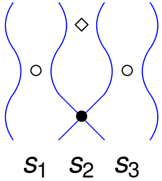
called defects.

Deodhar's result

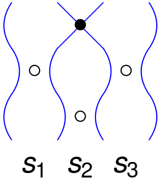
Example

$$P_{s_2, s_2 s_1 s_3 s_2} = q^{d\left(\begin{bmatrix} 2 & 1 & 3 & 2 \\ 1 & 0 & 0 & 0 \end{bmatrix}\right)} + q^{d\left(\begin{bmatrix} 2 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)} = q + 1$$

$=$



$+$



Deodhar elements

Theorem

Deodhar, Billey–Warrington *Let W be the symmetric group. The following are equivalent:*

(1) $E_w = \{0,1\}^{l(w)}$.

(2) $C'_w = C'_{s_{i_1}} C'_{s_{i_2}} \cdots C'_{s_{i_p}}$ *for* $w = s_{i_1} \cdots s_{i_p}$.

(3) *The Bott–Samelson resolution* $f_w : Z_w \to X_w$ *is small (so* $IH_*(X_w) \cong H_*(Z_w)$*).*

(4) w *is a 321-hexagon avoiding permutation (i.e. avoids* $[321], [46718235], [46781235], [56718234], [56781234]$*).*

Definition

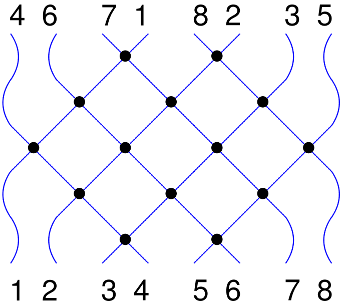
For any finite Weyl group, we say w is *Deodhar* if any of (1)-(3) hold.

Deodhar elements

Theorem

(Billey–Warrington, 2001) *A permutation is Deodhar if and only if it is* **fully-commutative** *(no $s_i s_{i+1} s_i$ factors) and it heap-avoids the* **hexagon pattern***:*

$w = s_5 s_6 s_7 s_3 s_4 s_5 s_6 s_2 s_3 s_4 s_5 s_1 s_2 s_3$

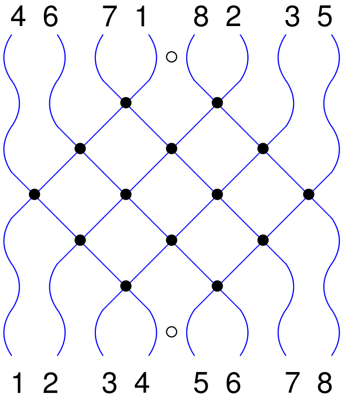


Deodhar elements

Theorem

(Billey–Warrington, 2001) *A permutation is Deodhar if and only if it is* **fully-commutative** *(no $s_i s_{i+1} s_i$ factors) and it heap-avoids the* **hexagon pattern***:*

$w = s_5 s_6 s_7 s_3 s_4 s_5 s_6 s_2 s_3 s_4 s_5 s_1 s_2 s_3$



Deodhar elements

Theorem

Billey–Jones, Billey–Warrington *Let W be a finite Weyl group. The following are equivalent:*

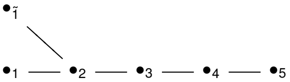
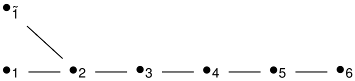
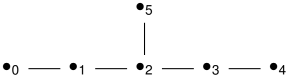
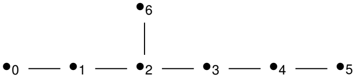
(1) *w is Deodhar.*

(2) *w avoids a list of* embedded factor patterns. *These patterns appear as factors of a fixed reduced expression for w, up to a Coxeter graph embedding of the generators that appear in the pattern.*

Theorem

If W is a finite Weyl group and w is a Deodhar element of W then $\mu(x, w) \in \{0, 1\}$ for all $x \in W$.

Minimal Non-Deodhar Embedded Factors for Weyl Groups

Type	Coxeter Graph	Embedded Factor Patterns
A_2	$\bullet_1$ — $\bullet_2$	121, 212 (short-braids)
A_7	$\bullet_1$ — $\bullet_2$ — $\bullet_3$ — $\bullet_4$ — $\bullet_5$ — $\bullet_6$ — $\bullet_7$	56734562345123 (Hexagon)
D_6		$3451234\tilde{1}231$ (HEX_5)
D_7		$34562345\tilde{1}234123$ (HEX_2) $4562345123 4\tilde{1}21$ (HEX_{3a}) $1456 2345\tilde{1}23412$ (HEX_{3b})
E_6		0125342312501 5123012543210 1253423125012 2512301254321
E_7		012346523412301 346123012543210 123465234123012 234612301254321 523461253423012 5

Also, for each $n \geq 8$, there is one additional embedded factor pattern in D_n denoted $FLHEX_n$. These all contain the 1-line pattern $[\bar{1}, 6, 7, 8, \bar{5}, 2, 3, 4]$.

Some K–L polynomials for permutations

(complexity of $P_{x,w}$) ↓

smooth = $\{[3412], [4231]\}$-avoiding, $P_{x,w} = 1$

co-Grassmannian ⊂ covexillary

Boolean elements

freely-braided hexagon-avoiding ⊂ maximally-clustered hexagon-avoiding

Deodhar elements = fully-commutative hexagon-avoiding, $E_w = \{0, 1\}^{l(w)}$

↑ (complexity of E_w)

fully-commutative ⊂ freely-braided ⊂ maximally-clustered

Let W be simply-laced and let $w \in W$. Every reduced expression $s_{i_1} \cdots s_{i_p}$ for w determines a *root sequence*. (The **set** of vectors in the root sequence is the *inversion set* of w.)

Example

$[321] = s_1 s_2 s_1 = \{\alpha_1, s_1(\alpha_2) = \alpha_2 + \alpha_1, s_1 s_2(\alpha_1) = \alpha_2\}$.

A **consecutive subsequence** $\alpha, \alpha + \beta, \beta$ of a root sequence is called a *contractible triple*.
We say w is:

- fully-commutative $\iff$ there are no contractible triples.
- freely-braided $\iff$ all contractible triples are pairwise disjoint.
- maximally-clustered $\iff$ whenever T and T' are contractible triples with non-empty intersection, then the highest (i.e. middle) roots of T and T' agree.

Example

$w = [4231]$ is maximally-clustered but not freely-braided.
Consider

$$w = s_3 s_2 s_1 s_2 s_3 \leftrightarrow \{\alpha_3, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1, \alpha_1 + \alpha_2\}$$

versus

$$w = s_1 s_3 s_2 s_3 s_1 \leftrightarrow \{\alpha_1, \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3\}$$

There are also pattern-avoidance criteria.

- fully-commutative $\iff$ [321]-avoiding $\iff$ avoids $s_1 s_2 s_1$ as embedded factor.
- freely-braided $\iff$ $\{[3421], [4231], [4312], [4321]\}$-avoiding $\iff$ avoids $s_3 s_1 s_2 s_1 s_3$ and $s_2 s_3 s_2 s_1 s_2$.
- maximally-clustered $\iff$ $\{[3421], [4312], [4321]\}$-avoiding $\iff$ avoids $s_2 s_3 s_2 s_1 s_2$.

Theorem

(Green, Losonczy, 2004) *Let w be a freely-braided (or maximally-clustered) permutation. Then, there exists a reduced expression for w of the form*

$$w = u_0 b_1 u_1 \cdots b_{N(w)} u_{N(w)}$$

where u_i are fully-commutative, and b_i are braids $s_i s_{i+1} s_i$ (or braid-clusters $s_i s_{i+1} \cdots s_{i+j-1} s_{i+j} s_{i+j-1} \cdots s_{i+1} s_i$, *respectively).*

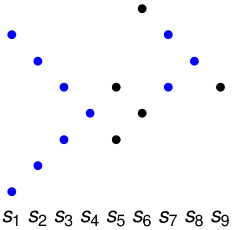
We call such a reduced expression *contracted*.

Example

For example, suppose w is given by the contracted reduced expression

$$(s_5)(s_1 s_2 s_3 s_4 s_3 s_2 s_1)(s_6 s_5 s_9)(s_7 s_8 s_7)(s_6).$$

Then the heap of w is drawn below. The braid clusters are shown in blue.



Definition

Let w be a contracted expression for a **maximally-clustered hexagon-avoiding permutation**, where every braid cluster is consecutive. We say that a mask σ on w has a $\texttt{10}\star$*-instance* if it has the values

$$\begin{bmatrix} \dots & s_i & s_{i+1} & s_i & \dots \\ * & 1 & 0 & * & * \end{bmatrix}$$

on any **central braid instance** $s_i s_{i+1} s_i$ of any **braid cluster** in w, where $*$ denotes an arbitrary mask value. Otherwise, we say that σ is a $\texttt{10}\star$*-avoiding* mask for w.

Theorem

For any $x \in S_n$*,*

$$P_{x,w}(q) = \sum_{\substack{\texttt{10}\star\text{-avoiding masks } \sigma \\ \mathrm{w}^\sigma = x}} q^{d(\sigma)}$$

Axioms for Deodhar's formula

We say that a set of masks E_{w} is *admissible for* w if:

- E_{w} contains $(1, 1, \cdots, 1)$.
- E_{w} contains $(\sigma_1, \sigma_2, \cdots, 1 - \sigma_k)$ whenever E_{w} contains $(\sigma_1, \sigma_2, \cdots, \sigma_k)$.
- $C'(E_{\mathrm{w}})$ is invariant under the Hecke algebra involution.

The set of masks E_{w} is *bounded for* w if:

$$\text{degree } P_x(E_{\mathrm{w}}) \leq \frac{1}{2}(l(w) - l(x) - 1) \text{ for all } x < w.$$

Proposition

E_{w} computes $P_{x,w}$ in Deodhar's formula if and only if E_{w} is admissible and bounded.

Proof

The main difficulty is to show $C'(E_{\mathrm{w}})$ is **invariant** under the Hecke algebra involution. Use that

$$C'(E_{\mathrm{w}}) = \overline{C'(E_{\mathrm{w}})} \iff C'(\{0,1\}^{l(w)} \setminus E_{\mathrm{w}}) = \overline{C'(\{0,1\}^{l(w)} \setminus E_{\mathrm{w}})}$$

Induct on the number of **short-braid instances** in the contracted expression w.

Miraculously,

$$C'\left(\begin{bmatrix} \cdots & s_i & s_{i\pm 1} & s_i & \cdots \\ \cdots & 1 & 0 & * & \cdots \end{bmatrix}\right) = C'\left(\begin{bmatrix} \cdots & s_i & \cdots \\ \cdots & (1-*) & \cdots \end{bmatrix}\right).$$

References

- Embedded factor patterns for Deodhar elements in Kazhdan-Lusztig theory (with Sara Billey) (Ann. Combin. 11 (3/4) (2007) 285-333. `arXiv:math/0612043`)
- Kazhdan–Lusztig polynomials for maximally-clustered hexagon-avoiding permutations (`arXiv:0704.3067`)
- Leading coefficients of Kazhdan–Lusztig polynomials for Deodhar elements (to appear in J. Algebraic Combin. `arXiv:0711.1391`)