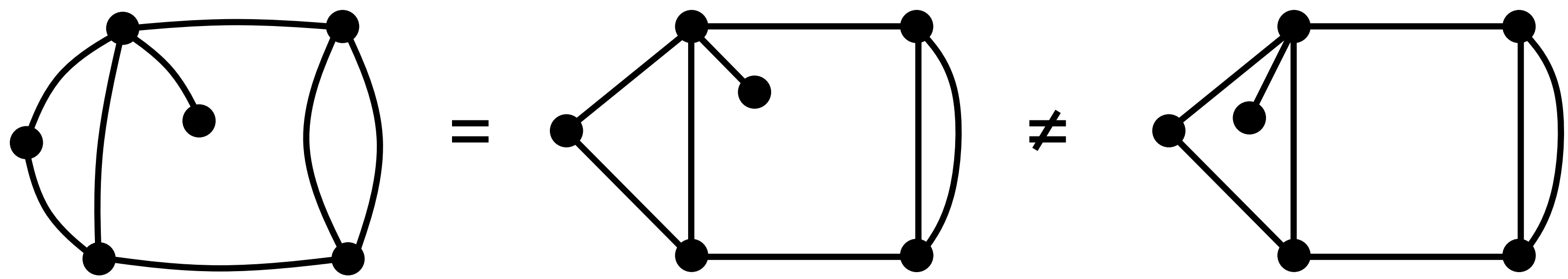
New bijective links on planar maps

Éric Fusy

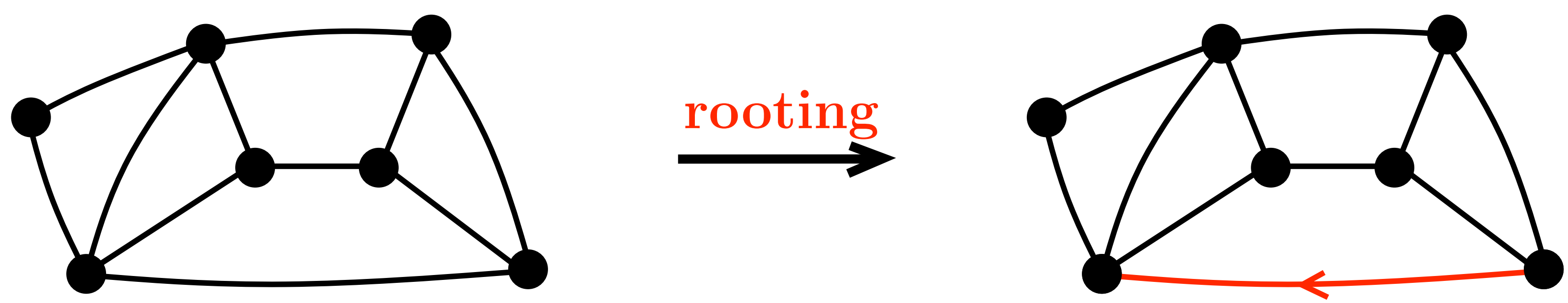
Dept. Math, Simon Fraser University (Vancouver)

Planar maps

- Planar map = graph drawn in the plane without edge-crossing, taken up to isotopy (continuous structure-preserving transformation)

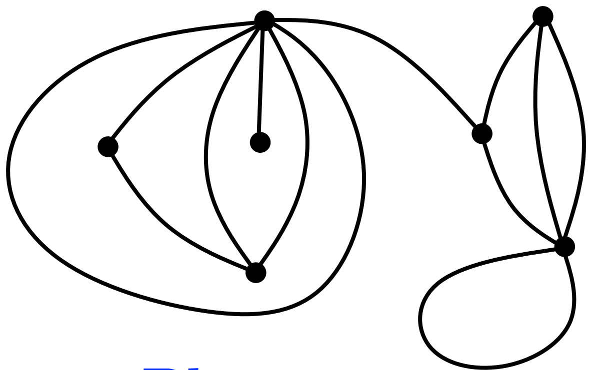


- Rooted map = map + root edge

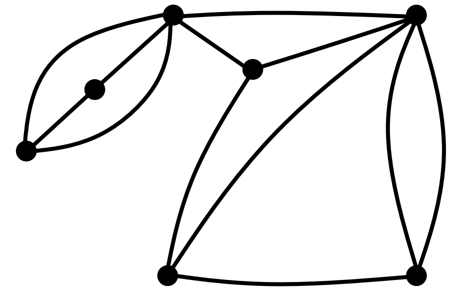


- Motivations: mesh compression, graph drawing + nice combinatorial properties

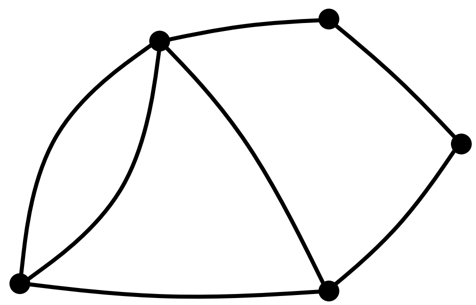
Families of planar maps



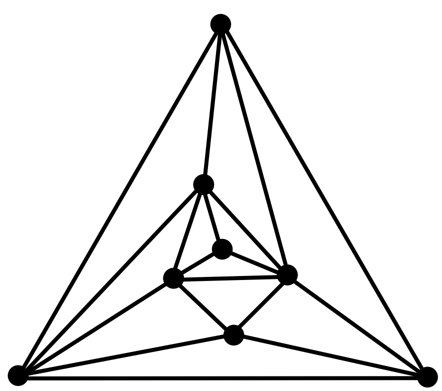
Planar map



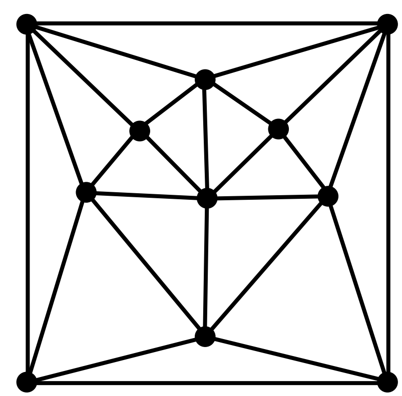
Loopless map



Nonseparable map
(no separating vertex)

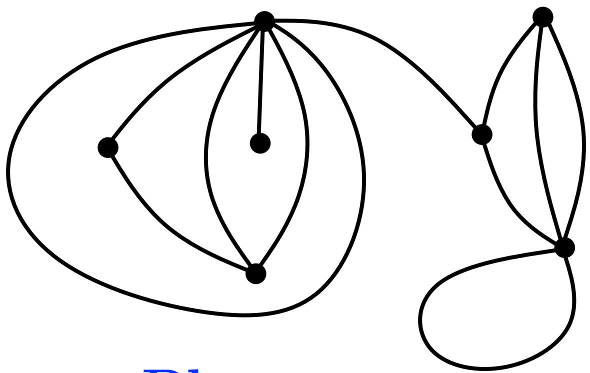


Triangulation

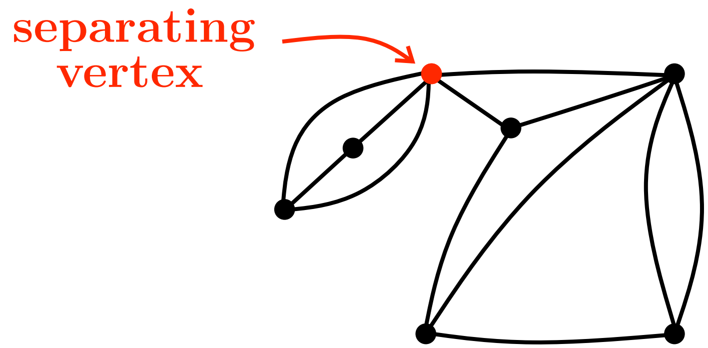


Irreducible triangulation
(no separating triangle)

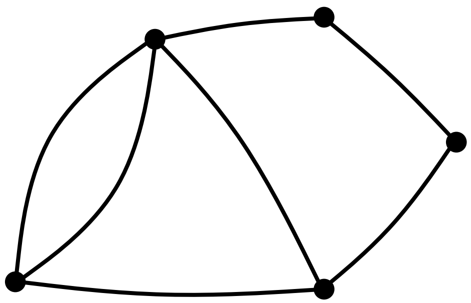
Families of planar maps



Planar map



Loopless map



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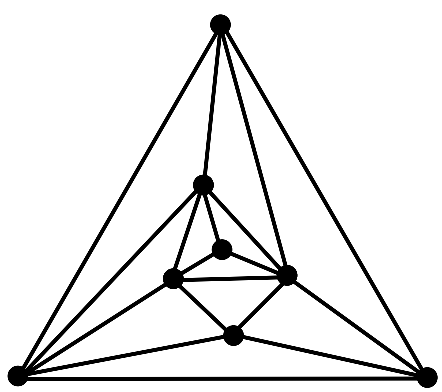
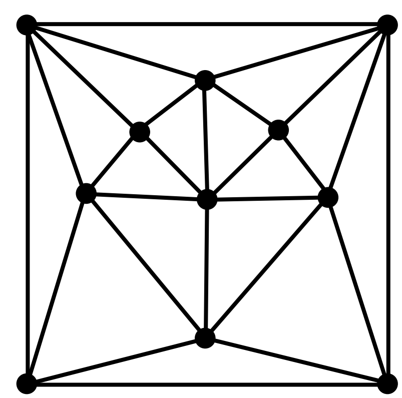


Triangulation



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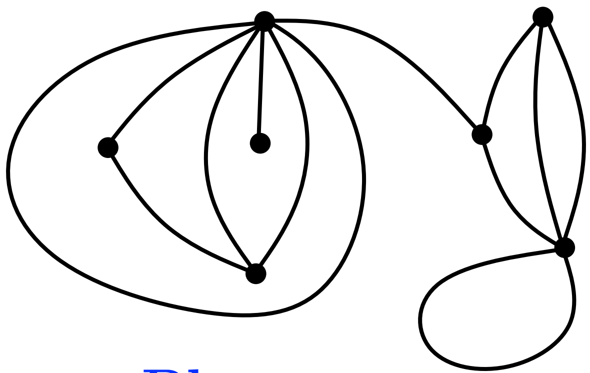
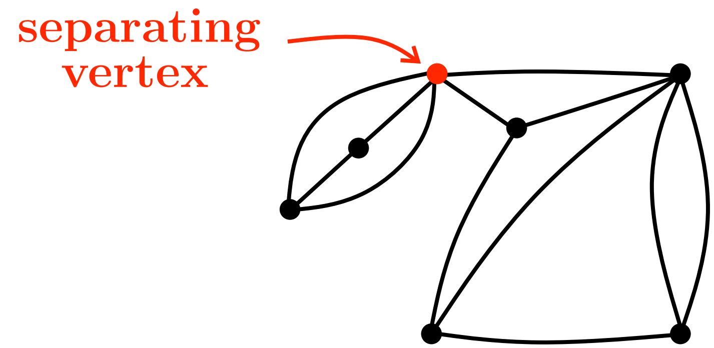
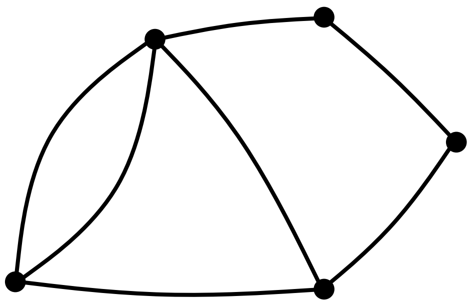


Planar map



Loopless map



Nonseparable map
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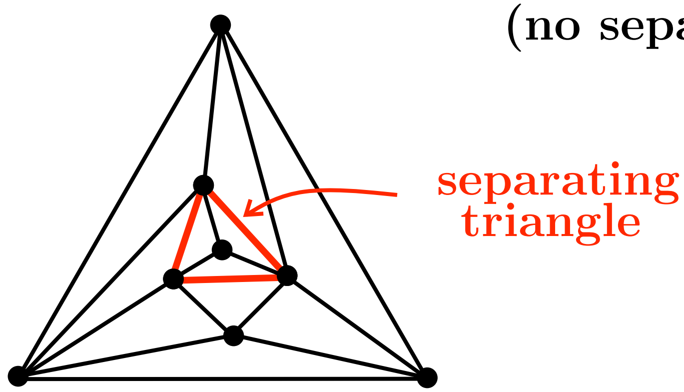
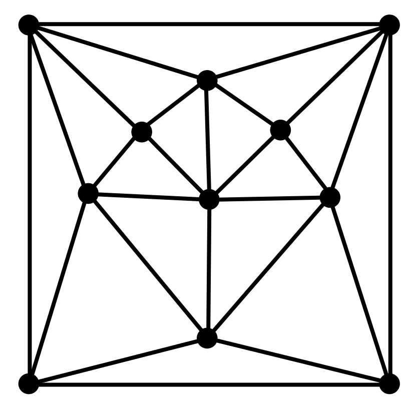


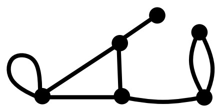
Triangulation



Irreducible triangulation
(no separating triangle)

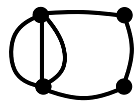
Enumeration of planar maps

- **Symbolic approach: Tutte, Brown**
- **Bijective approach: Cori, Schaeffer, Bouttier-Di Francesco-Guitter**



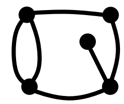
Planar maps

$\#(\mathbf{n}\text{ edges}) = \frac{2\cdot 3^n (2n)!}{(n+2)!n!}$



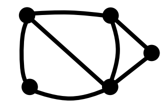
Eulerian

$\#(\mathbf{n}\text{ edges}) = \frac{3\cdot 2^{n-1} (2n)!}{(n+2)!n!}$



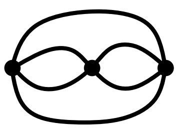
Loopless

$\#(\mathbf{n}\text{ edges}) = \frac{2(4n+1)!}{(n+1)!(3n+2)!}$



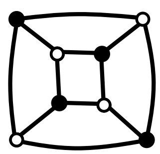
Nonseparable

$\#(\mathbf{n}\text{ edges}) = \frac{4(3n-3)!}{(n-1)!(2n)!}$



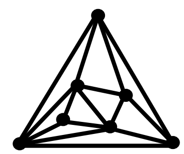
4-regular

$\#(\mathbf{n}\text{ vert.}) = \frac{2\cdot 3^n (2n)!}{(n+2)!n!}$



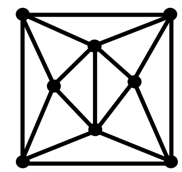
Bicubic

$\#(\mathbf{2n}\text{ vert.}) = \frac{3\cdot 2^{n-1} (2n)!}{(n+2)!n!}$



Triangulations

$\#(\mathbf{n+3}\text{ vert.}) = \frac{2(4n+1)!}{(n+1)!(3n+2)!}$



Irreducible

$\#(\mathbf{n+3}\text{ vert.}) = \frac{4(3n-3)!}{(n-1)!(2n)!}$

Enumeration of planar maps

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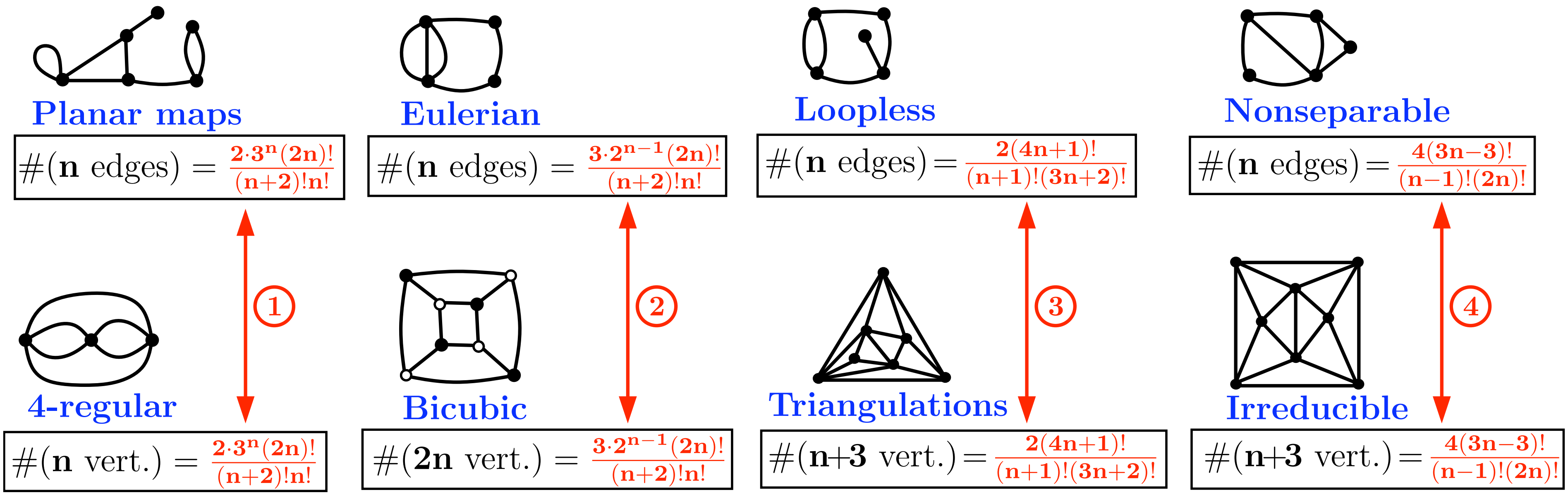

Enumeration of planar maps

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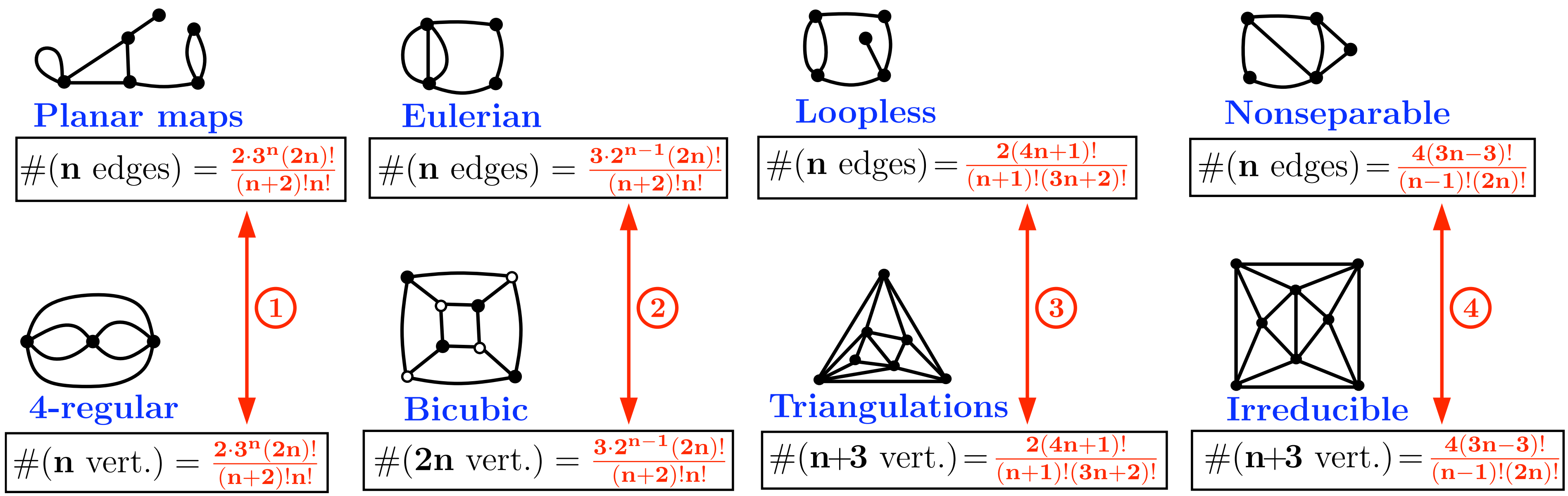
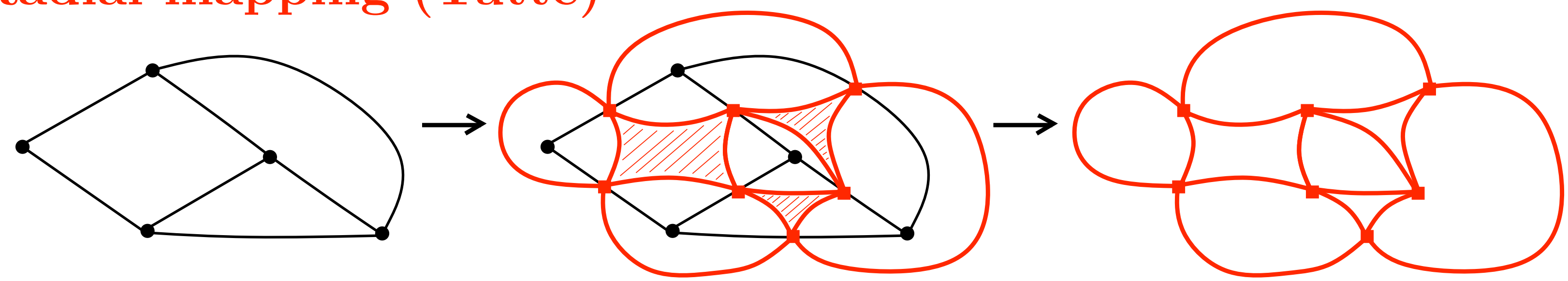


(1), (2) well known bijections (Tutte) (3) recursive bijection (Wormald)

This talk:
- **new** bijective construction for (3)
- **first** bijective construction for (4)

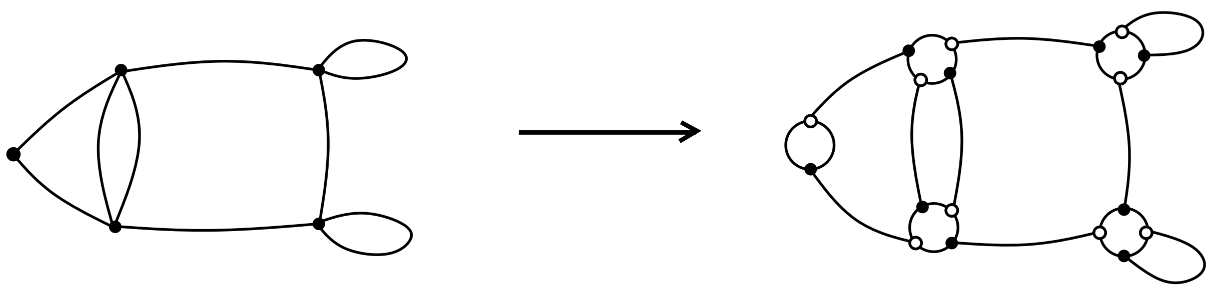
Well known bijections

① Radial mapping (Tutte)



Planar maps n edges **4-regular n vertices**

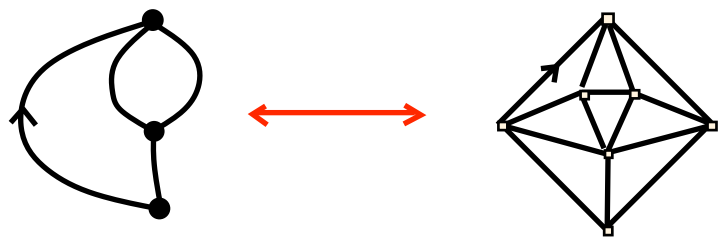
② Trinity mapping (Tutte)



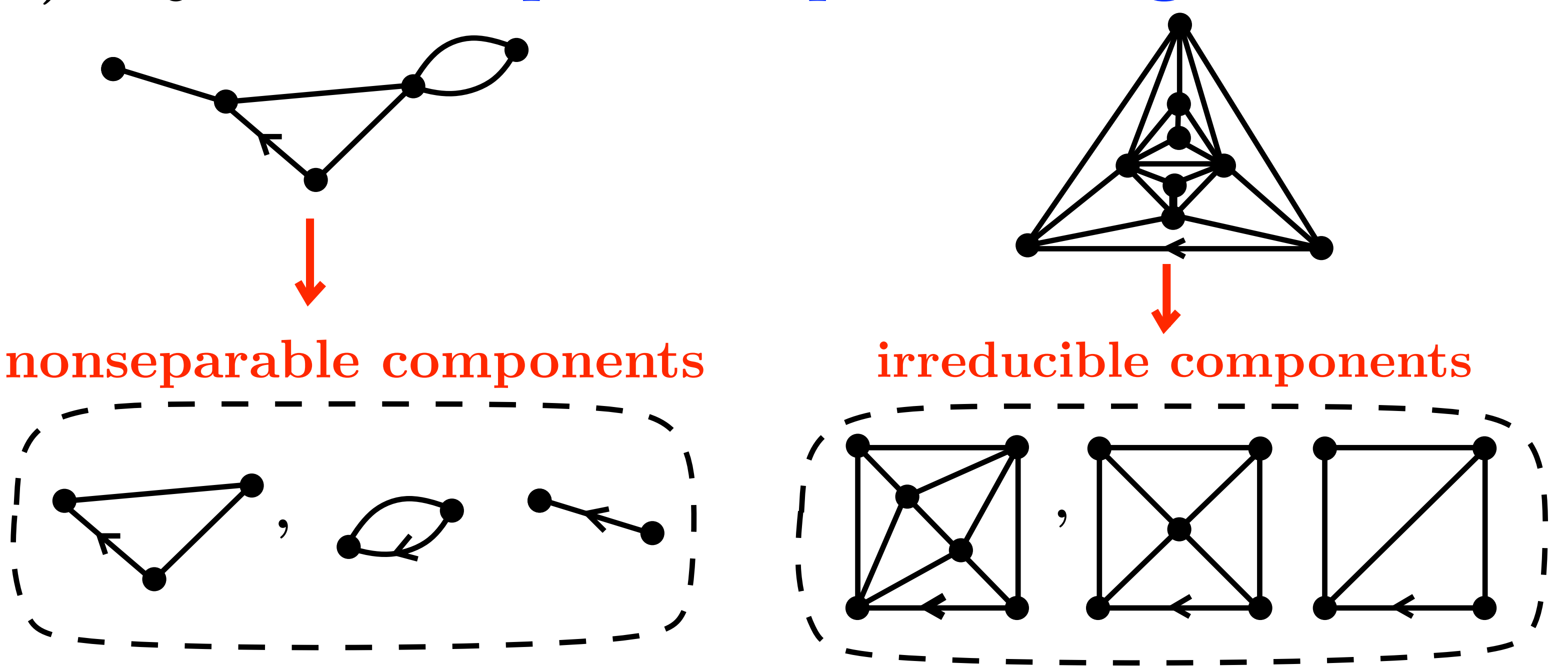
Eulerian maps n edges **Bicubic maps $2n$ vertices**

Overview of the talk

1) Bijection nonseparable maps $\simeq$ irreducible triang
+ new duality relation for bipolar orientations



2) Bijection loopless maps $\simeq$ triangulations

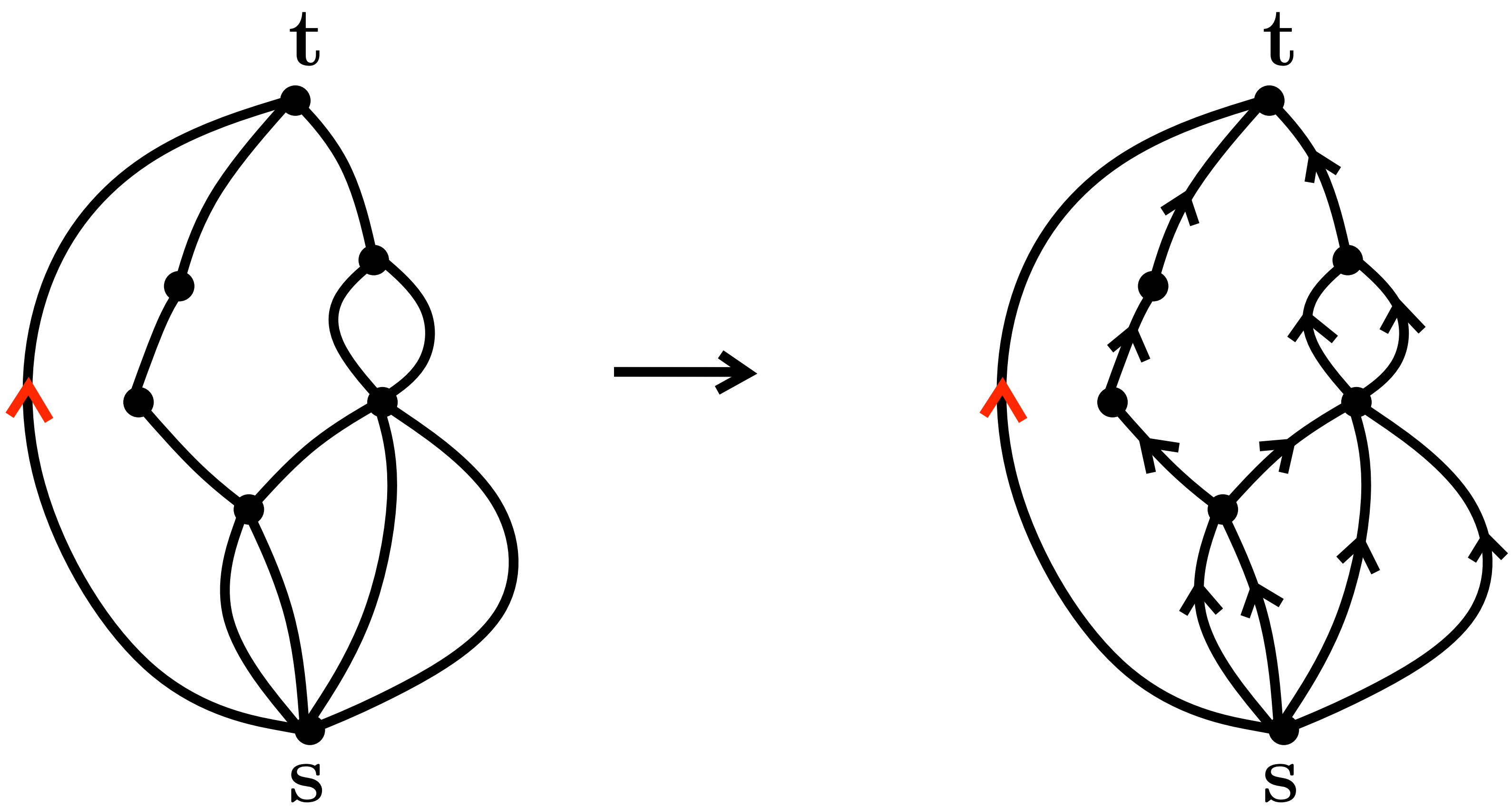


3) Applications to random generation and encoding

Bijection between nonseparable maps and irreducible triangulations

Bipolar orientations

Bipolar orientation = **acyclic orientation** with a unique source and a unique sink



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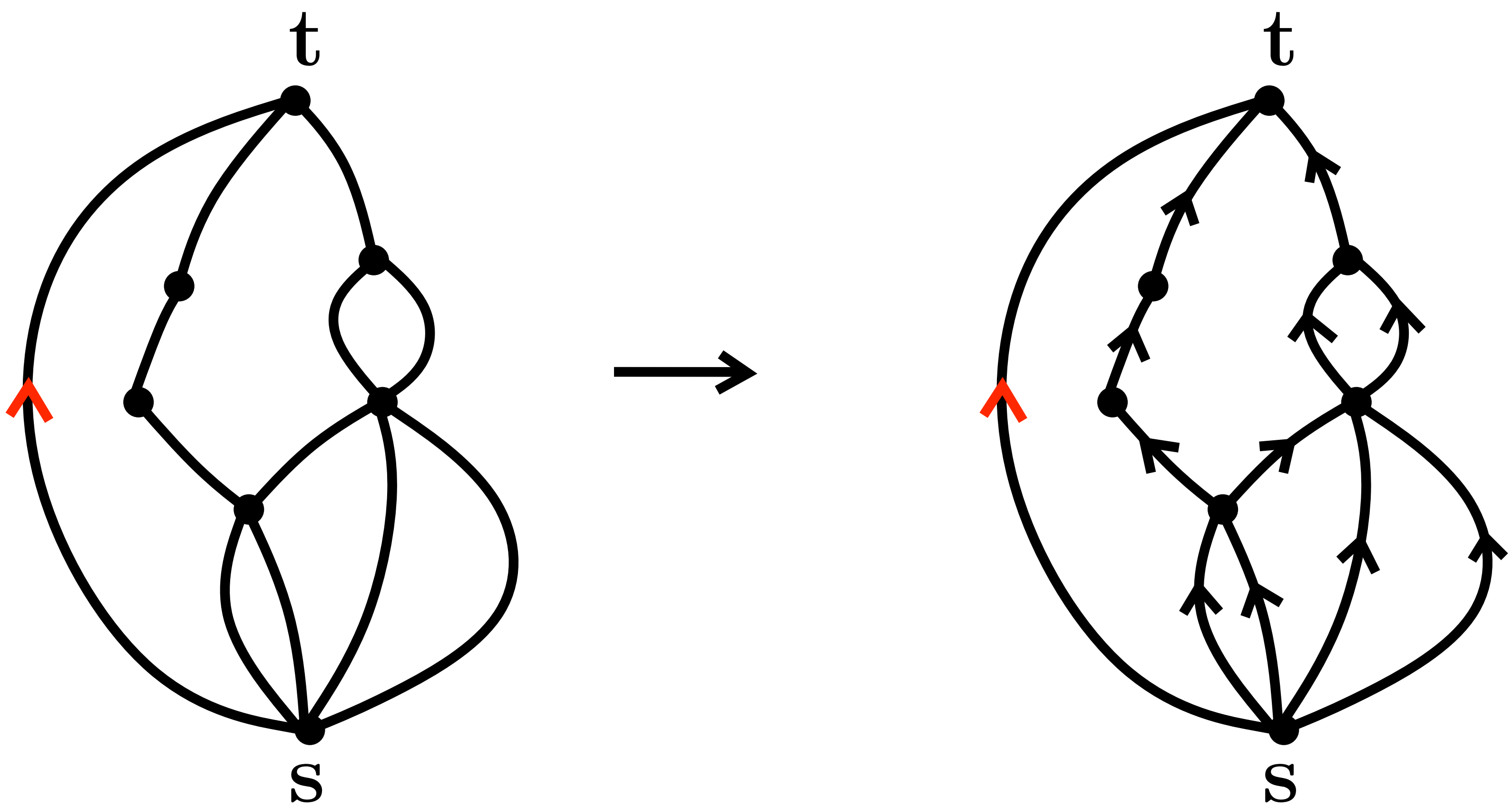
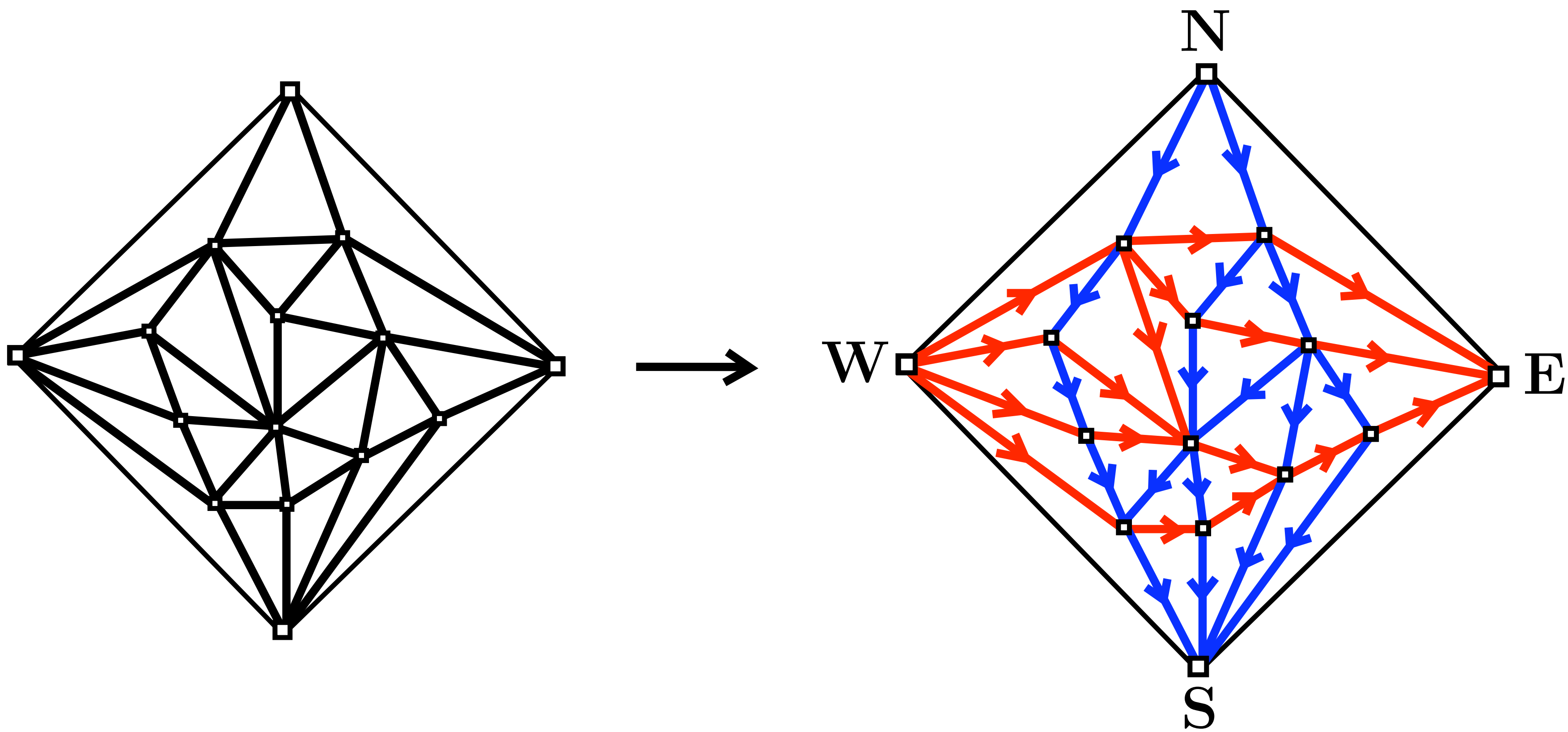


A map admits a bipolar orientation iff there is no separating vertex (nonseparable)

Transversal structures

Transversal structure = partition of inner edges into a red and a blue bipolar orientations that are transversal (introduced by Xin He'93)



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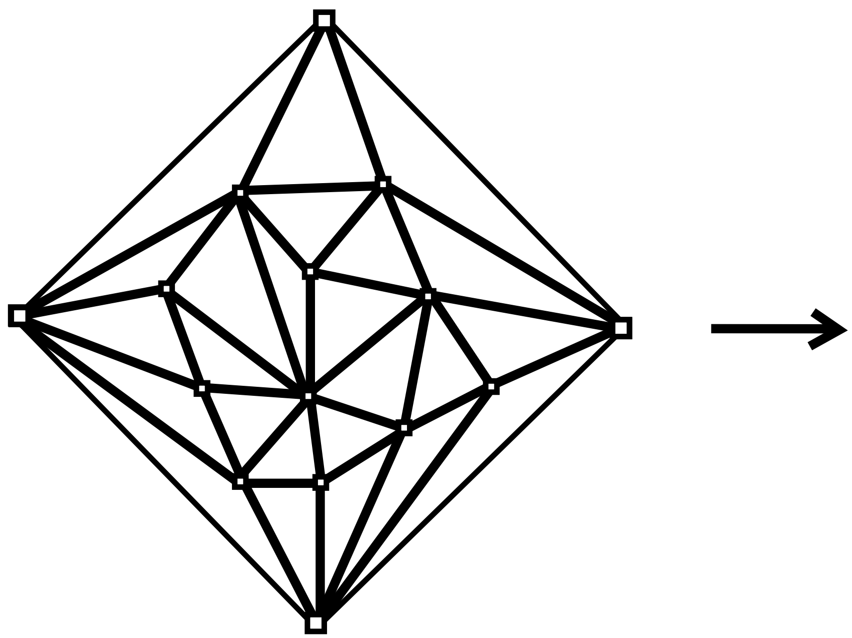
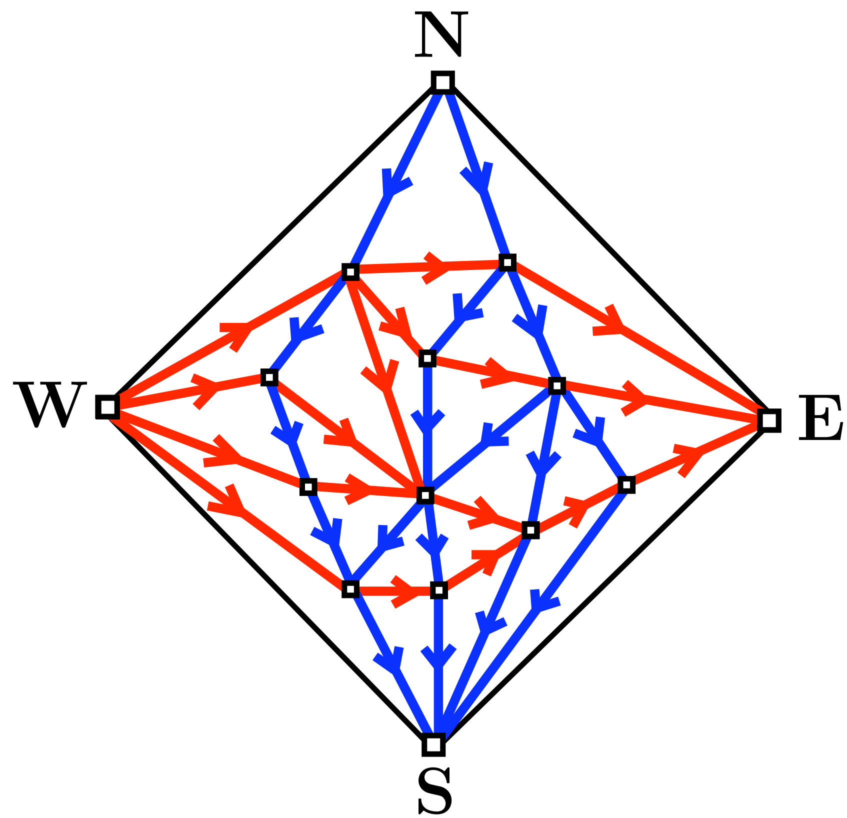
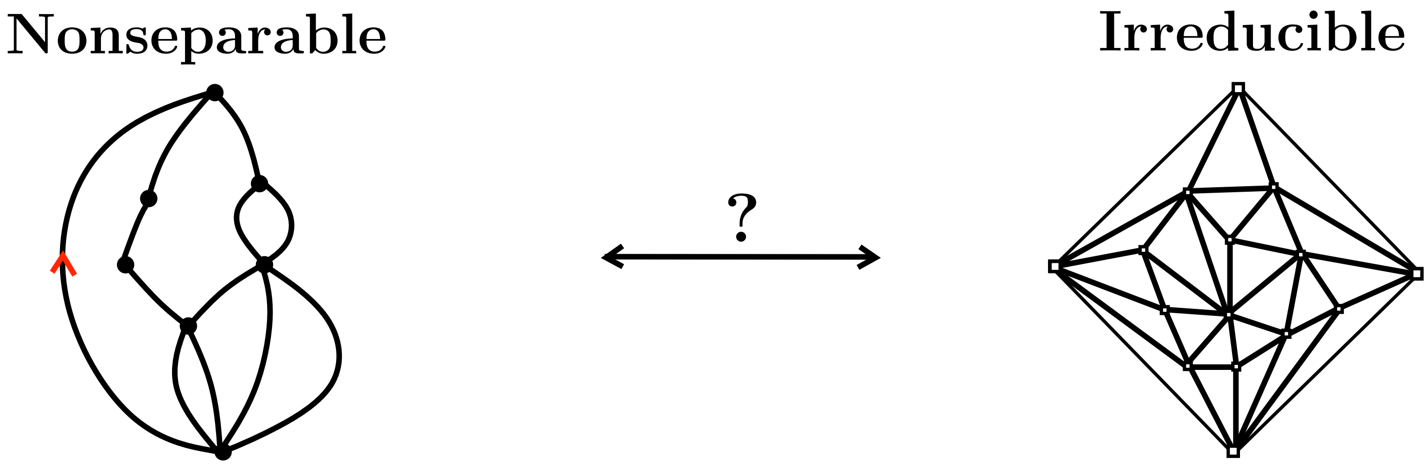


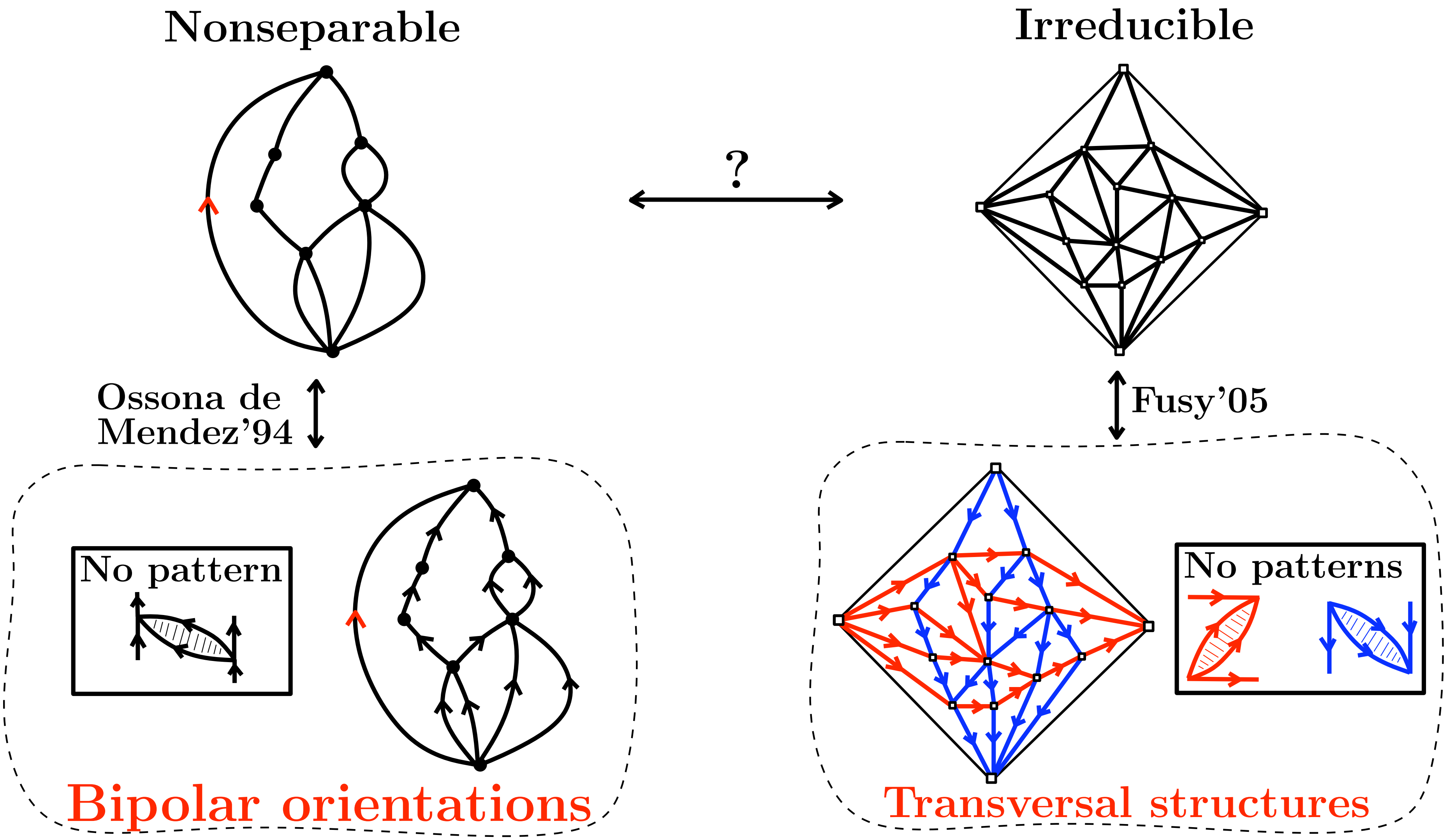


A triangulation of the 4-gon admits a transversal structure iff there is no separating triangle (irreducible)

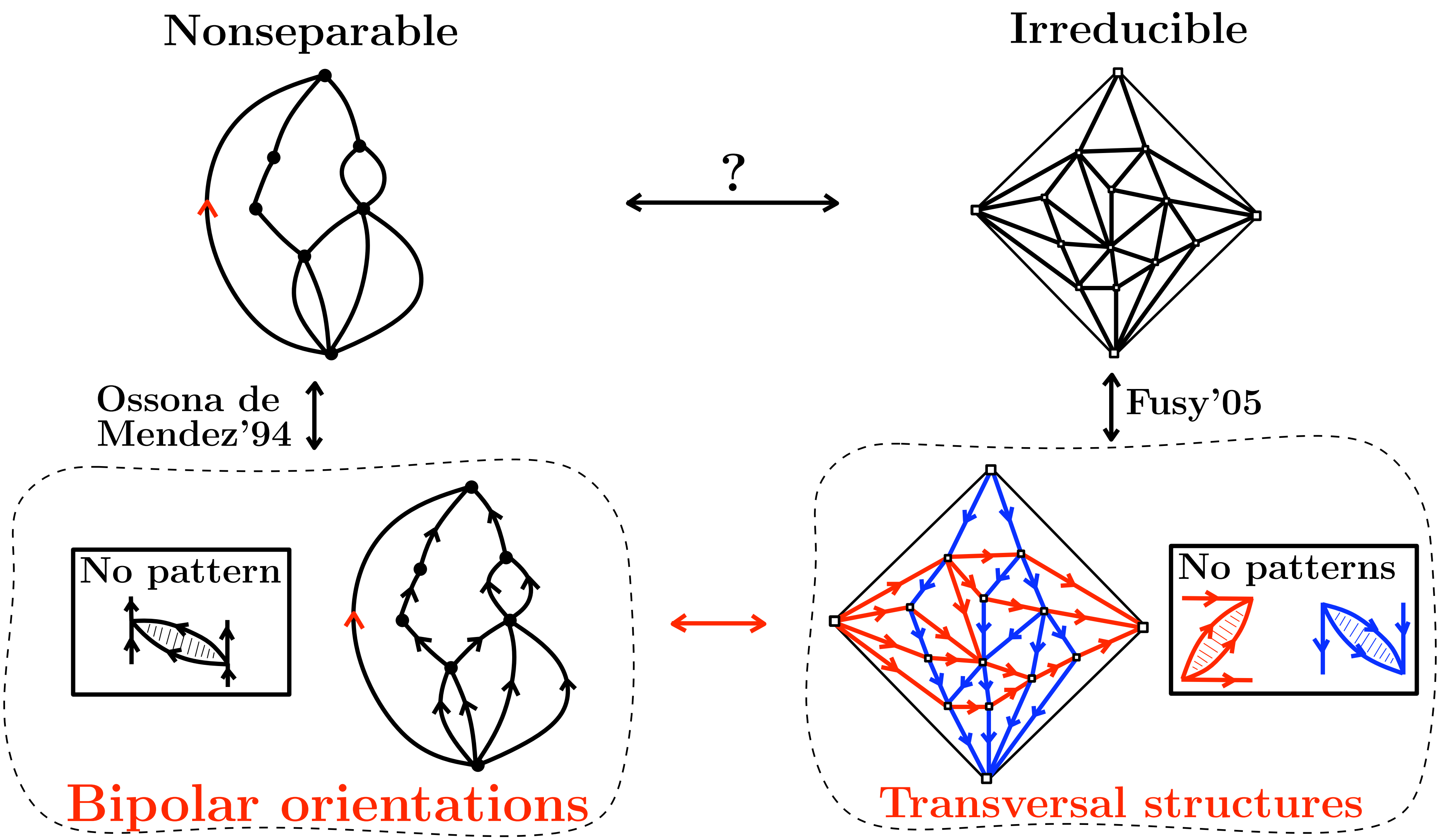
Reformulating the bijection



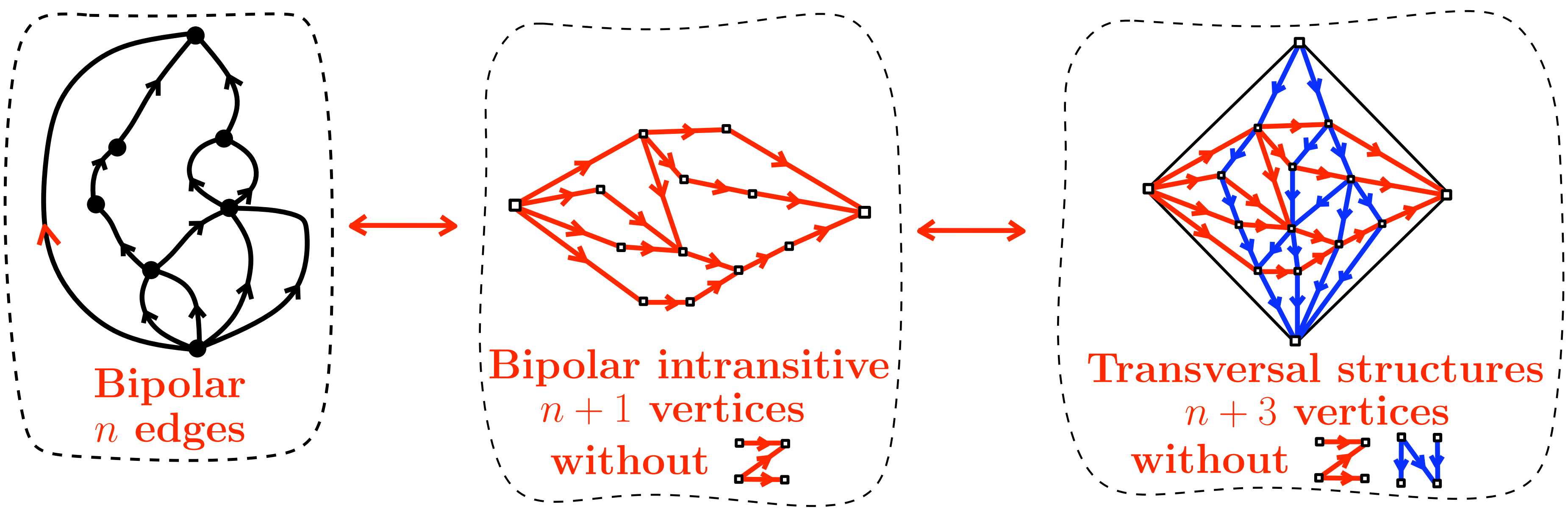
Reformulating the bijection



Reformulating the bijection



How the bijection works



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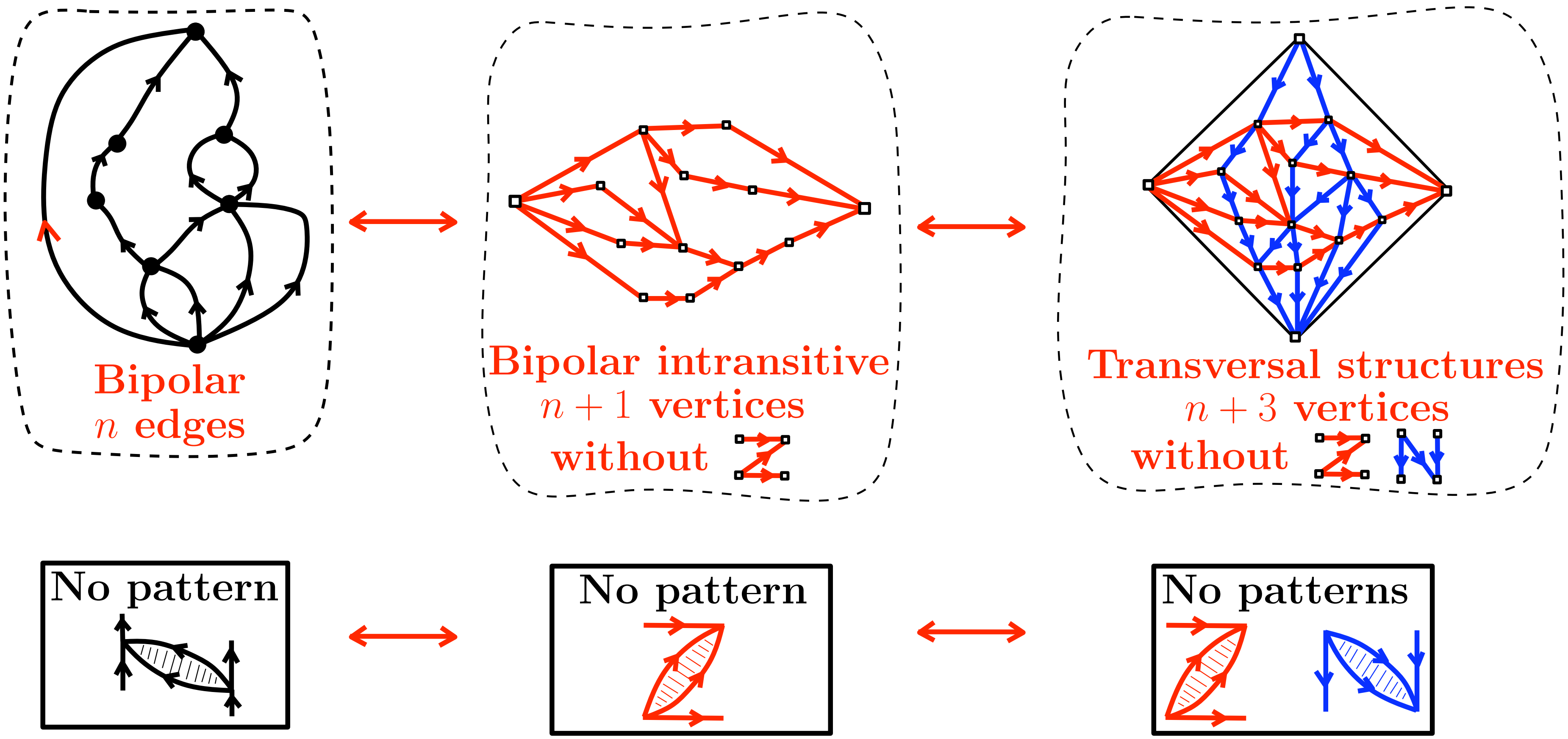
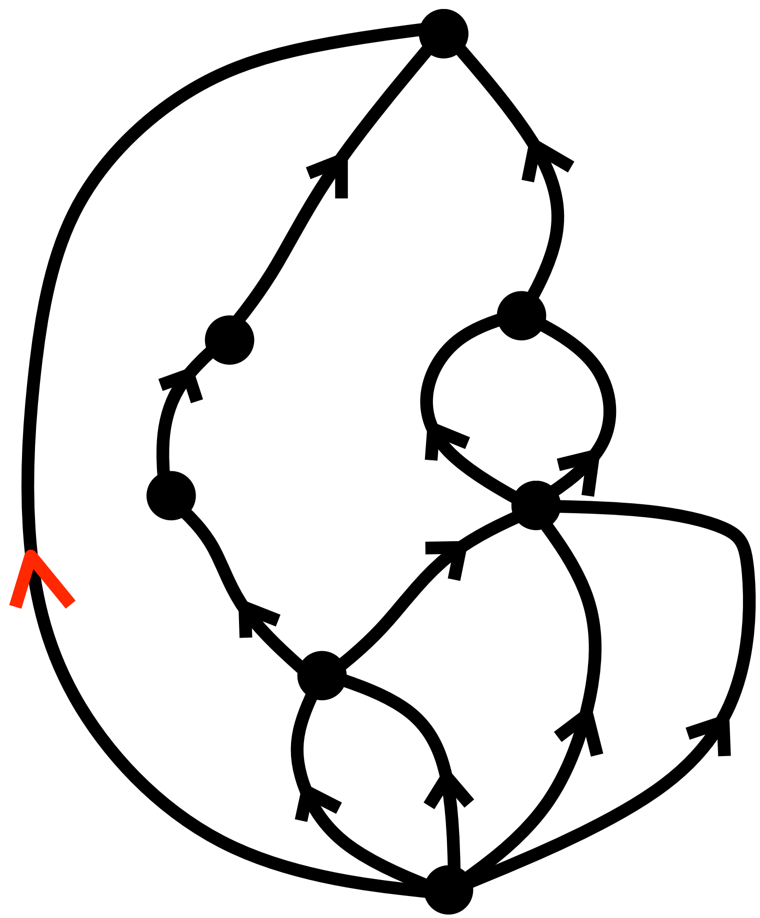


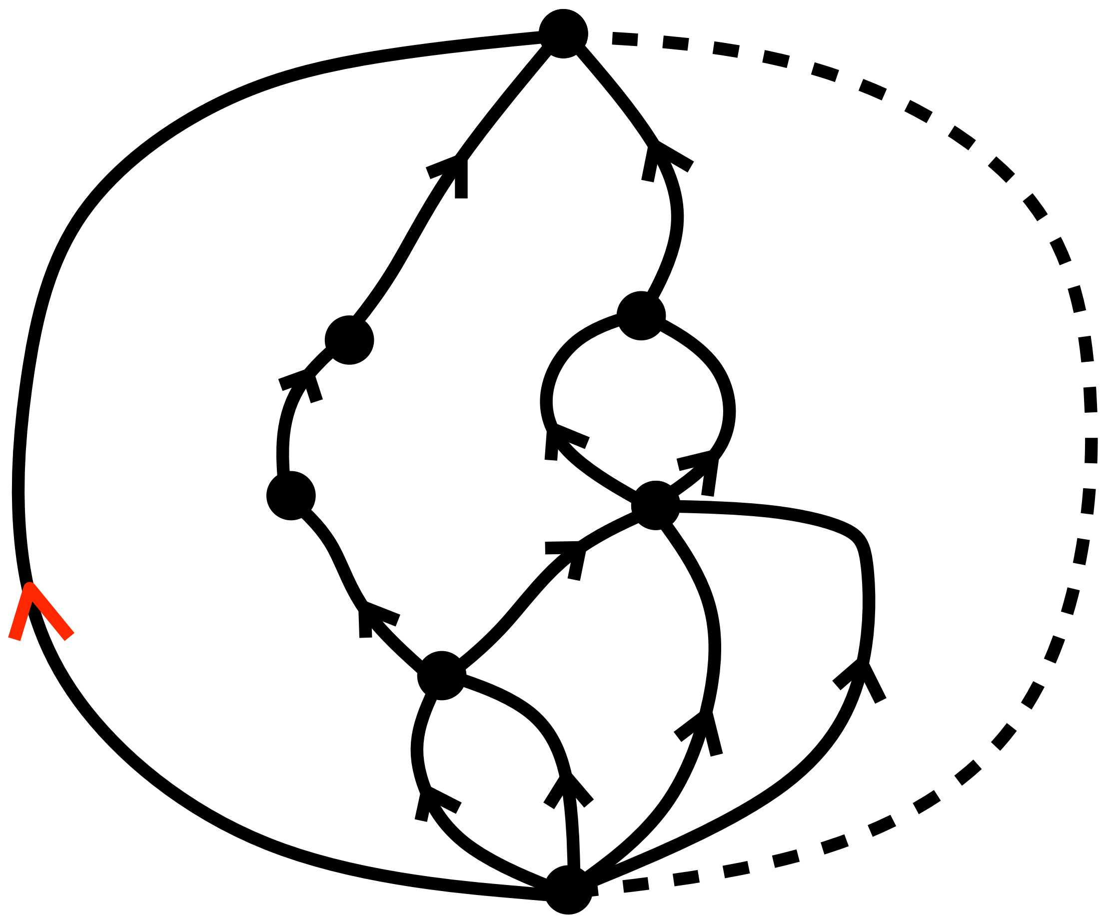
Bipolar→Bipolar intransitive

Start with a plane bipolar orientation



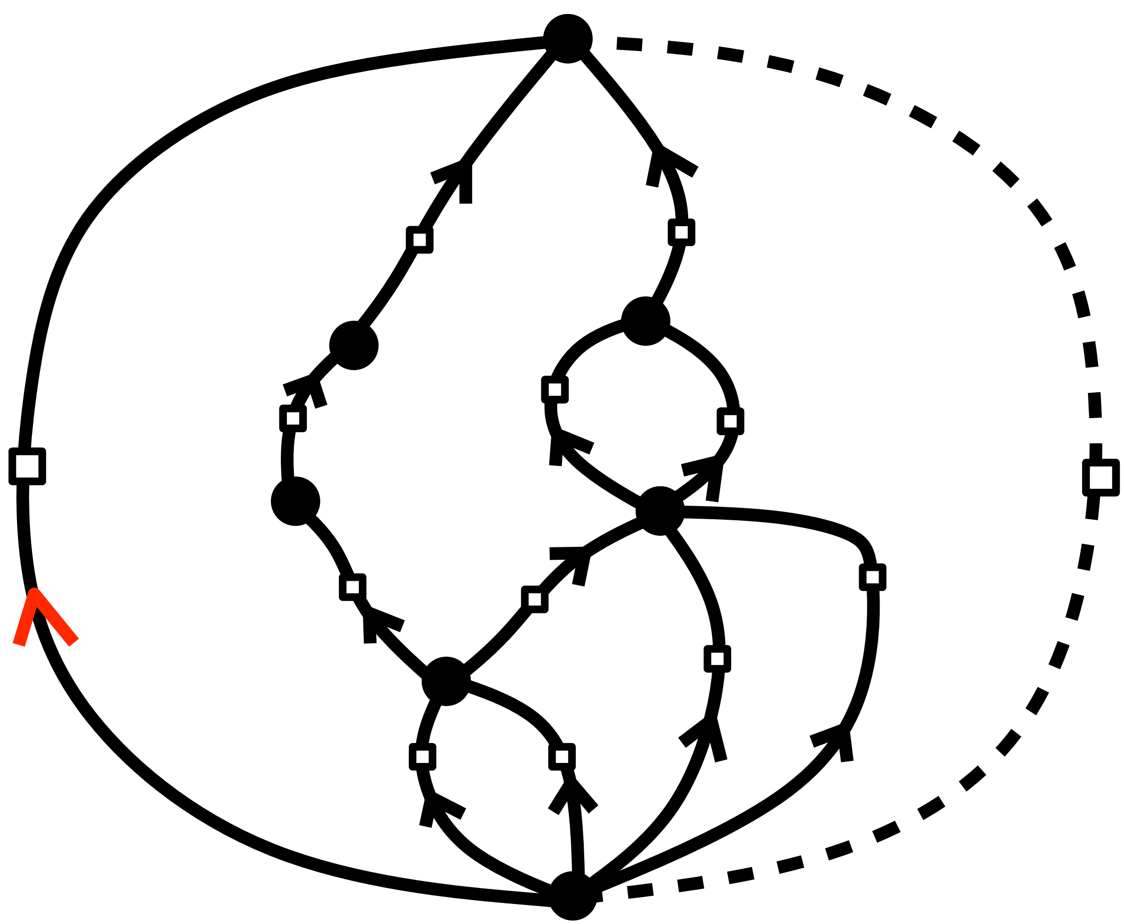
Bipolar→Bipolar intransitive

Double the root edge



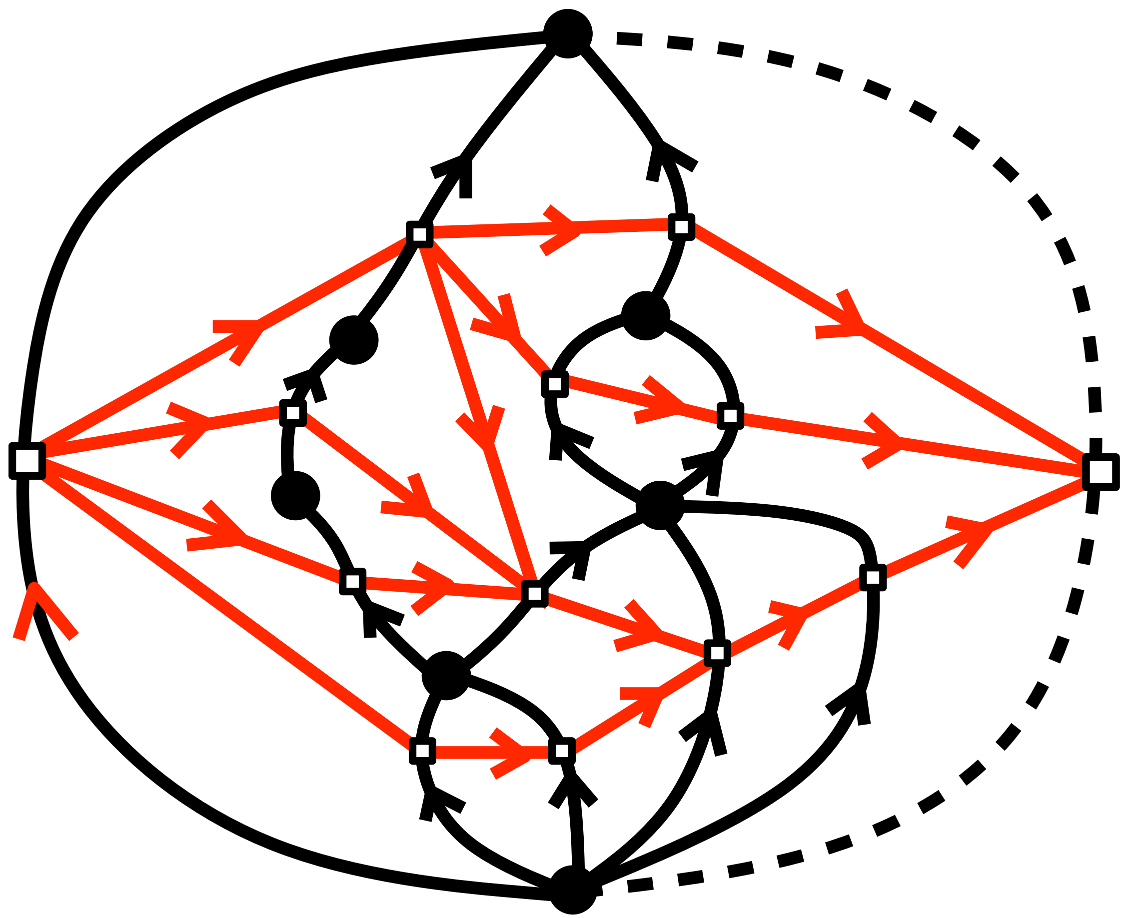
Bipolar→Bipolar intransitive

Insert a white vertex in each edge



Bipolar→Bipolar intransitive

Triangulate the faces by red edges



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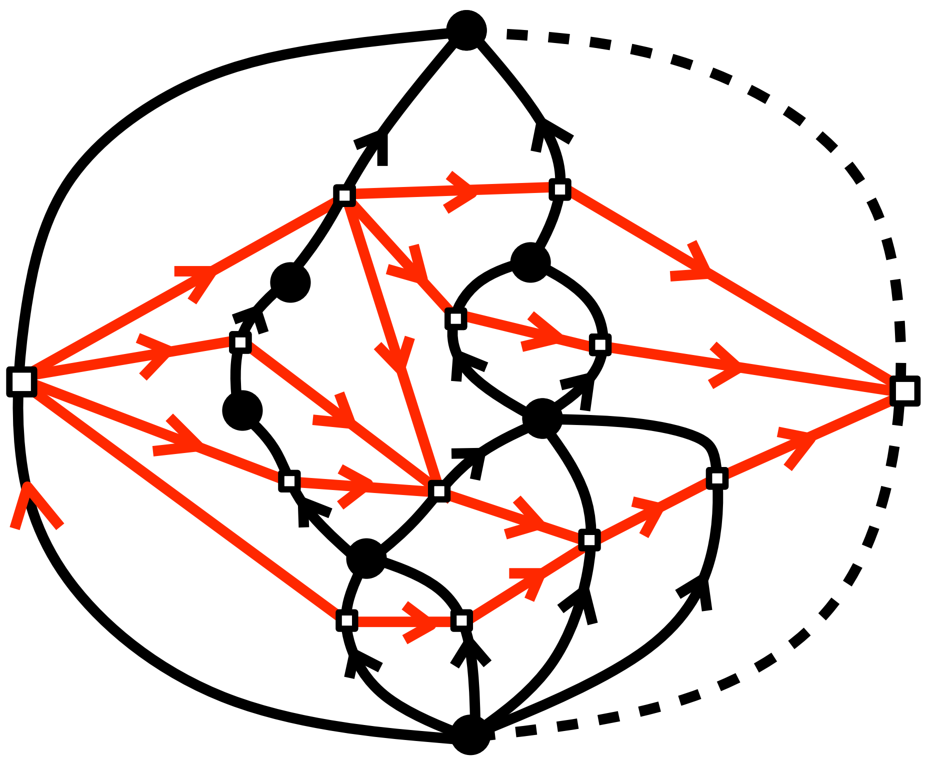

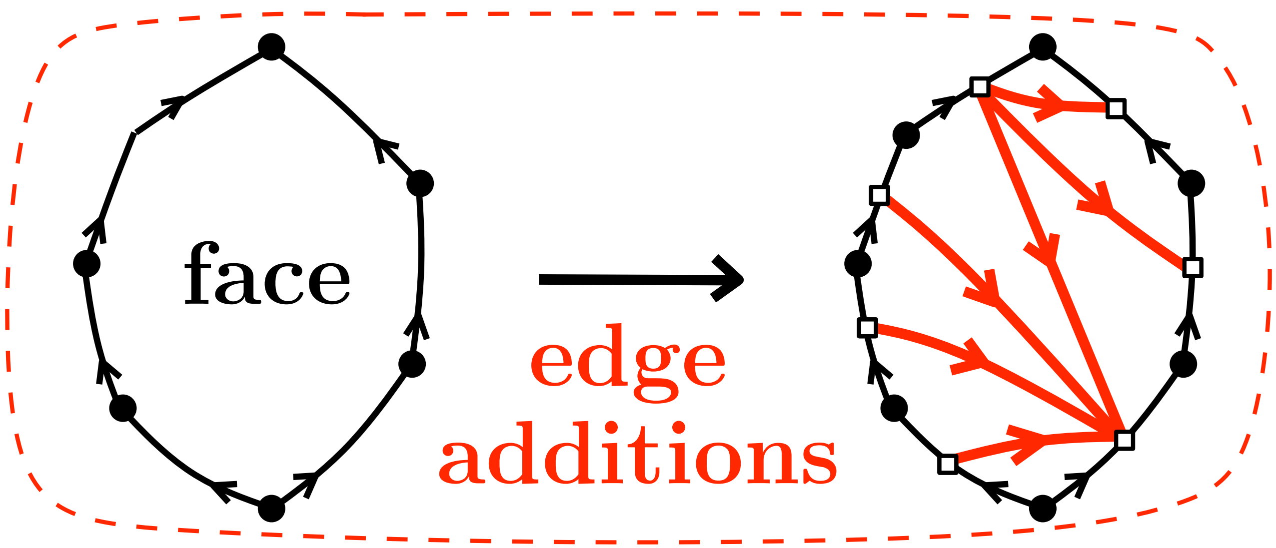

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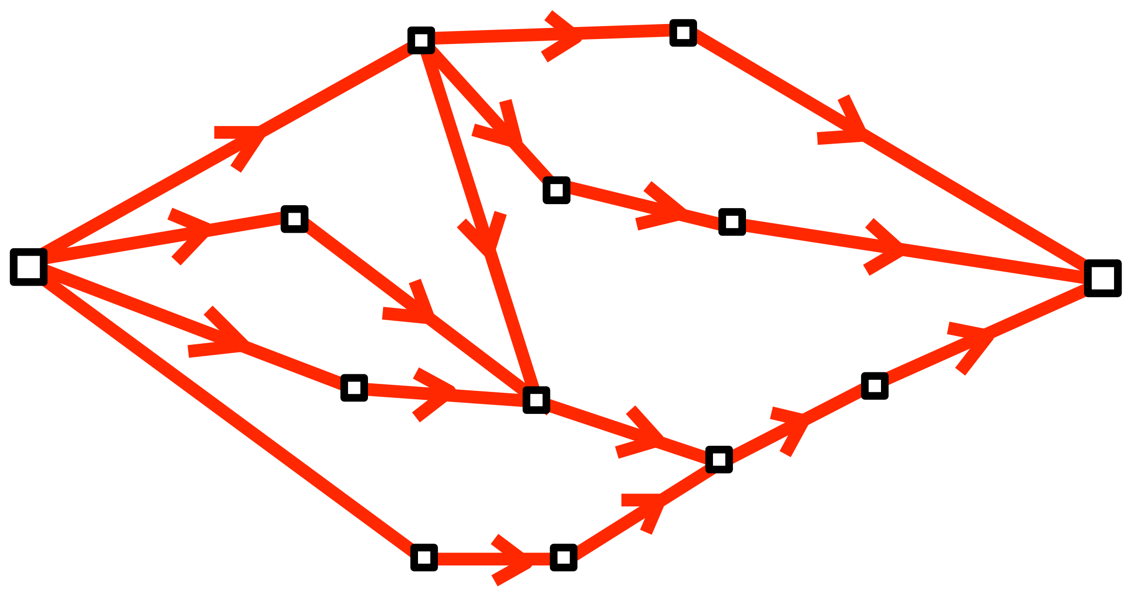
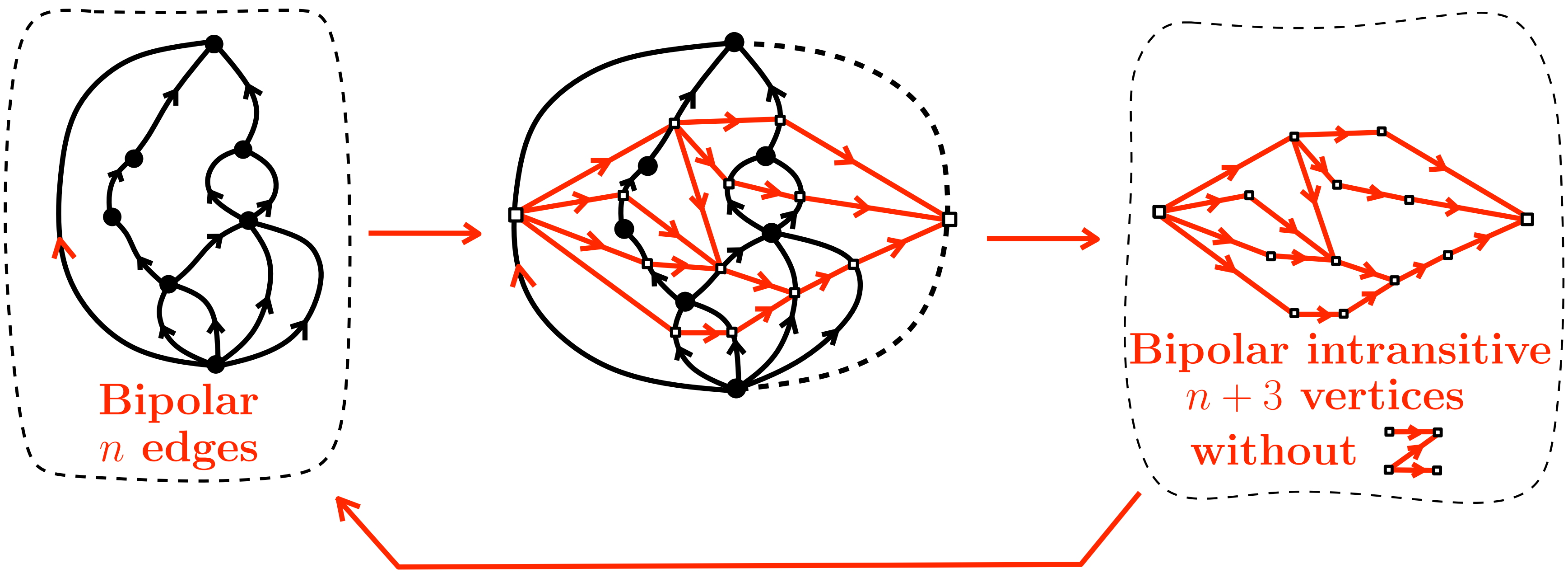
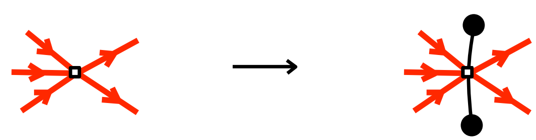


Bipolar⟶Bipolar intransitive

A first bijection:



1) place a black vertex in each face
2) insert one black edge for each white vertex



Bipolar→Bipolar intransitive

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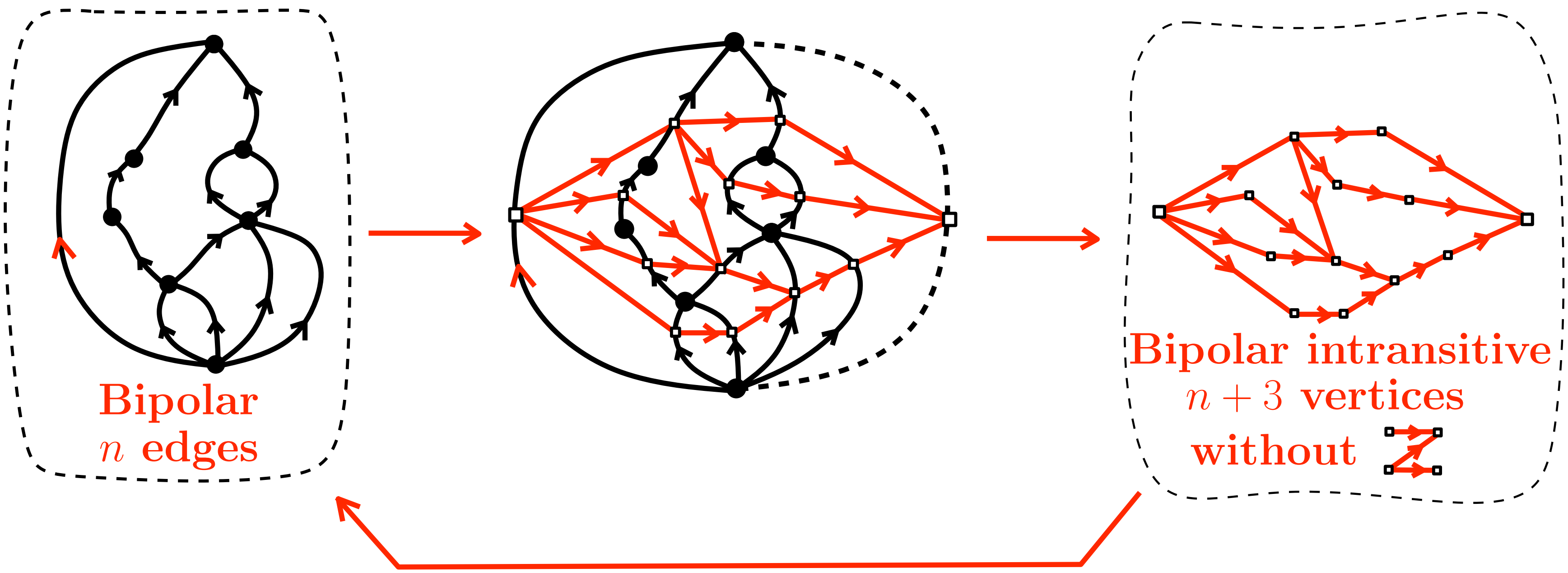

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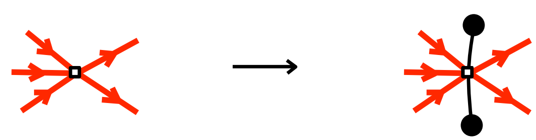
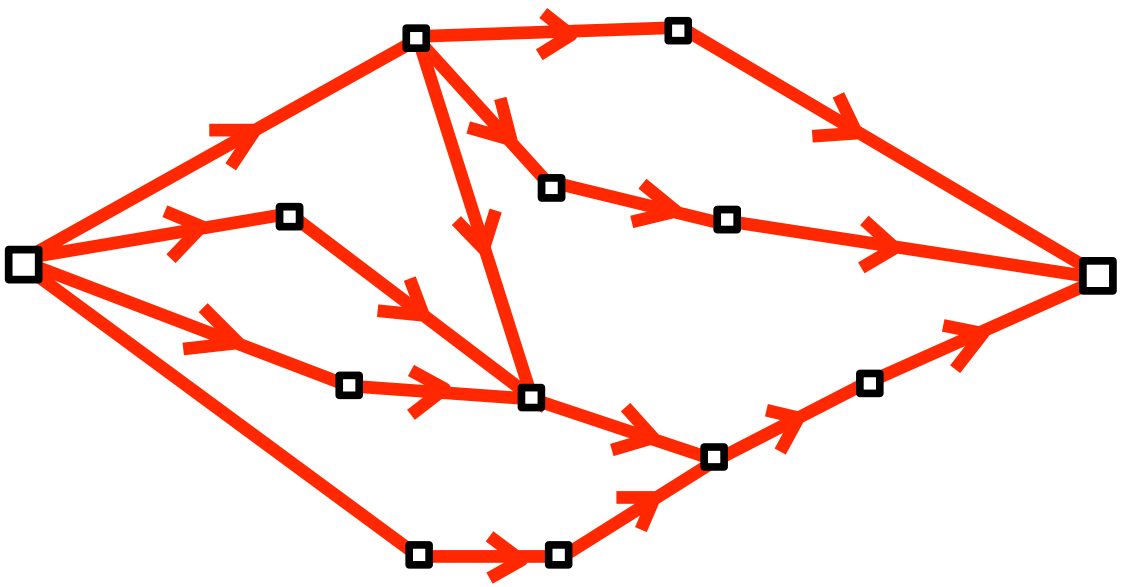


Remark: These are counted by the Baxter number: $B_n = \frac{2}{n(n+1)^2}\sum_{k=0}^{n-1}\binom{n+1}{k}\binom{n+1}{k+1}\binom{n+1}{k+2}$

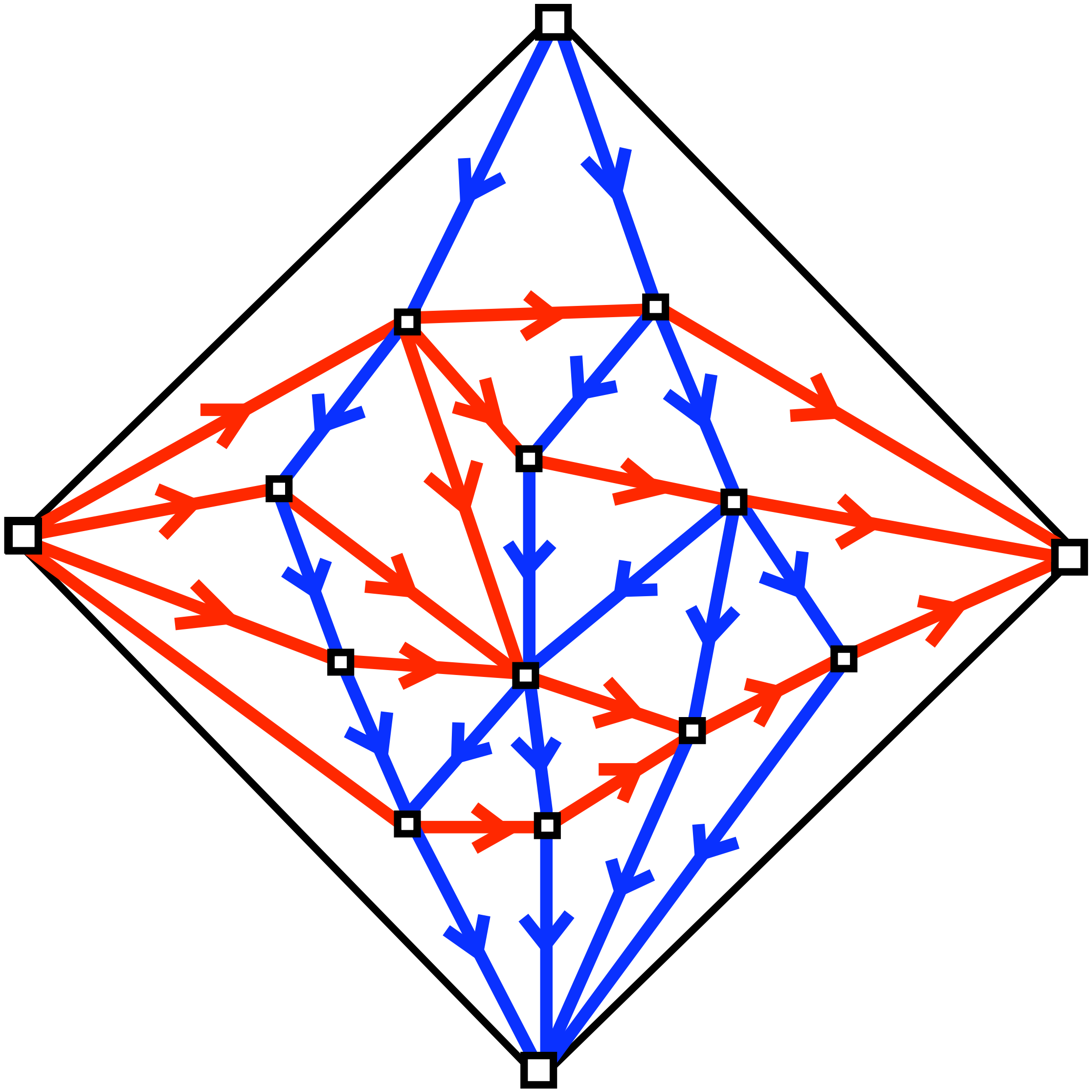
Bipolar intransitive→Trans. struct.

Start with an intransitive bipolar orientation

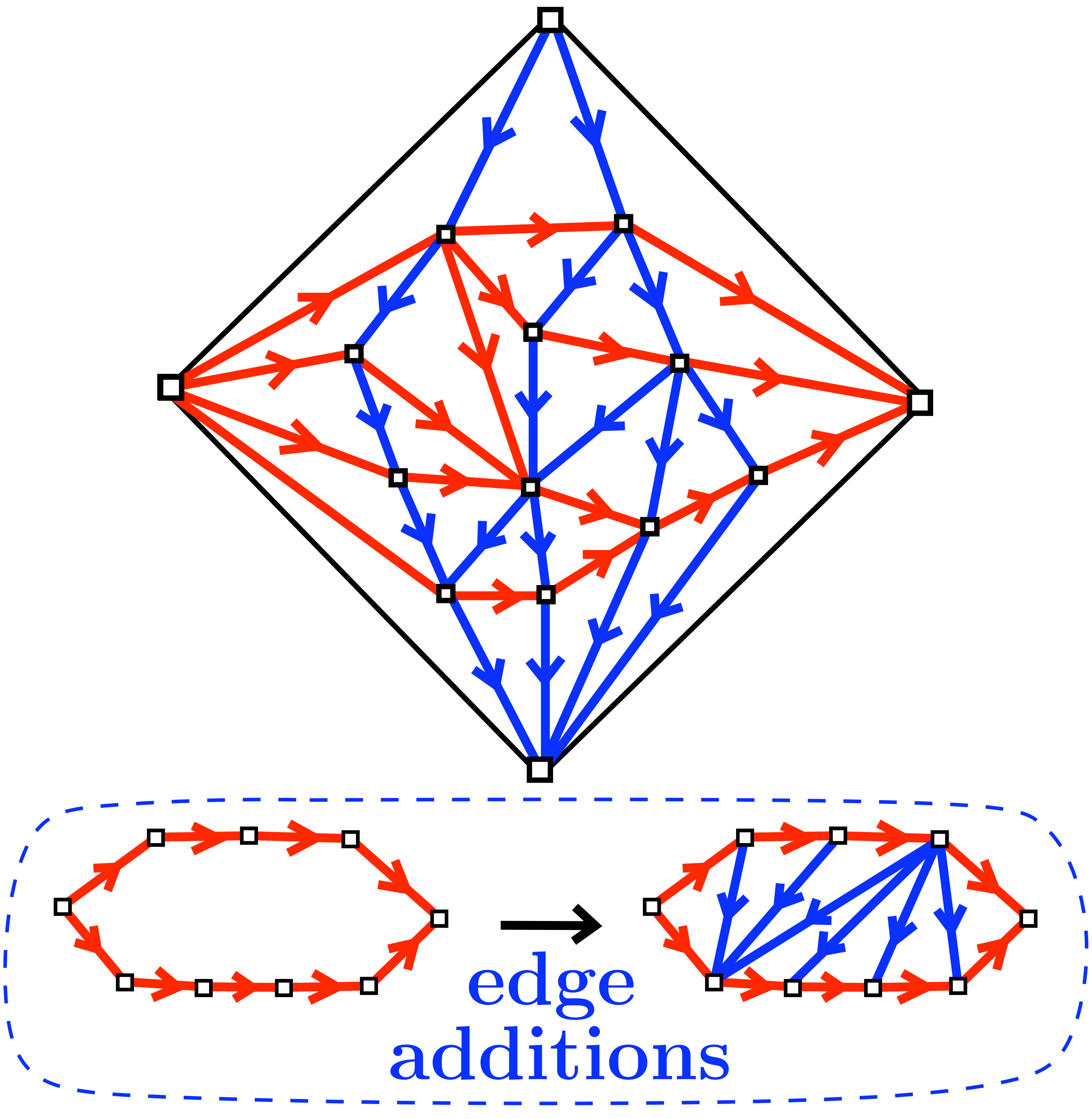


Bipolar intransitive⟶Trans. struct.

Triangulate the faces by blue edges

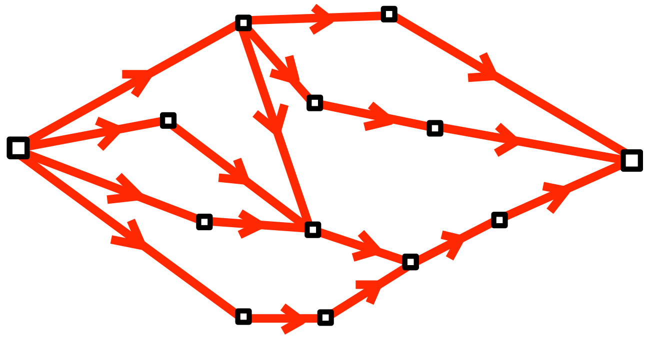


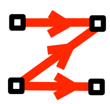
Bipolar intransitive→Trans. struct.

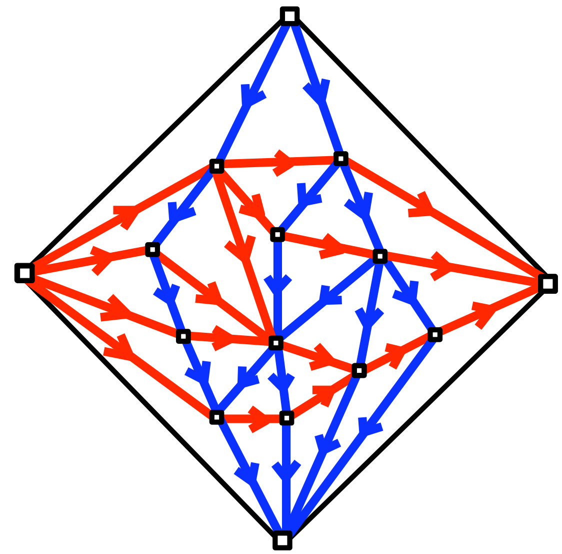


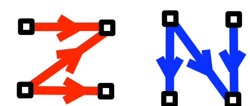
Bipolar intransitive$\longrightarrow$Trans. struct.

A second bijection:

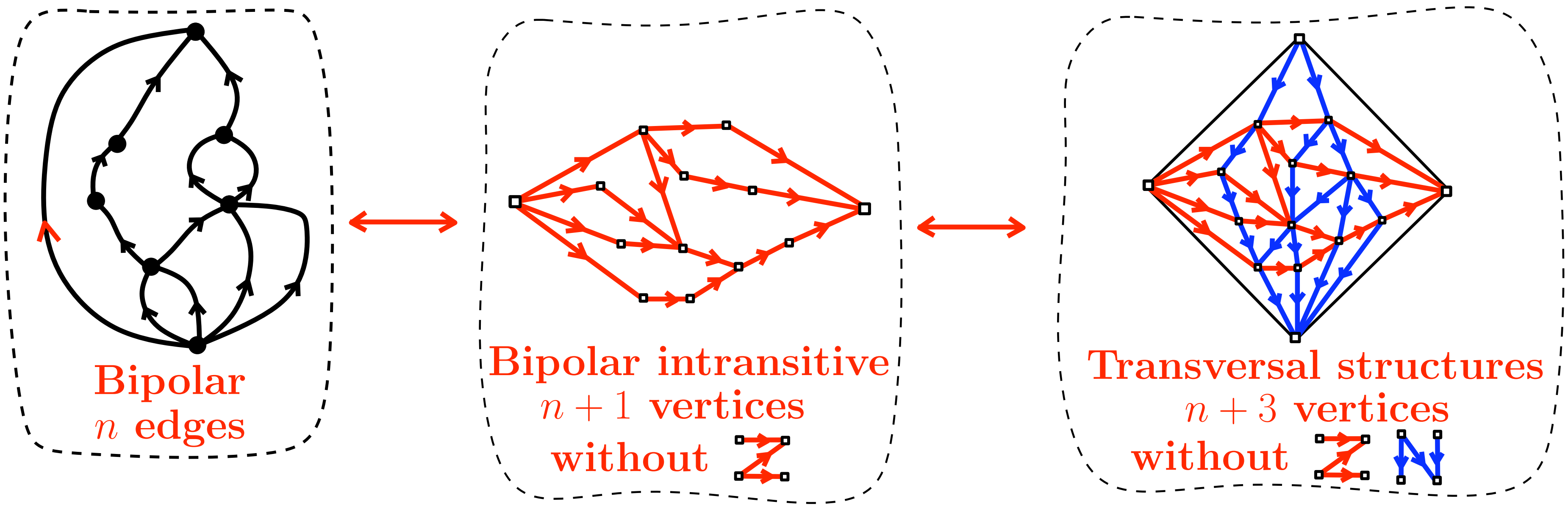


Bipolar intransitive
$n+1$ vertices
without 



Transversal structures
$n+3$ vertices
without 

The bijection



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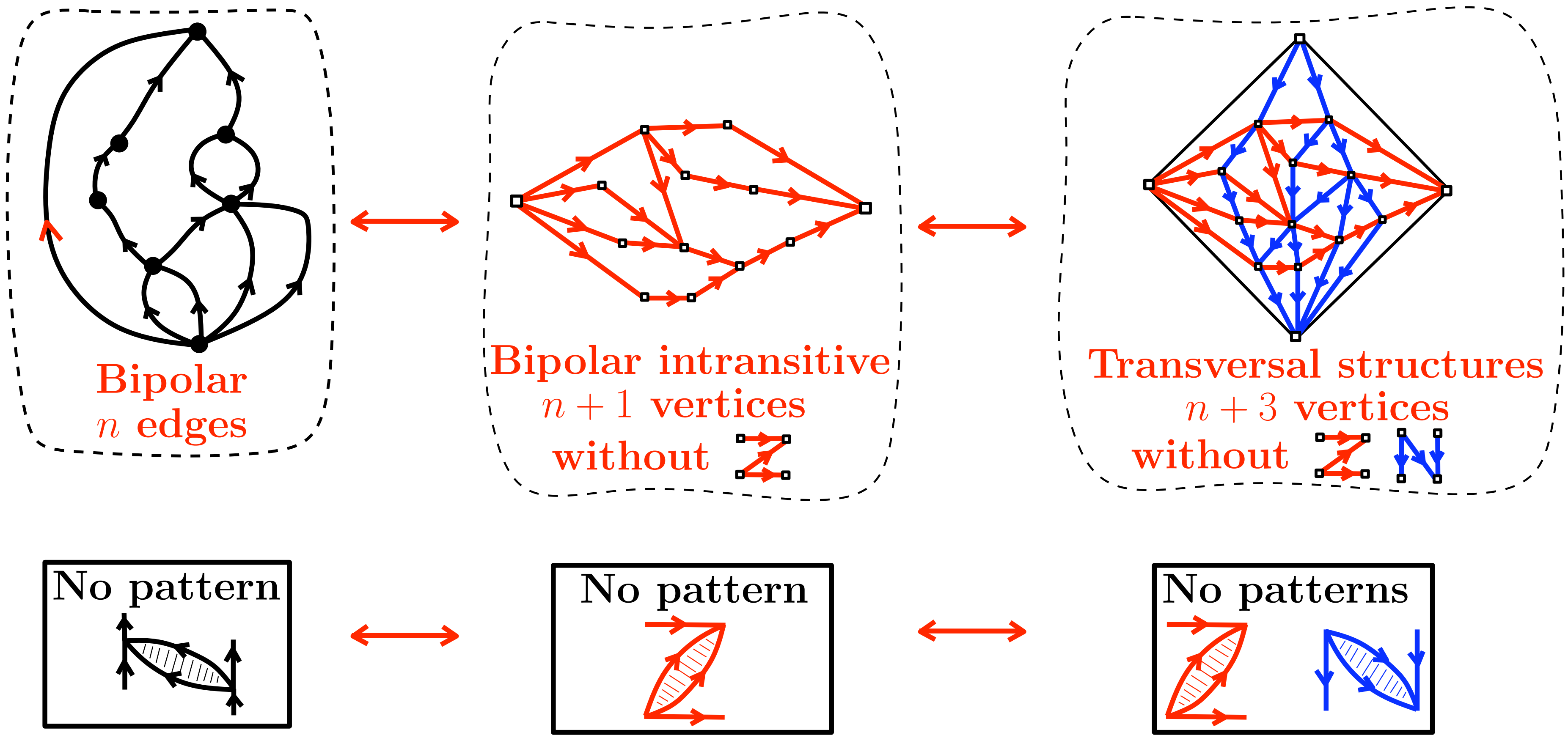

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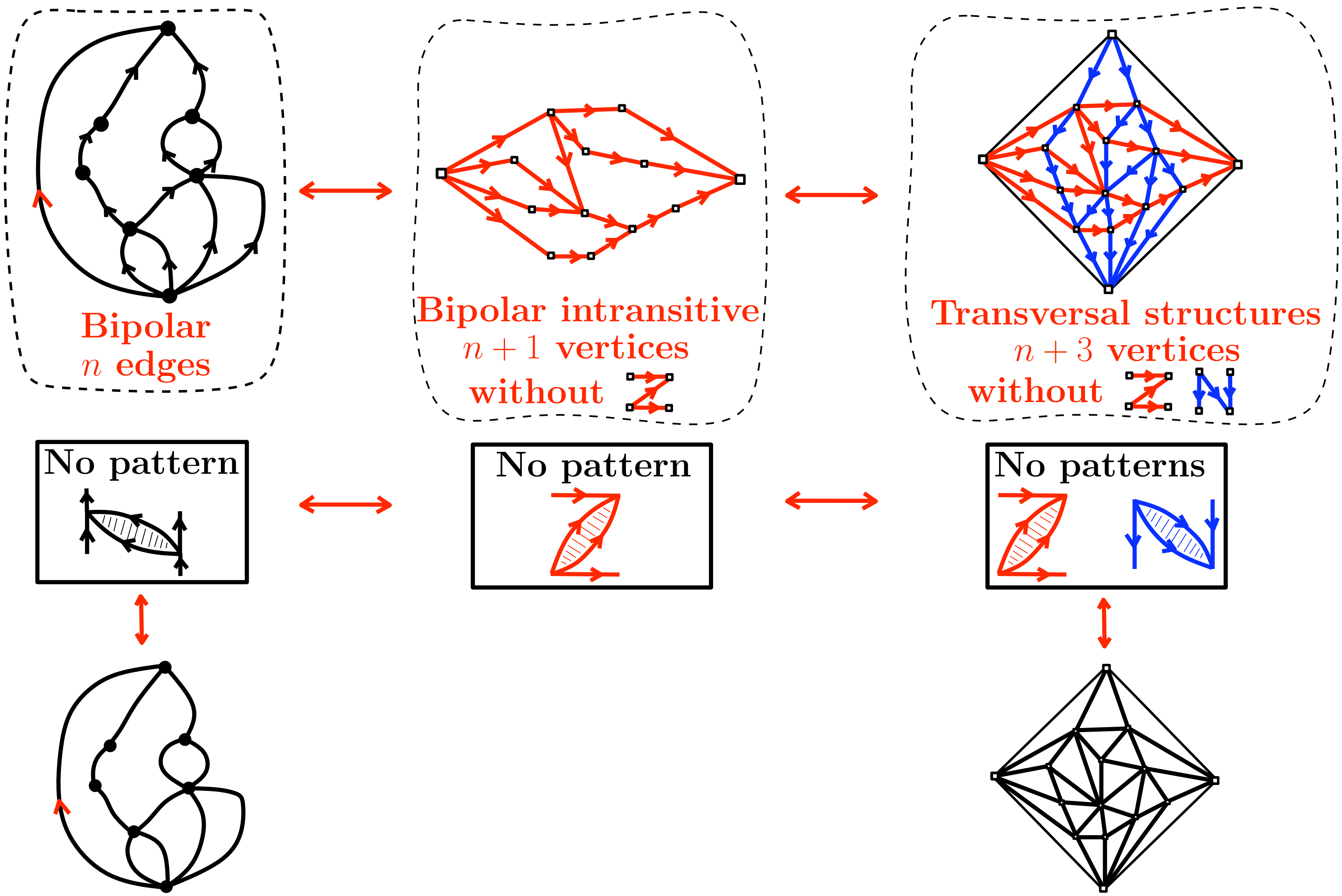

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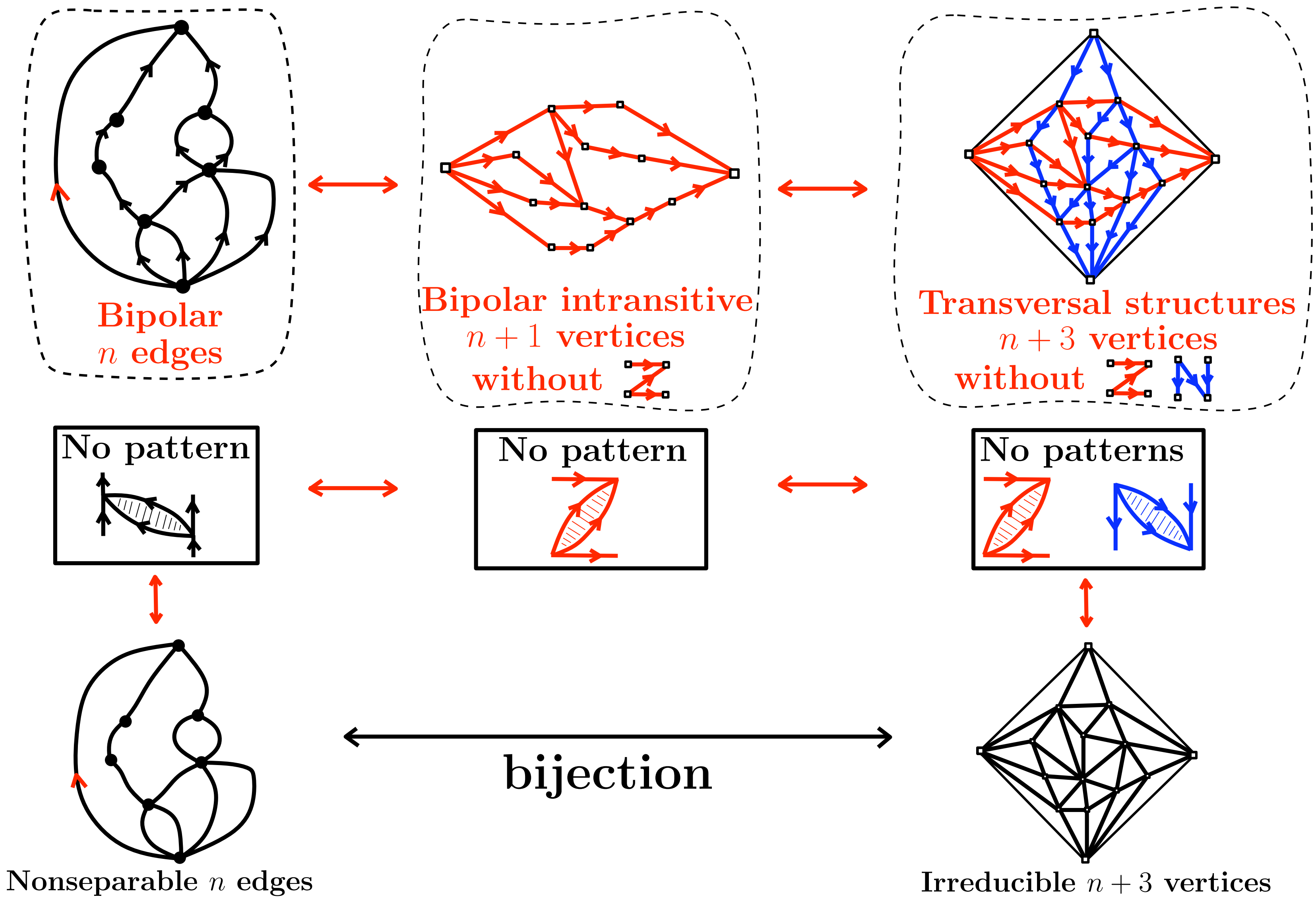
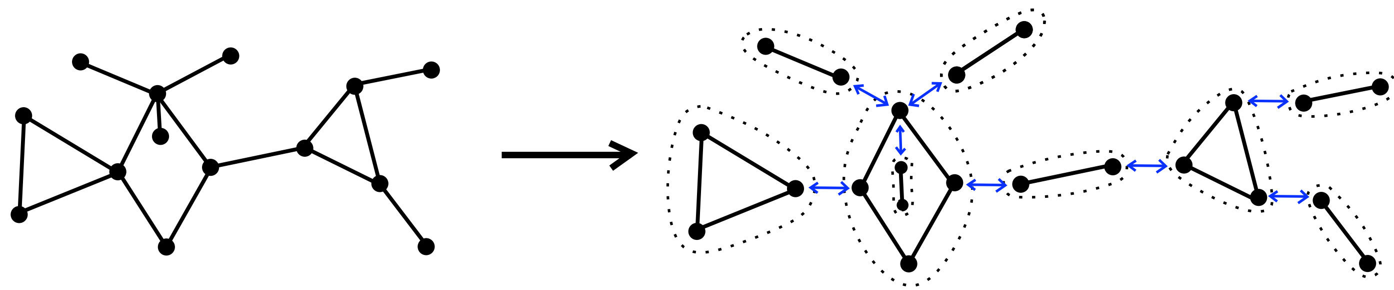


Bijection between loopless maps and triangulations

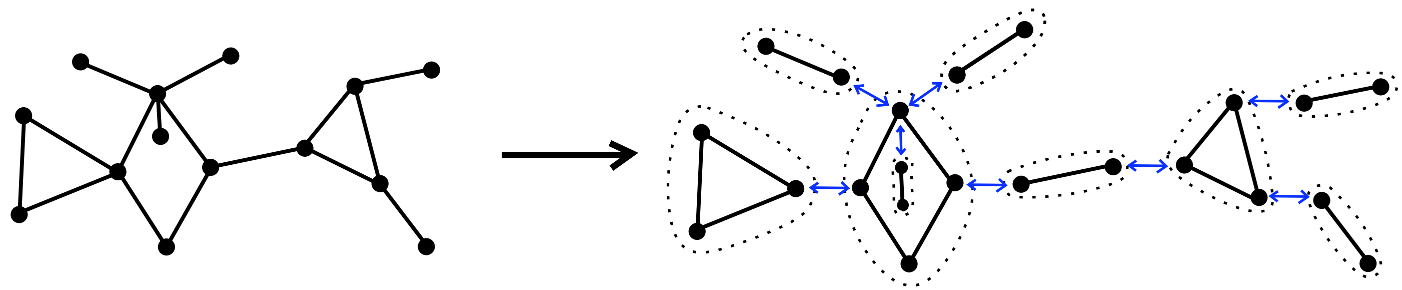
Decomposing a loopless map

- **Block decomposition**

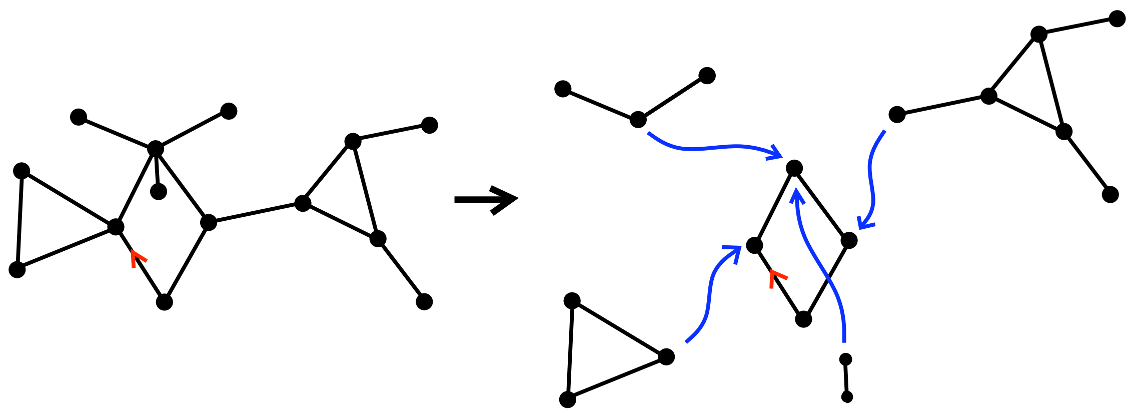


Decomposing a loopless map

- **Block decomposition**



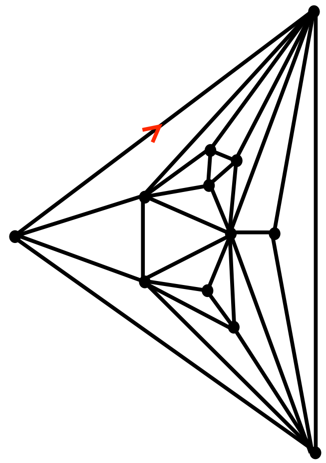
- **For rooted loopless maps:**



Nonseparable core where each corner is possibly occupied by a loopless map

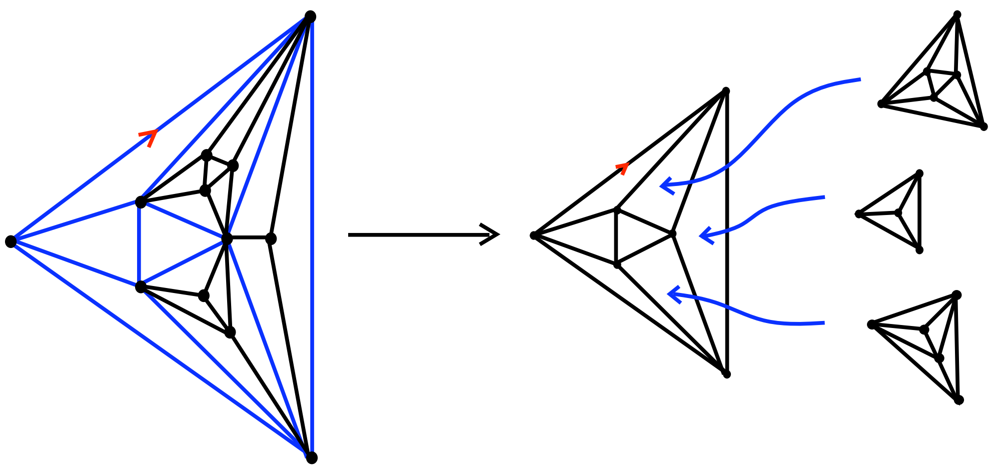
Decomposing a triangulation

- **Classical decomposition at separating triangles**



Decomposing a triangulation

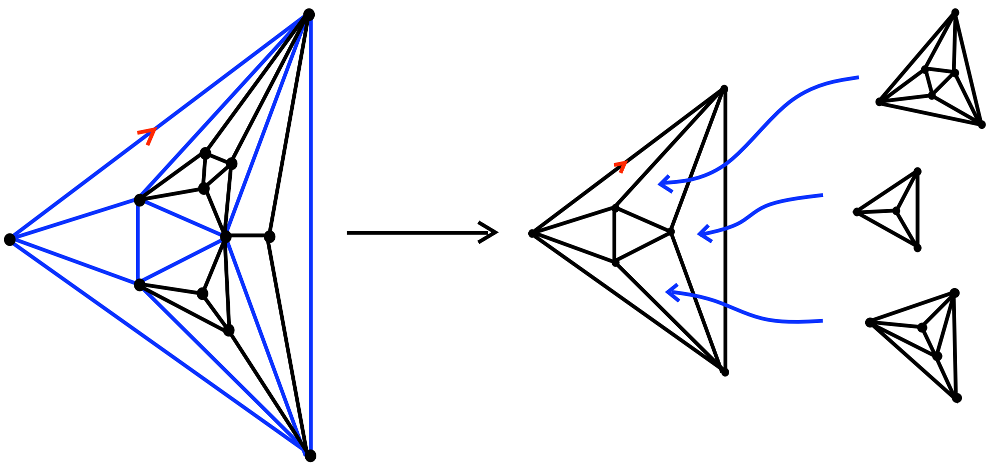
- **Classical decomposition at separating triangles**



4-connected triangulation where each face is possibly occupied by a triangulation

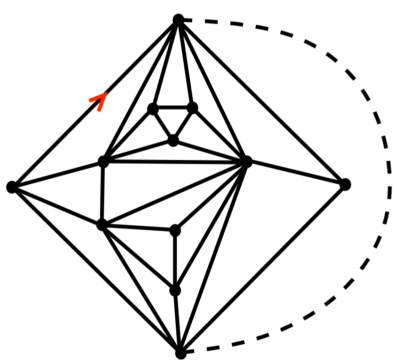
Decomposing a triangulation

- Classical decomposition at separating triangles



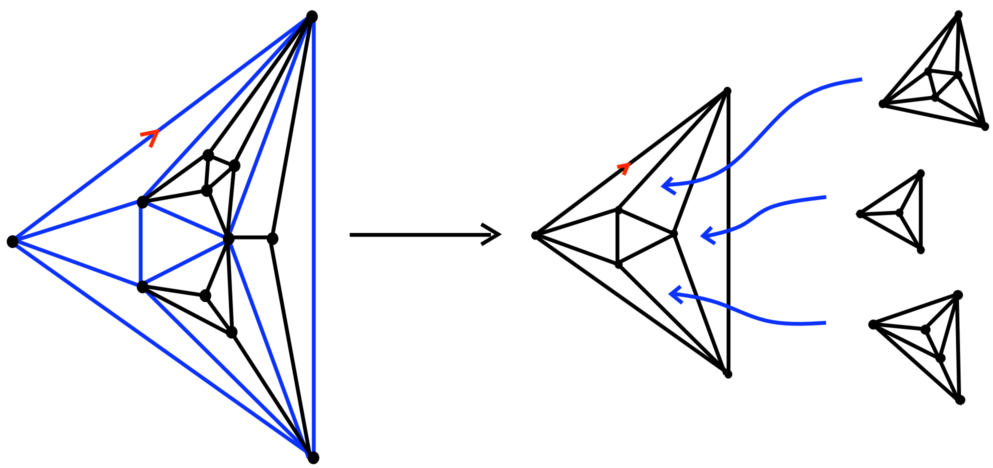
4-connected triangulation
where each face is
possibly occupied
by a triangulation

- Here: the same after deleting an outer edge



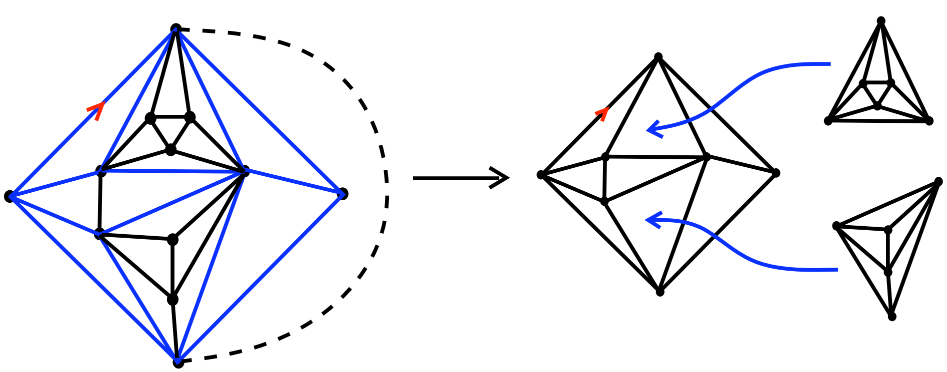
Decomposing a triangulation

- **Classical decomposition at separating triangles**



4-connected triangulation where each face is possibly occupied by a triangulation

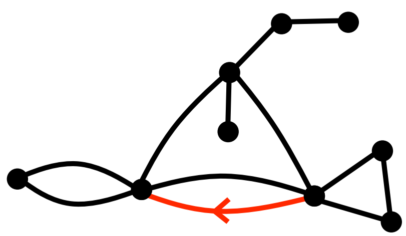
- **Here: the same after deleting an outer edge**



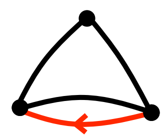
Irreducible triangulation where each face is possibly occupied by a triangulation

The decompositions are parallel

A loopless map M ($|M| = \#$ edges) is
i) the vertex-map •
or
ii) $|\mathbf{M}| \geq 1$

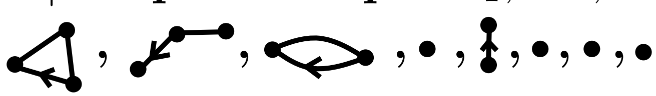


=

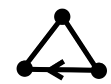
Nonseparable core C 

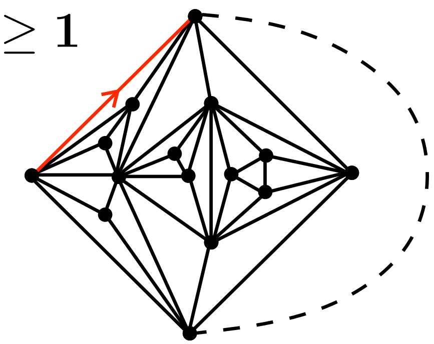
+

$2|C|$ loopless maps $M_1, \ldots, M_{2|C|}$

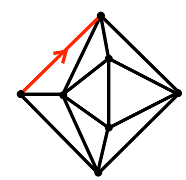


$$|\mathbf{M}| = |\mathbf{C}| + \sum_{\mathbf{i=1}}^{\mathbf{2|C|}} |\mathbf{M_i}|$$

A triangulation T ($|T| = \#$ vert. -3) is
i) the triangle-map 
or
ii) $|\mathbf{T}| \geq 1$

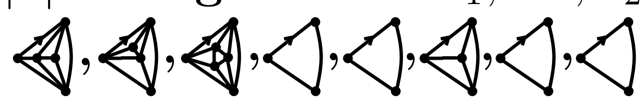


=

Irreducible core I 

+

$2|I|$ triangulations $T_1, \ldots, T_{2|I|}$

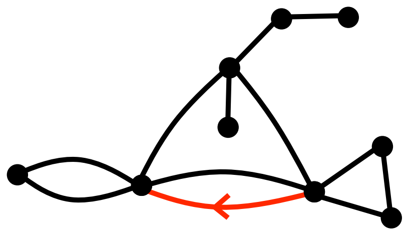


$$|\mathbf{T}| = |\mathbf{I}| + \sum_{\mathbf{i=1}}^{\mathbf{2|I|}} |\mathbf{T_i}|$$

The decompositions are parallel

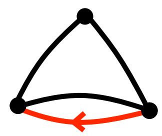
A **loopless map** M **($|M| = \#$ edges)** is
- i) the vertex-map $\bullet$

  or
- ii) $|\mathbf{M}| \geq \mathbf{1}$



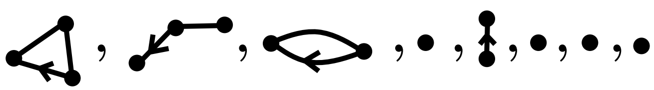
$=$

Nonseparable core C

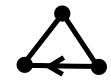


+

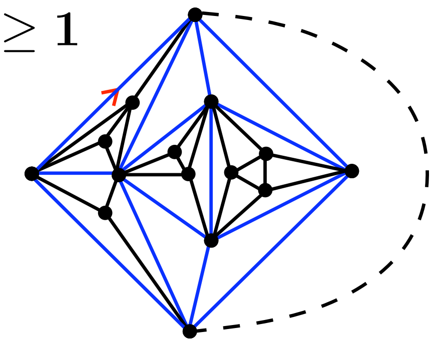
$2|C|$ **loopless maps** $M_1, \ldots, M_{2|C|}$



$$|\mathbf{M}| = |\mathbf{C}| + \sum_{\mathbf{i=1}}^{\mathbf{2|C|}} |\mathbf{M_i}|$$

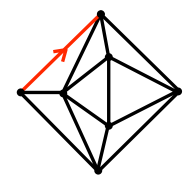
A **triangulation** T **($|T| = \#$ vert. -3)** is
- i) the triangle-map 

  or
- ii) $|\mathbf{T}| \geq \mathbf{1}$



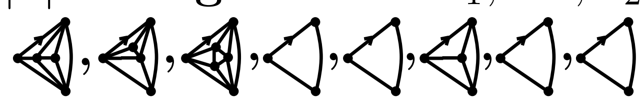
$=$

Irreducible core I



+

$2|I|$ **triangulations** $T_1, \ldots, T_{2|I|}$



$$|\mathbf{T}| = |\mathbf{I}| + \sum_{\mathbf{i=1}}^{\mathbf{2|I|}} |\mathbf{T_i}|$$

Nonseparable maps n **edges**	⟷ *bijection*	**Irreducible triang.** $n+3$ **vertices**
	↓	
Loopless maps n **edges**	⟷ *bijection*	**Triangulations** $n+3$ **vertices**

Results obtained so far

rooted nonseparable maps with n edges

↕ *bipolar orientations*

rooted irreducible triangulations with $n+3$ vertices

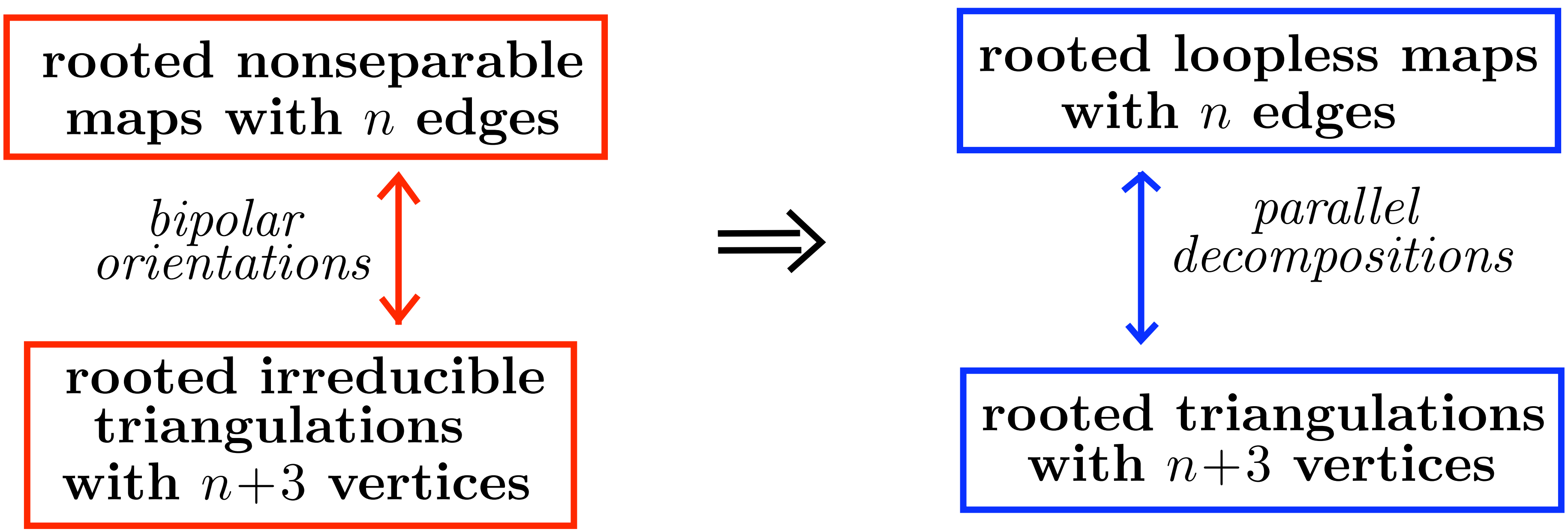
$\Longrightarrow$

rooted loopless maps with n edges

↕ *parallel decompositions*

rooted triangulations with $n+3$ vertices

Results obtained so far



$|\mathcal{N}_{\mathbf{n}}| = |\mathcal{I}_{\mathbf{n}}|$ $\qquad$ $|\mathcal{L}_{\mathbf{n}}| = |\mathcal{T}_{\mathbf{n}}|$

Results obtained so far

rooted nonseparable maps with n edges

bipolar orientations

rooted irreducible triangulations with $n+3$ vertices

$|\mathcal{N}_\mathbf{n}| = |\mathcal{I}_\mathbf{n}| = \mathbf{?}$

$\Longrightarrow$

rooted loopless maps with n edges

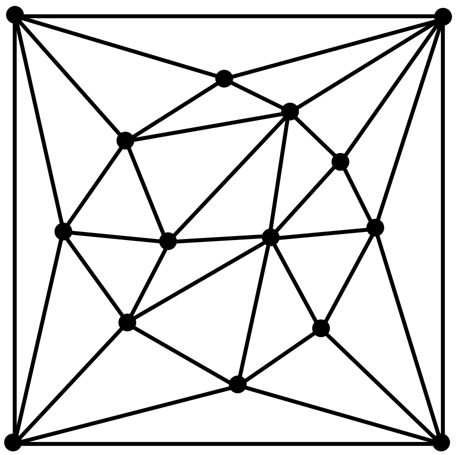
parallel decompositions

rooted triangulations with $n+3$ vertices

$|\mathcal{L}_\mathbf{n}| = |\mathcal{T}_\mathbf{n}| = \mathbf{?}$

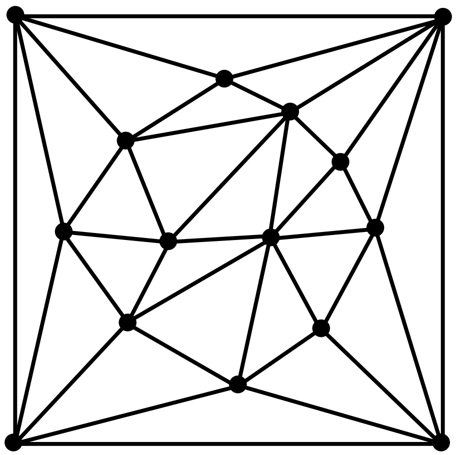
Counting the families

Irreducible triang. $\longleftrightarrow$ ternary trees

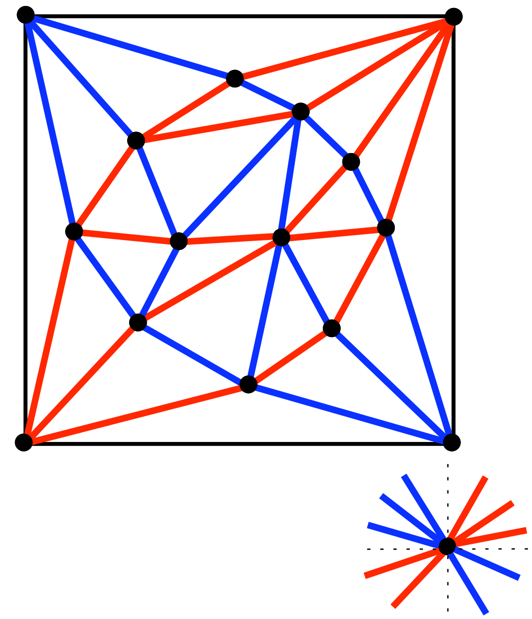


Fusy'05: irreductible triangulations are in bijection with ternary trees

Irreducible triang. $\longleftrightarrow$ ternary trees

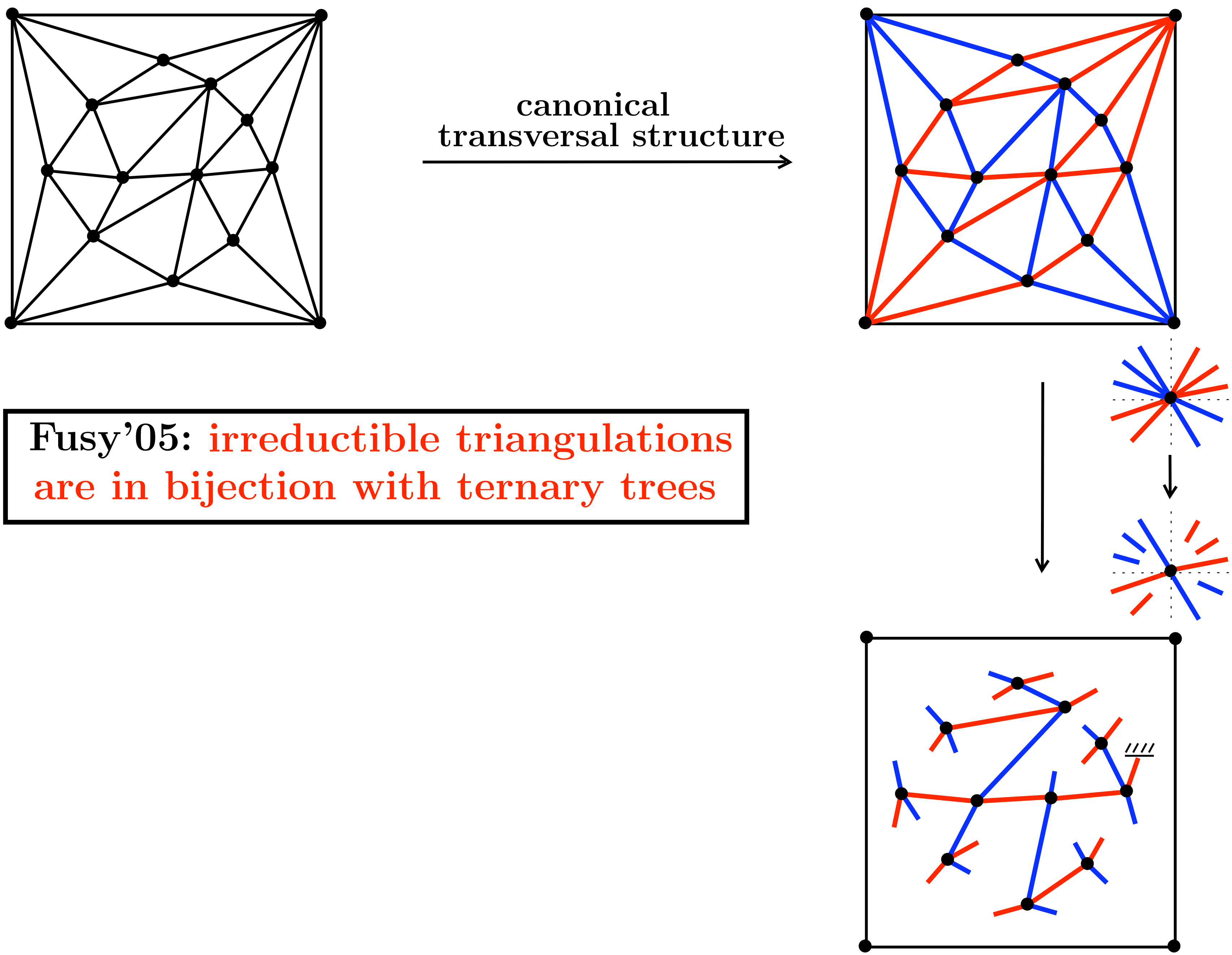


canonical
transversal structure
$\longrightarrow$



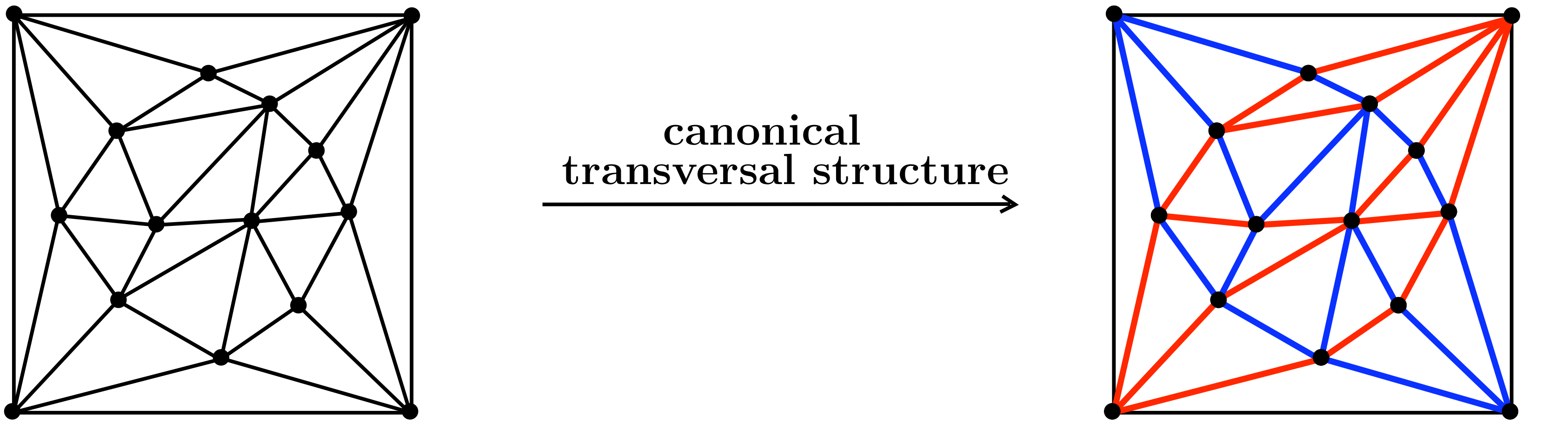
Fusy'05: irreductible triangulations
are in bijection with ternary trees

Irreducible triang. ↔ ternary trees

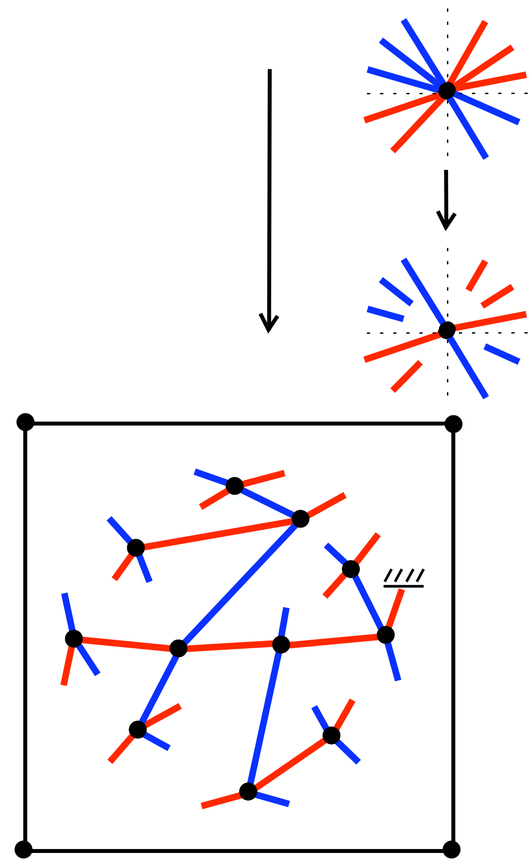


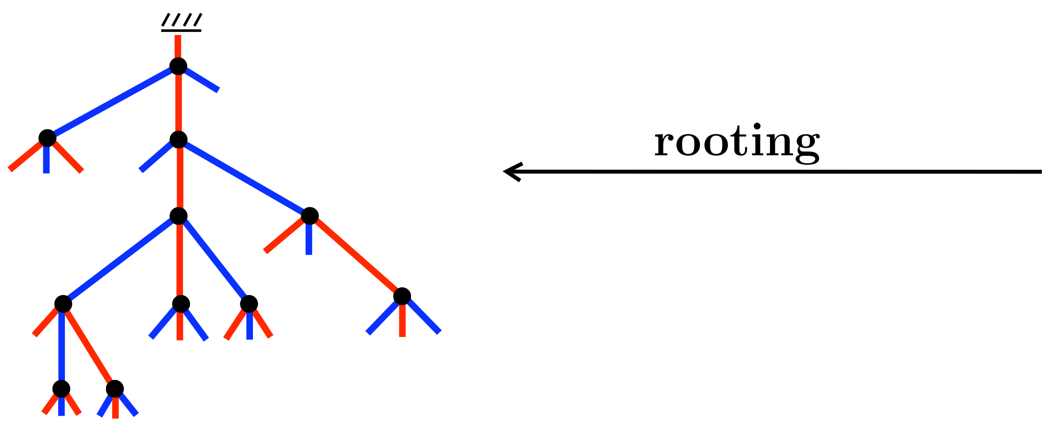
Fusy'05: irreductible triangulations are in bijection with ternary trees

Irreducible triang. $\longleftrightarrow$ ternary trees

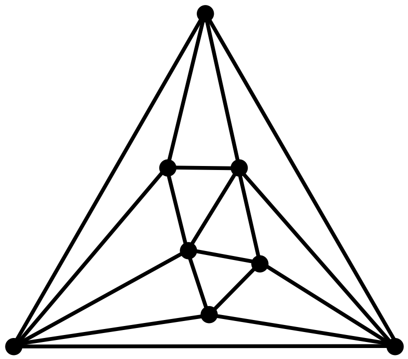


Fusy'05: irreductible triangulations are in bijection with ternary trees



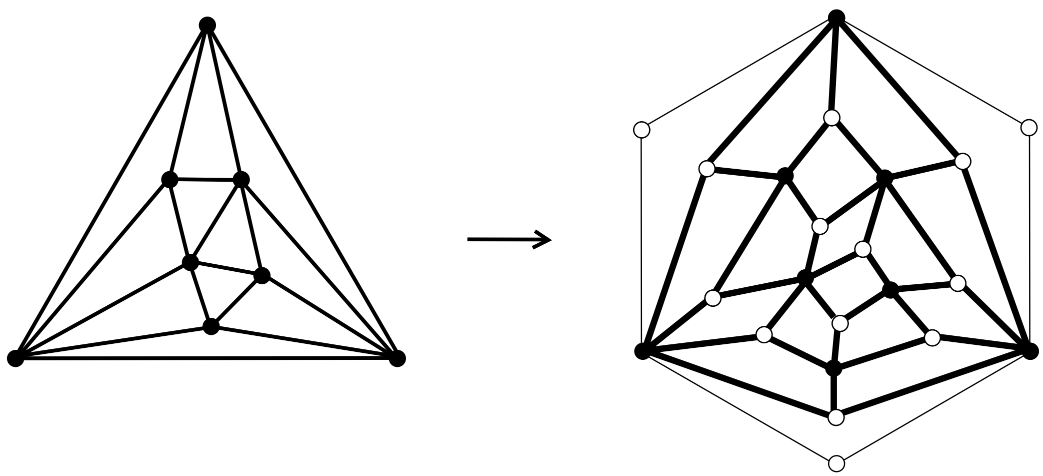


Triangulations↔quaternary trees



Poulalhon-Schaeffer'03, Fusy-Poulalhon-Schaeffer'05:
triangulations are in bijection with quaternary trees

Triangulations↔quaternary trees



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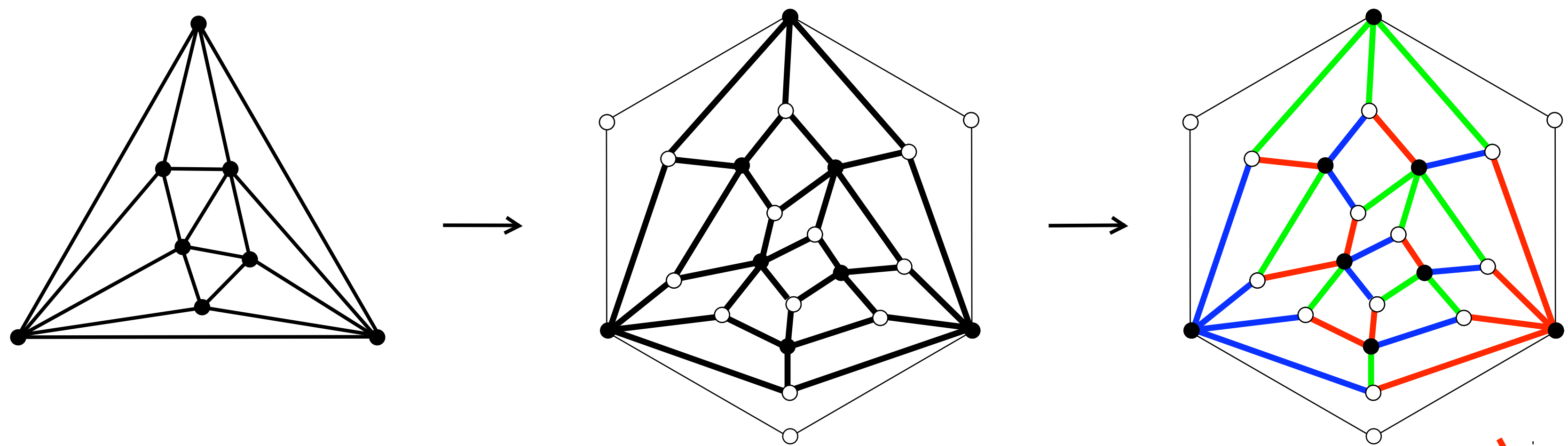
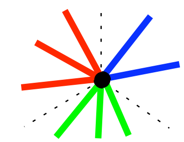
Triangulations↔quaternary trees





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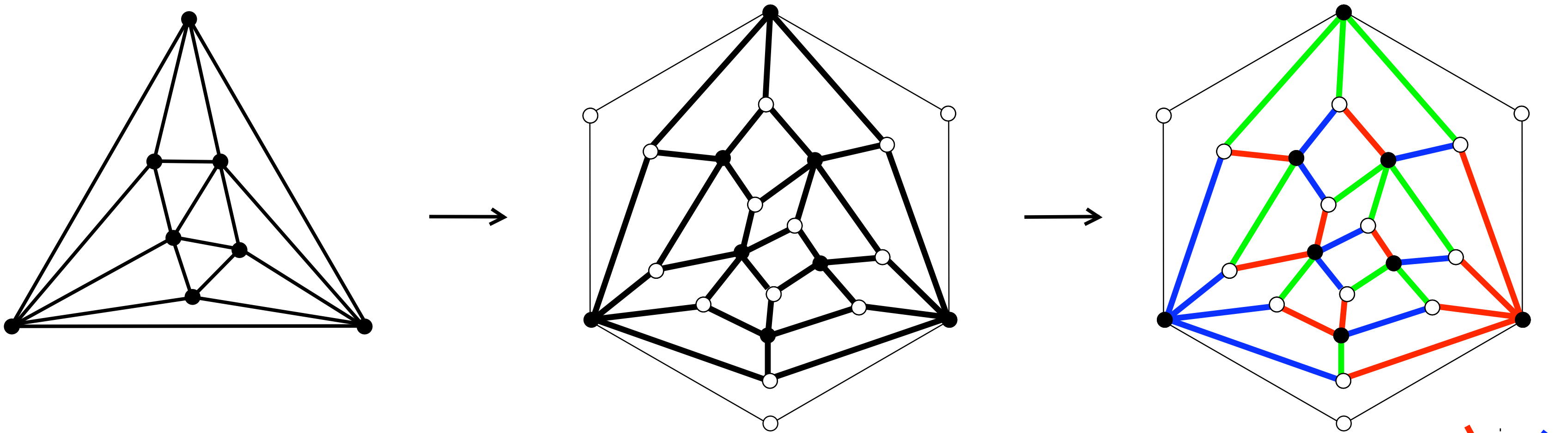
Triangulations↔quaternary trees



Poulalhon-Schaeffer’03, Fusy-Poulalhon-Schaeffer’05:
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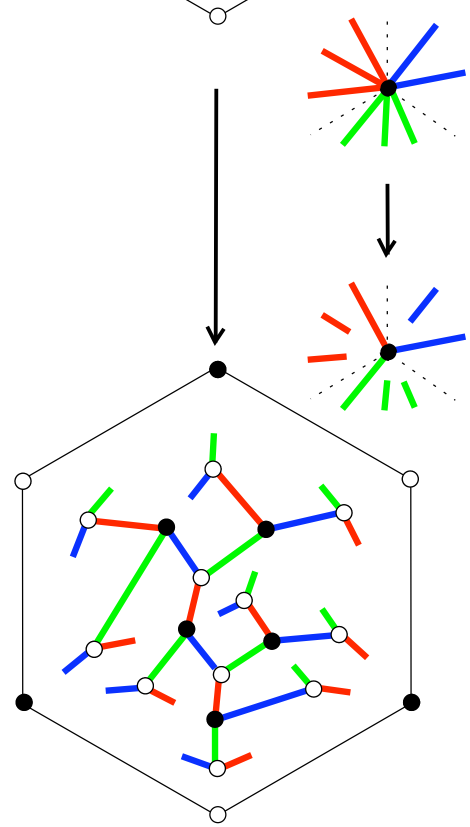
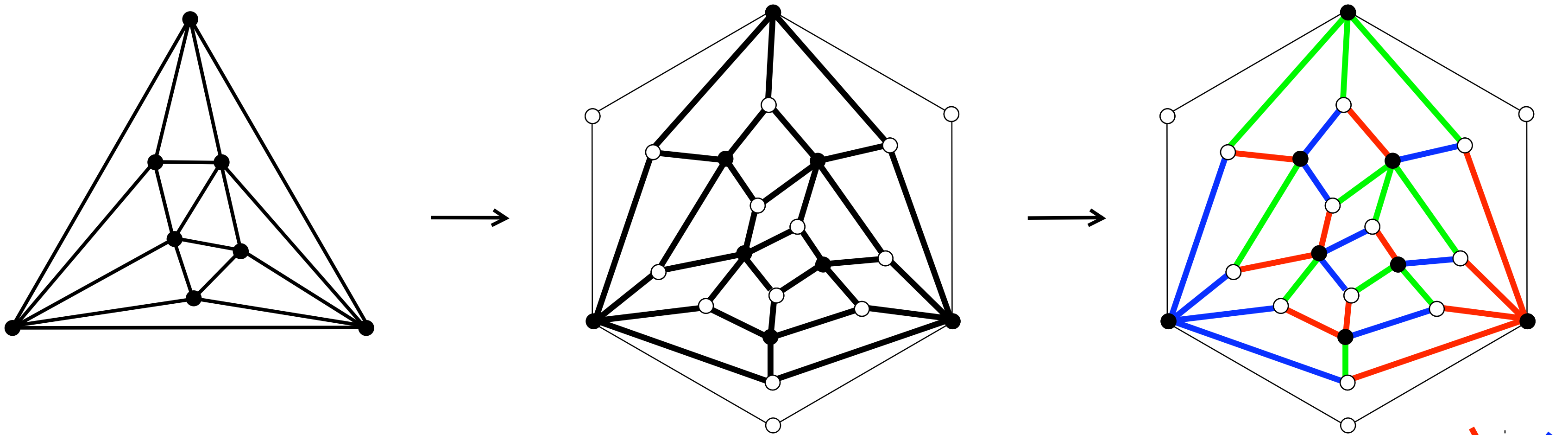


Triangulations↔quaternary trees



Poulalhon-Schaeffer'03, Fusy-Poulalhon-Schaeffer'05:
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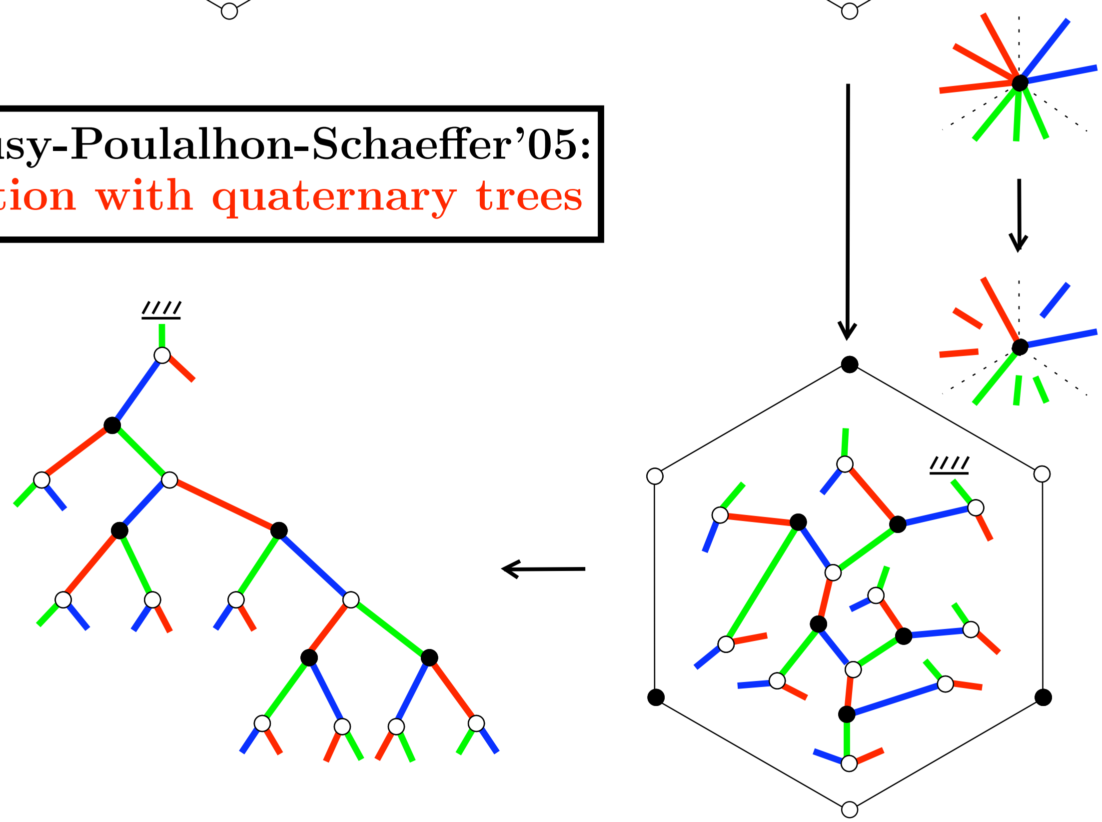
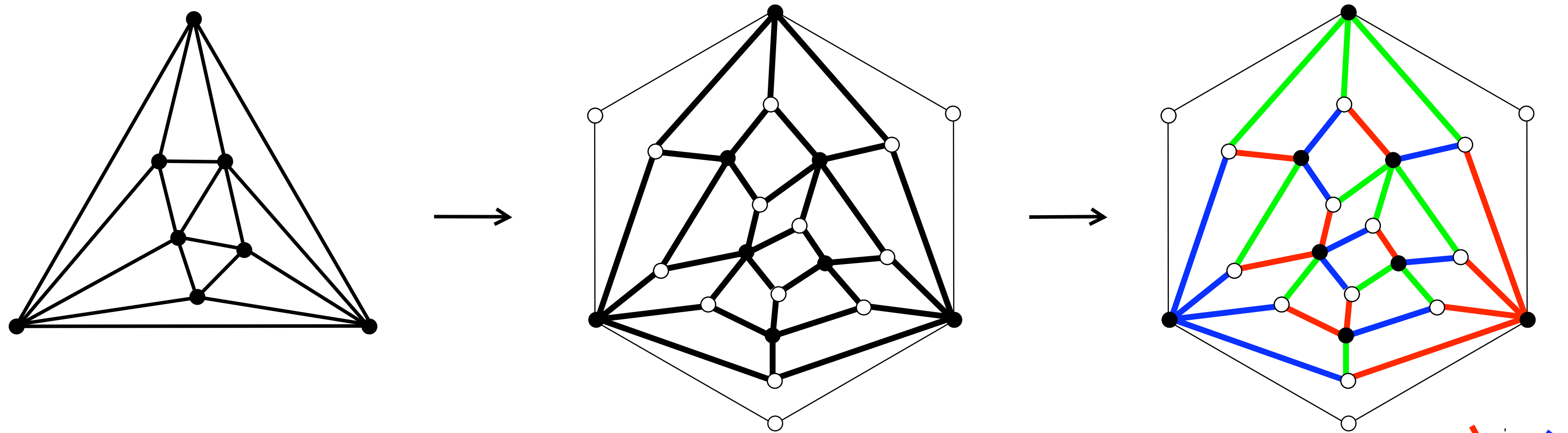
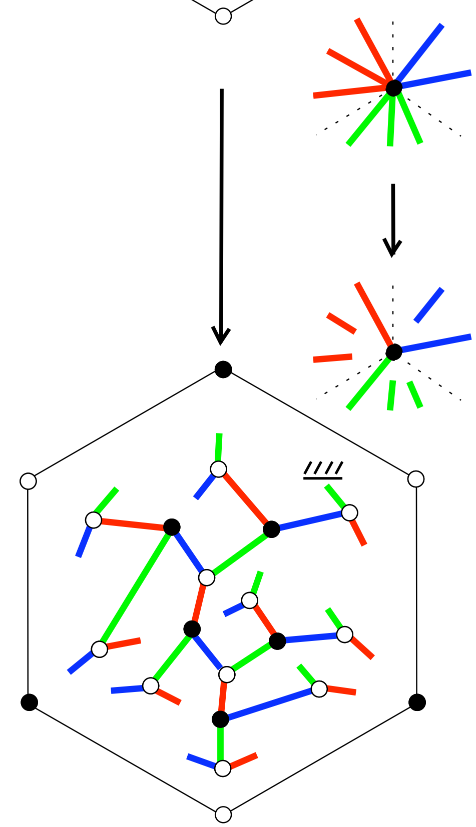


Triangulations↔quaternary trees





Poulalhon-Schaeffer'03, Fusy-Poulalhon-Schaeffer'05:
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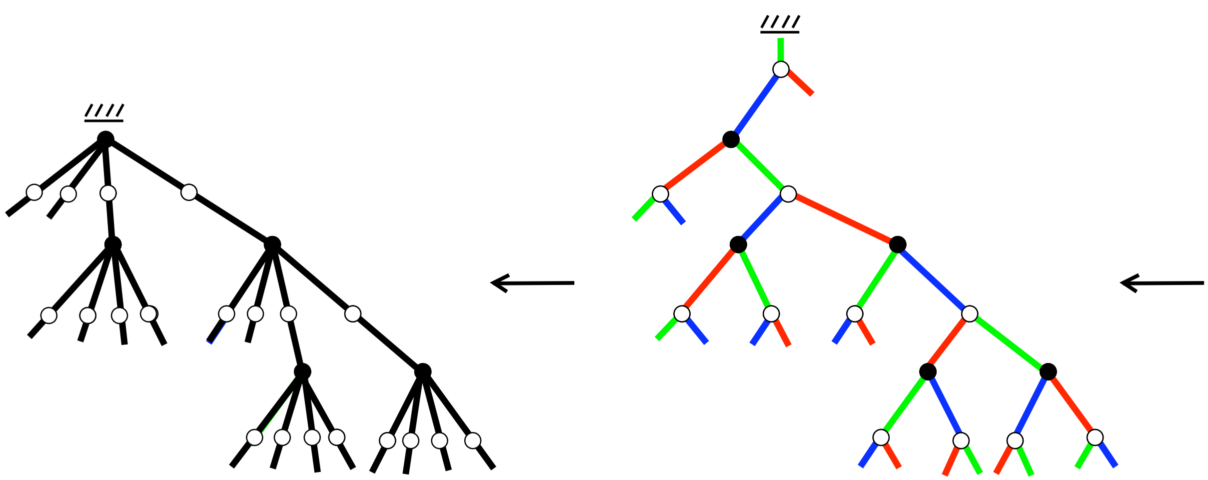


Enumerative Results

rooted nonseparable maps with n edges

$\updownarrow$

rooted irreducible triangulations with $n+3$ vertices

$\updownarrow$

rooted ternary trees with $n-1$ nodes

$\Longrightarrow$

rooted loopless maps with n edges

$\updownarrow$

rooted triangulations with $n+3$ vertices

$\updownarrow$

rooted quaternary trees with n nodes

Enumerative Results

rooted nonseparable maps with n edges

$\updownarrow$

rooted irreducible triangulations with $n+3$ vertices

$\updownarrow$

rooted ternary trees with $n-1$ nodes

$\Downarrow$

$$|\mathcal{N}_{\mathbf{n}}| = |\mathcal{T}_{\mathbf{n}}| = \frac{\mathbf{4(3n-3)!}}{\mathbf{(n-1)!(2n)!}}$$

$\Longrightarrow$

rooted loopless maps with n edges

$\updownarrow$

rooted triangulations with $n+3$ vertices

$\updownarrow$

rooted quaternary trees with n nodes

$\Downarrow$

$$|\mathcal{L}_{\mathbf{n}}| = |\mathcal{I}_{\mathbf{n}}| = \frac{\mathbf{2(4n+1)!}}{\mathbf{(n+1)!(3n+2)!}}$$

Enumerative Results

rooted nonseparable maps with n edges

$\Updownarrow$

rooted irreducible triangulations with $n+3$ vertices

$\Updownarrow$

rooted ternary trees with $n-1$ nodes

(Schaeffer'99)

$\Downarrow$

$$|\mathcal{N}_\mathbf{n}| = |\mathcal{T}_\mathbf{n}| = \frac{\mathbf{4(3n-3)!}}{\mathbf{(n-1)!(2n)!}}$$

$\Longrightarrow$

rooted loopless maps with n edges

$\Updownarrow$

rooted triangulations with $n+3$ vertices

$\Updownarrow$

rooted quaternary trees with n nodes

(?)

$\Downarrow$

$$|\mathcal{L}_\mathbf{n}| = |\mathcal{I}_\mathbf{n}| = \frac{\mathbf{2(4n+1)!}}{\mathbf{(n+1)!(3n+2)!}}$$

Applications

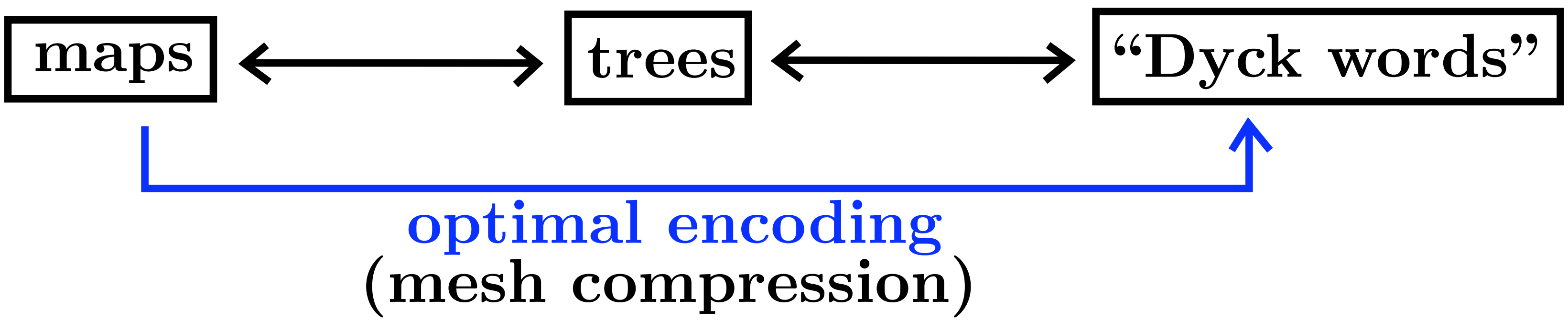
Algorithmic applications

- **General scheme: (Schaeffer'99, Poulalhon-Schaeffer'03)**

maps ⟷ **trees** ⟷ **"Dyck words"**

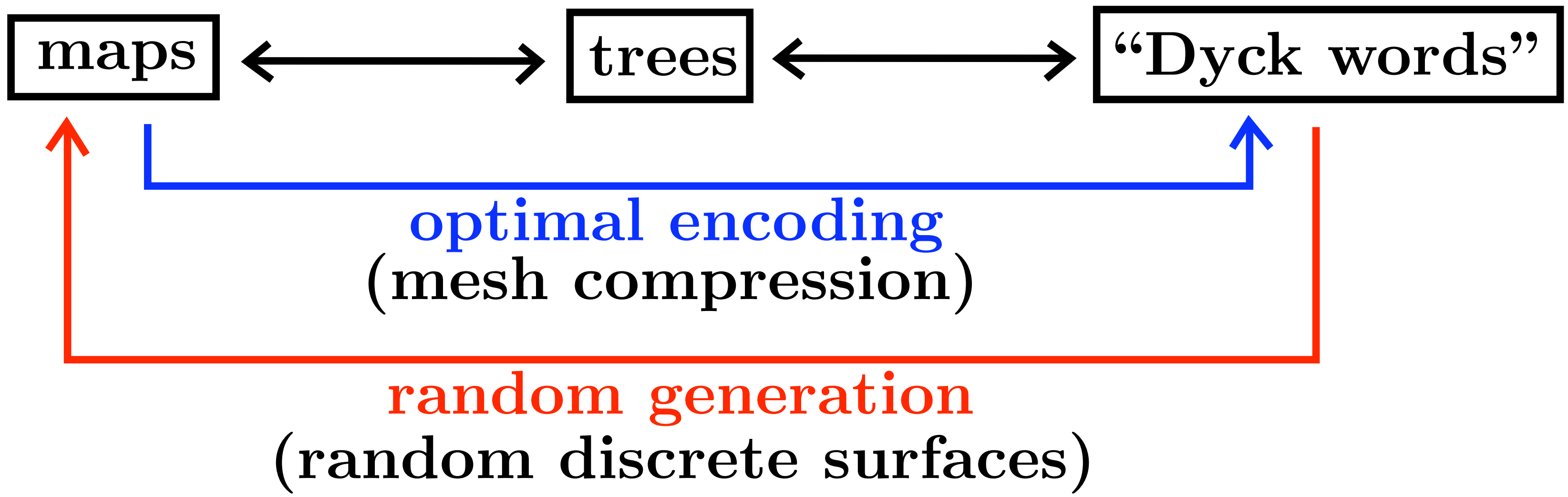
Algorithmic applications

- **General scheme: (Schaeffer'99, Poulalhon-Schaeffer'03)**



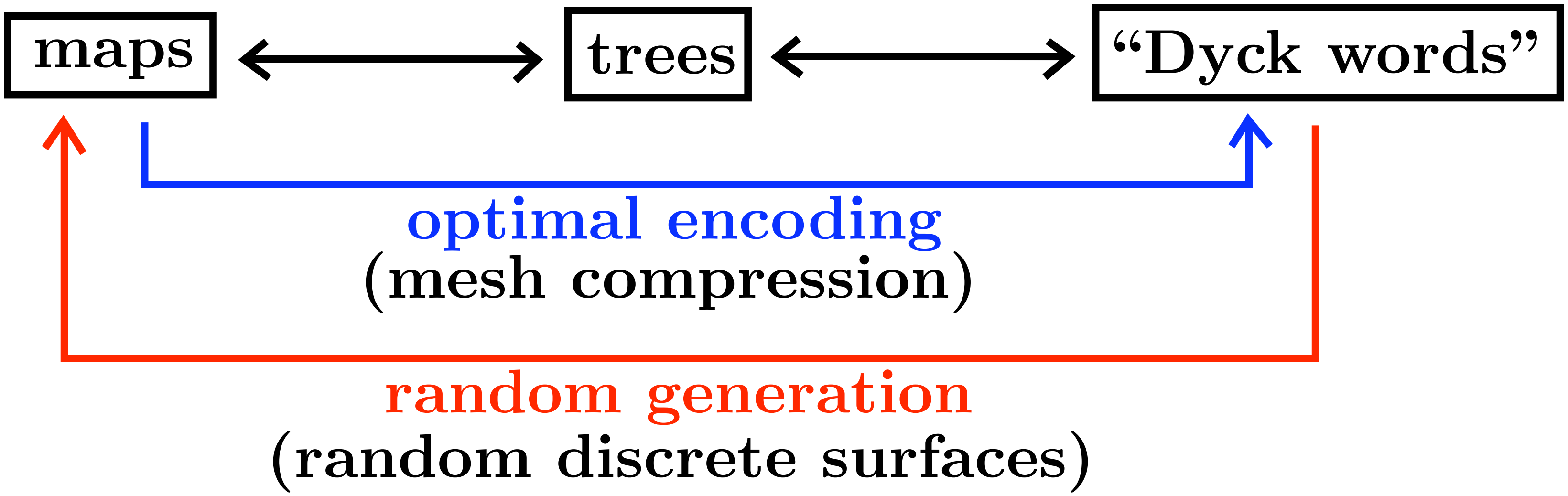
Algorithmic applications

- **General scheme**: (Schaeffer'99, Poulalhon-Schaeffer'03)



Algorithmic applications

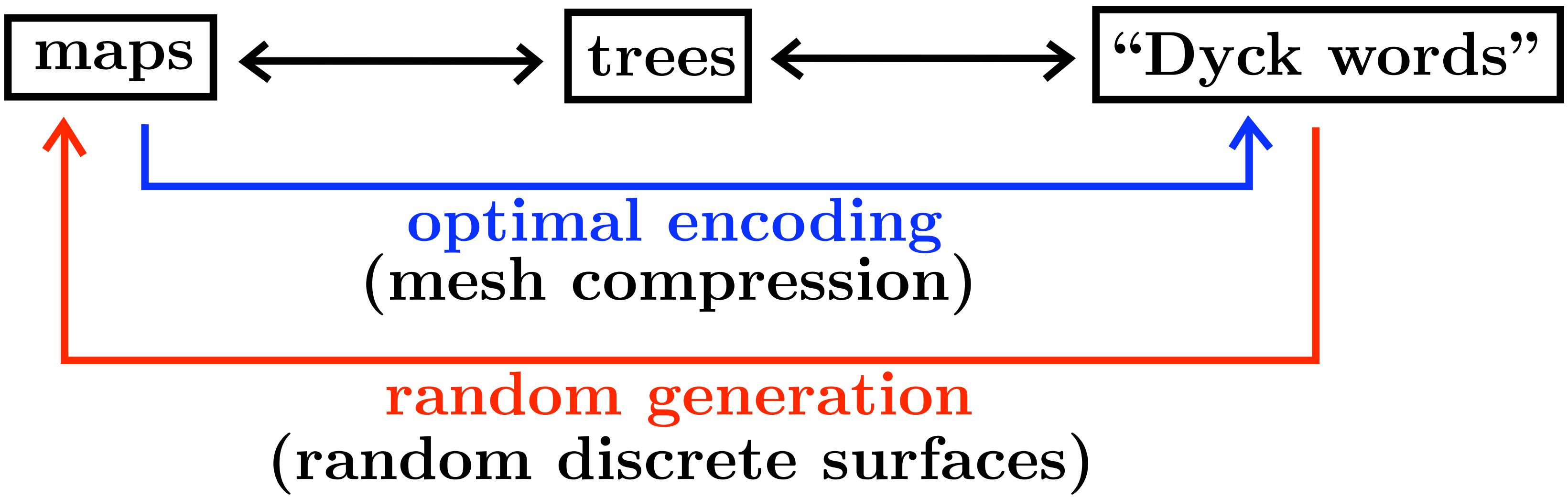
- **General scheme**: (Schaeffer'99, Poulalhon-Schaeffer'03)



- **Applies here to :**
  - irreducible triangulations
  - triangulations

Algorithmic applications

- **General scheme**: (Schaeffer'99, Poulalhon-Schaeffer'03)



- Applies here to :
  - irreducible triangulations
  - triangulations

  as well as :
  - loopless maps
  - nonseparable maps