Forbidden patterns and shift systems

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Joint work with José M. Amigó and Matt Kennel

Deterministic or random?

Two sequences of numbers in $[0, 1]$:

$.6416,\ .9198,\ .2951,\ .8320,\ .5590,\ .9861,\ .0550,\ .2078,\ .6584,\ .8996,$
$.3612,\ .9230,\ .2844,\ .8141,\ .6054, \ldots$

$.9129,\ .5257,\ .4475,\ .9815,\ .4134,\ .9930,\ .1576,\ .8825,\ .3391,\ .0659,$
$.1195,\ .5742,\ .1507,\ .5534,\ .0828, \ldots$

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Are they random? Are they deterministic?

The first one is deterministic: taking $f(x) = 4x(1-x)$, we have

$f(.6146) = .9198$,

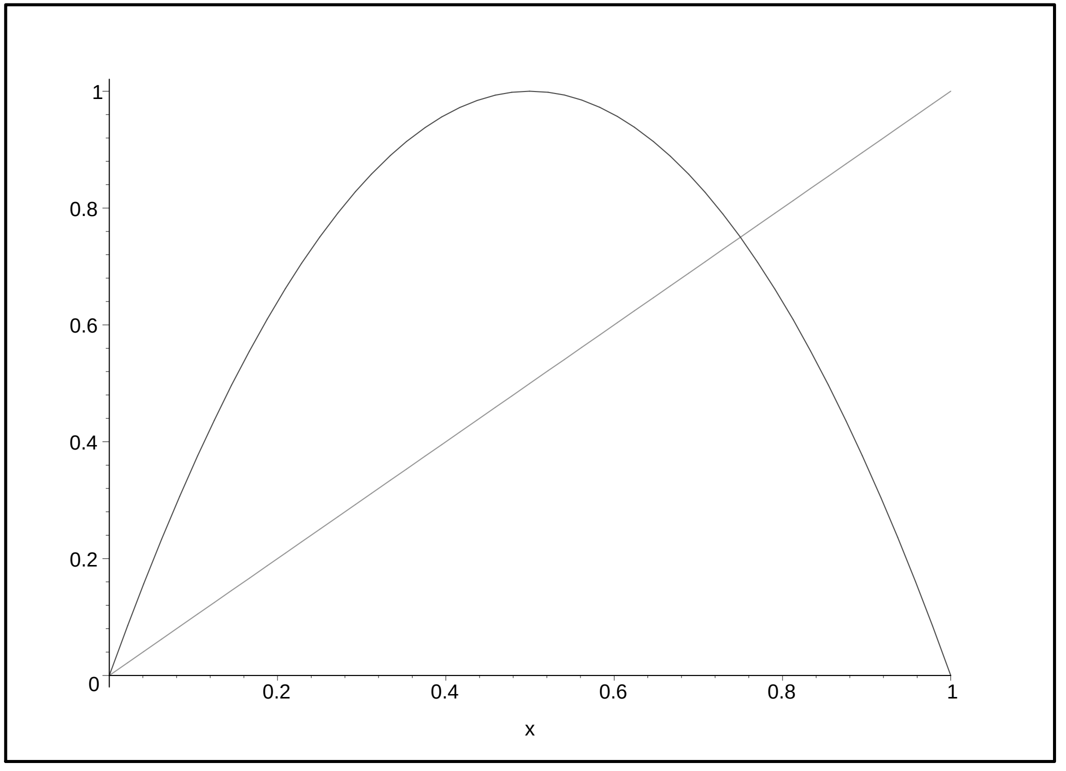
$f(.9198) = .2951$,

$f(.2951) = .8320$,

$\ldots$

Order patterns of a map

$$f : [0,1] \to [0,1], \quad f(x) = 4x(1-x)$$



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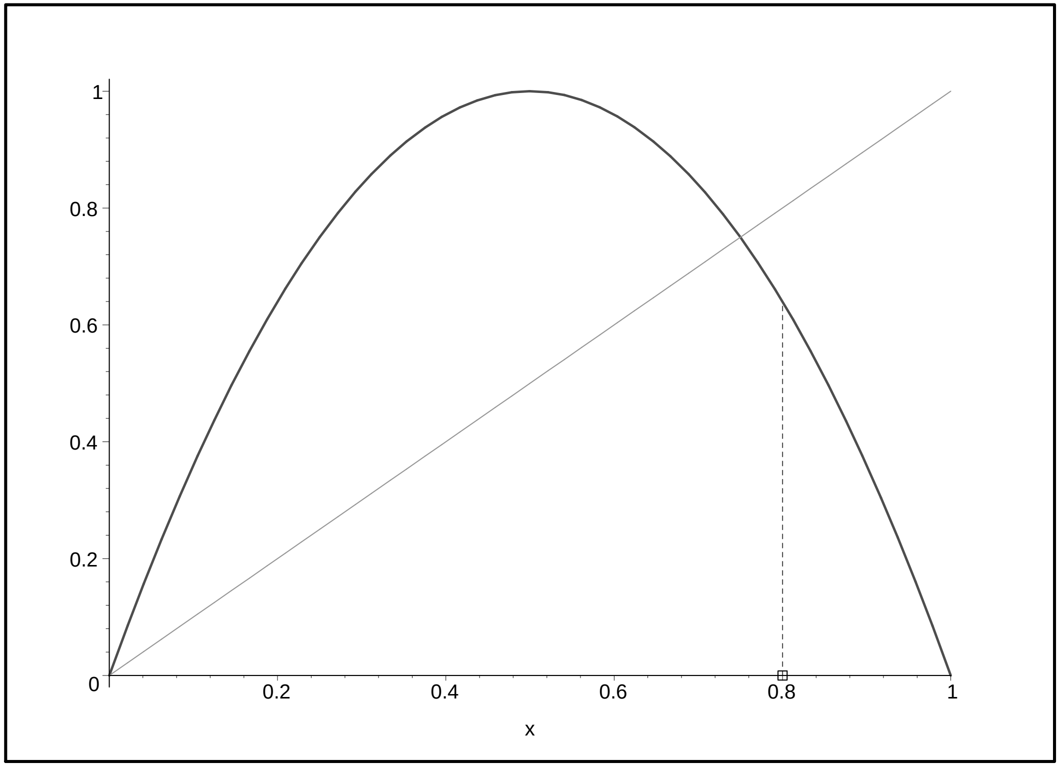

Given $x \in [0,1]$, consider the sequence

$$[x, f(x), f(f(x)), \ldots, f^{n-1}(x)].$$

For $x = 0.8$ and $n = 4$, $\qquad [0.8,$

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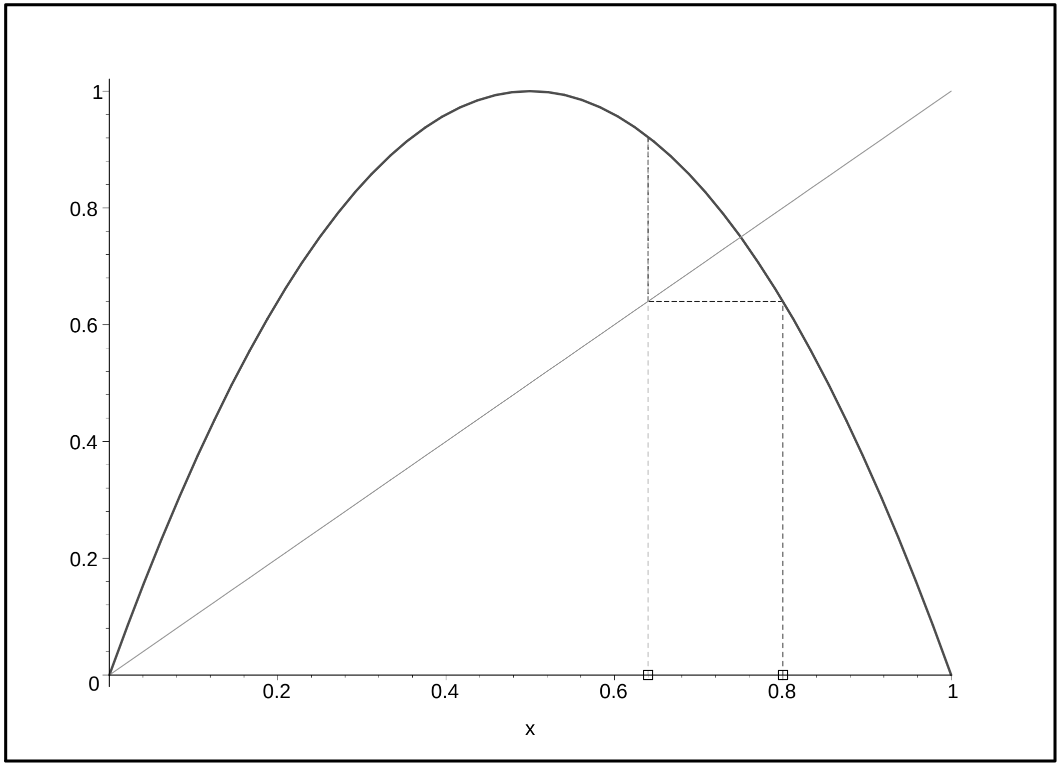

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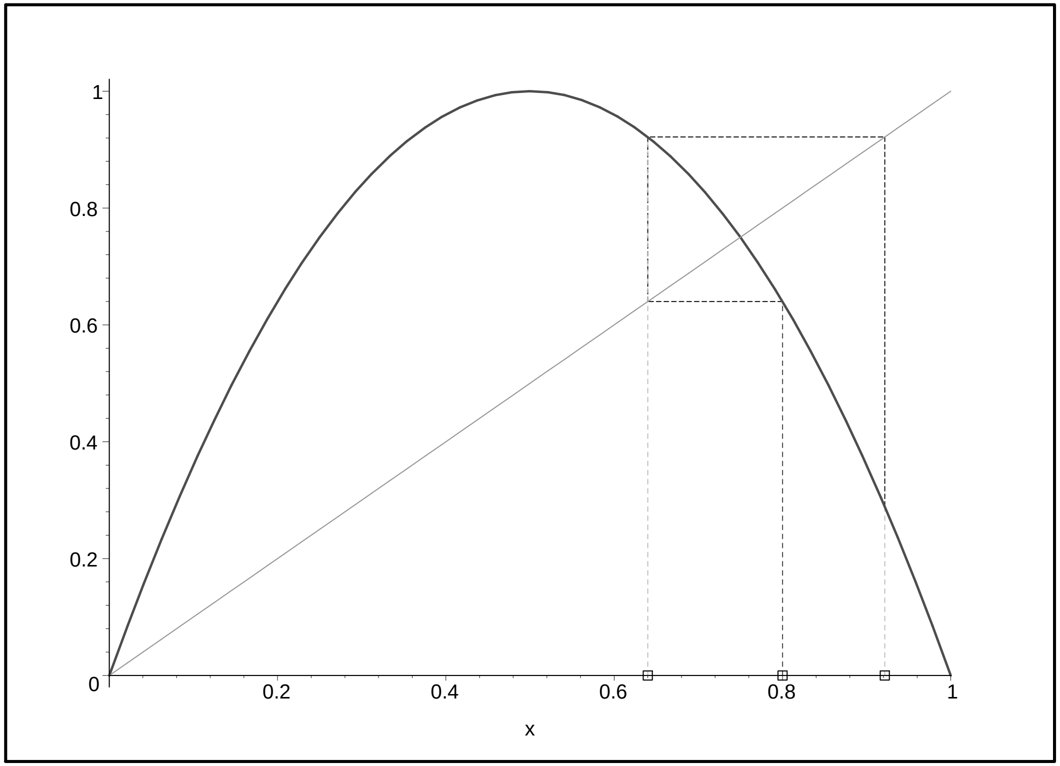

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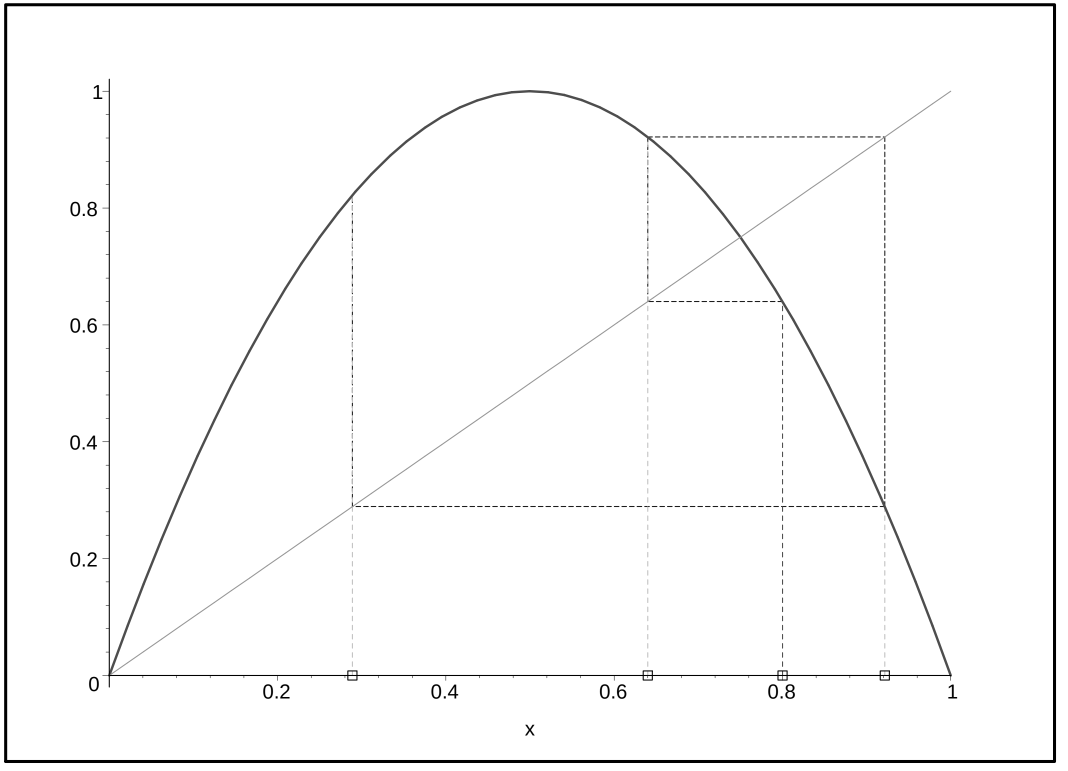

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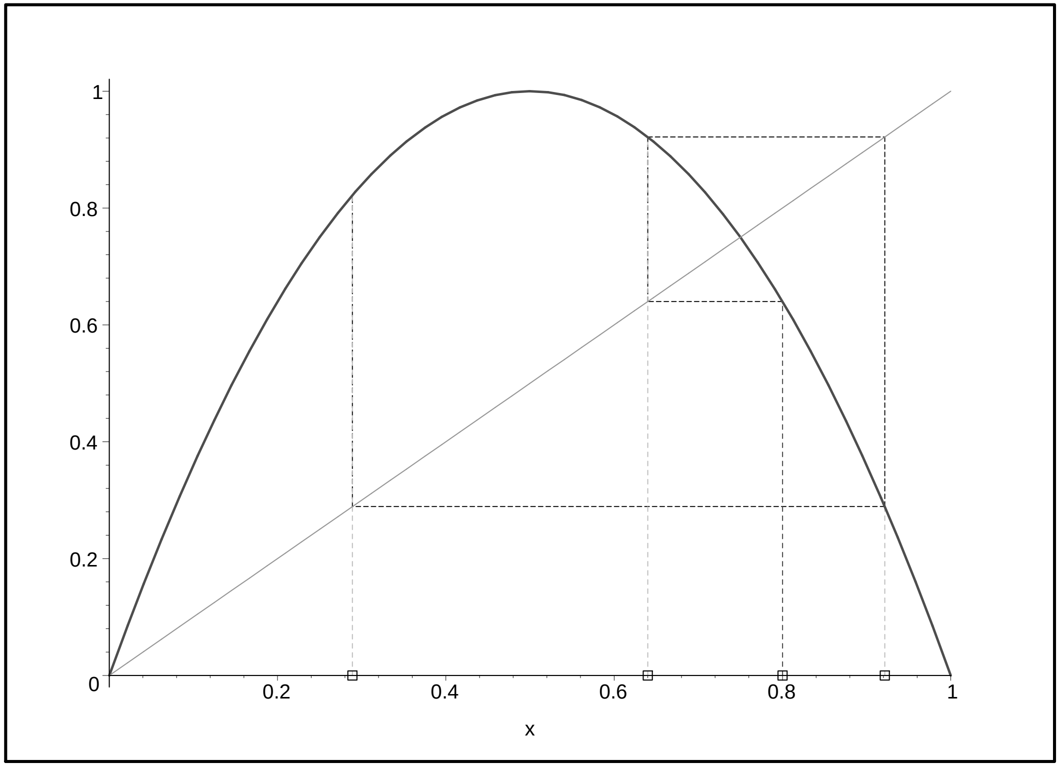

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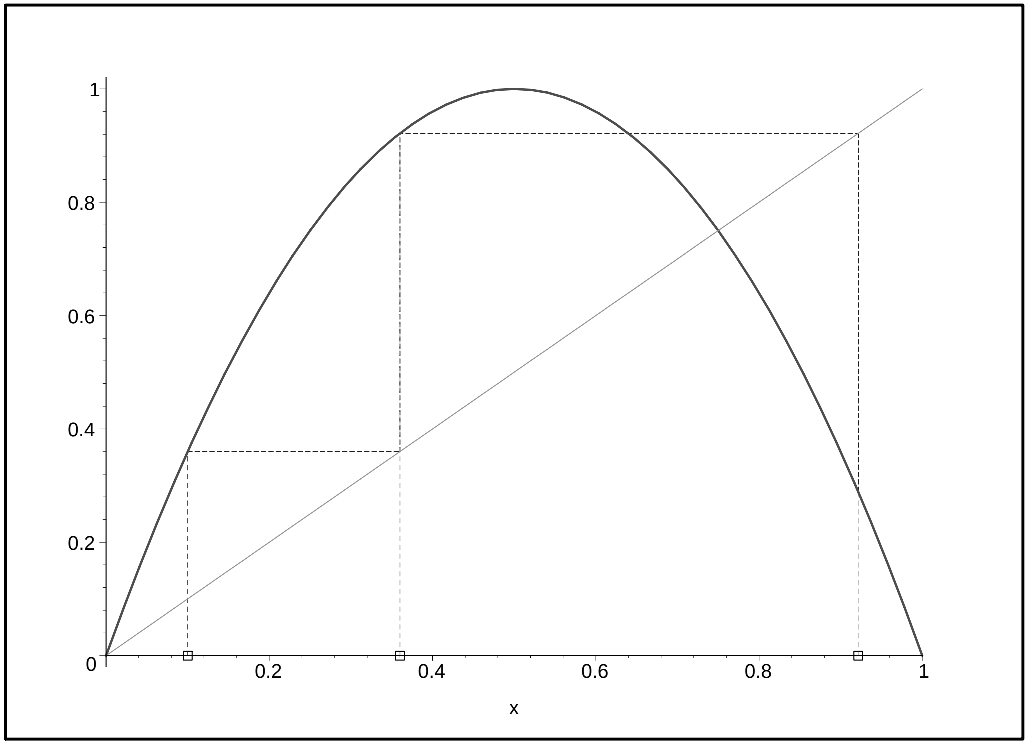
For $x = 0.8$ and $n = 4$, $\quad [0.8,\, 0.64,\, 0.9216,\, 0.2890]$

We say that x defines the pattern $[3, 2, 4, 1]$, and we write $\quad x \rightsquigarrow 3241$.

Let $n = 3$.



$$x = 0.1 \qquad \rightsquigarrow \qquad \color{blue}{123}$$

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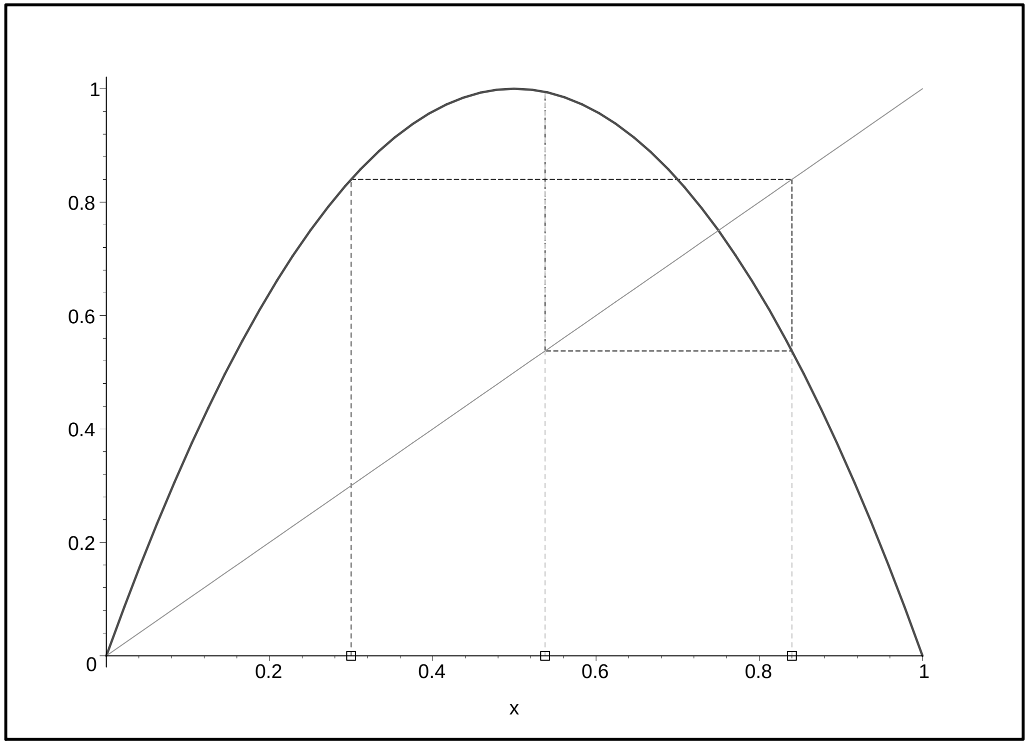

$$x = 0.1 \quad \rightsquigarrow \quad 123$$
$$x = 0.3 \quad \rightsquigarrow \quad 132$$

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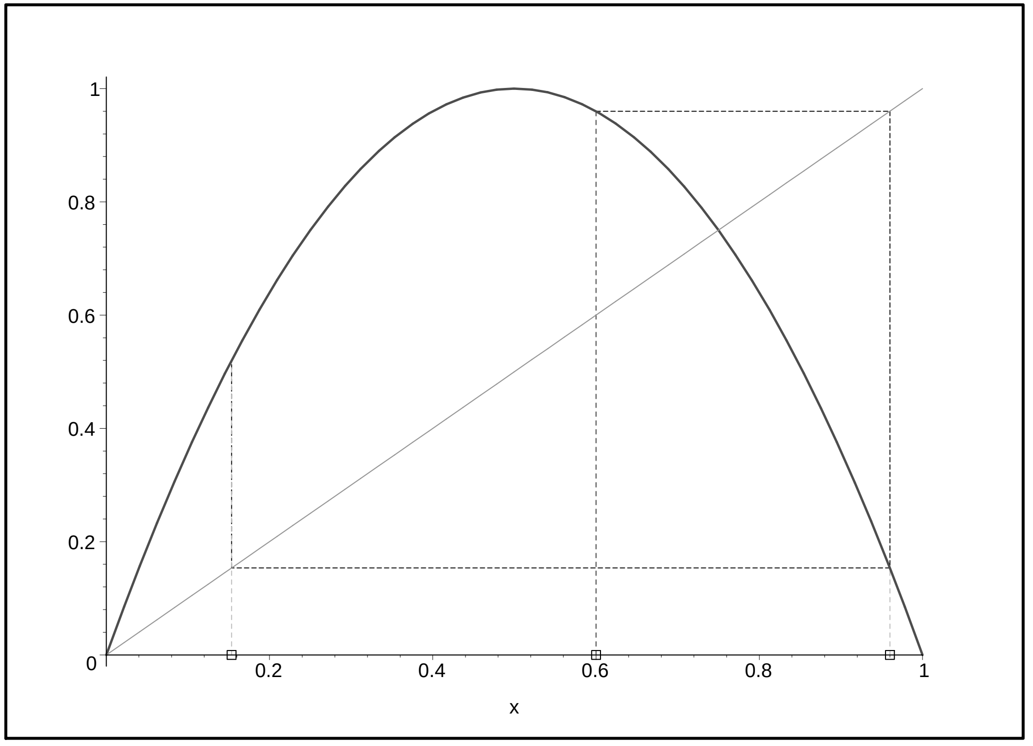

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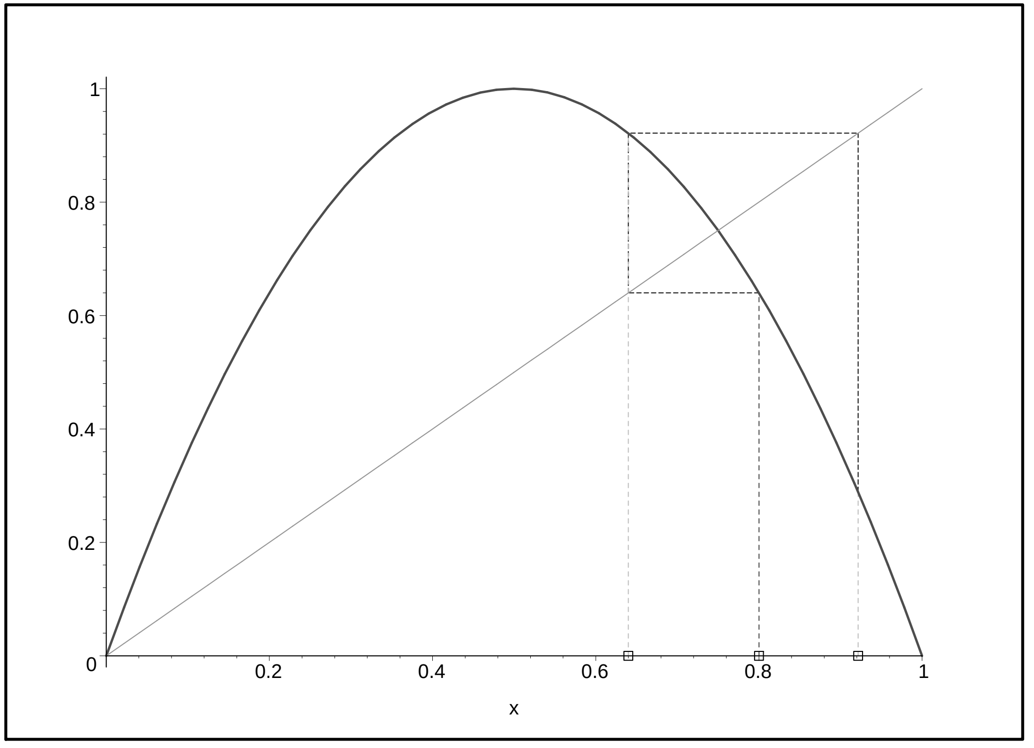

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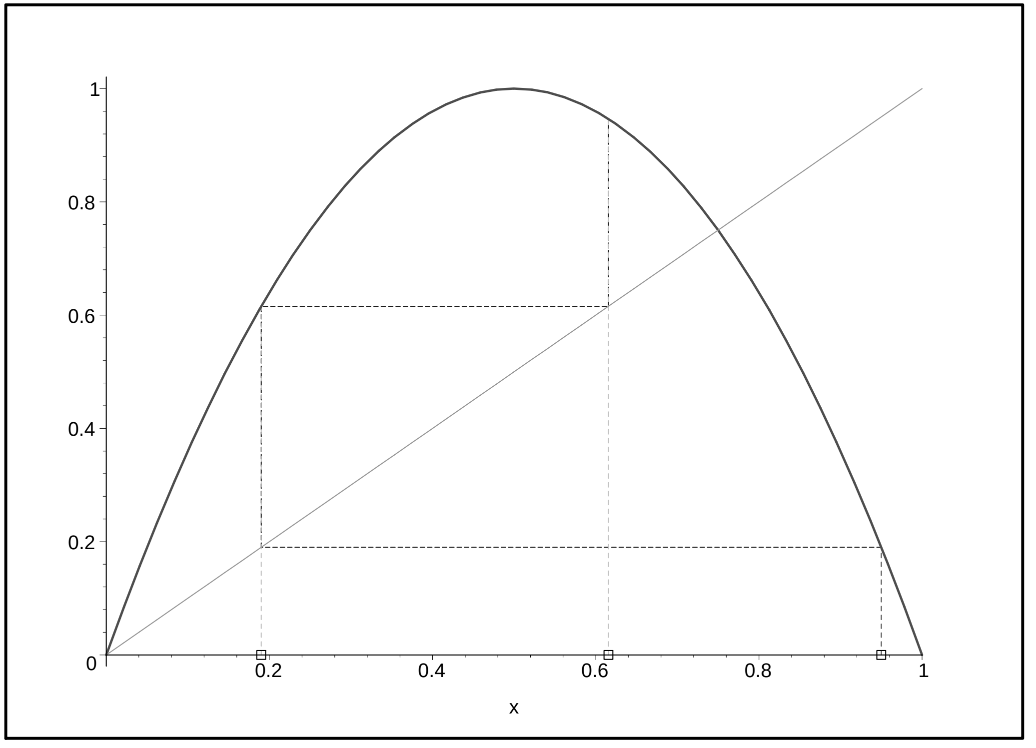

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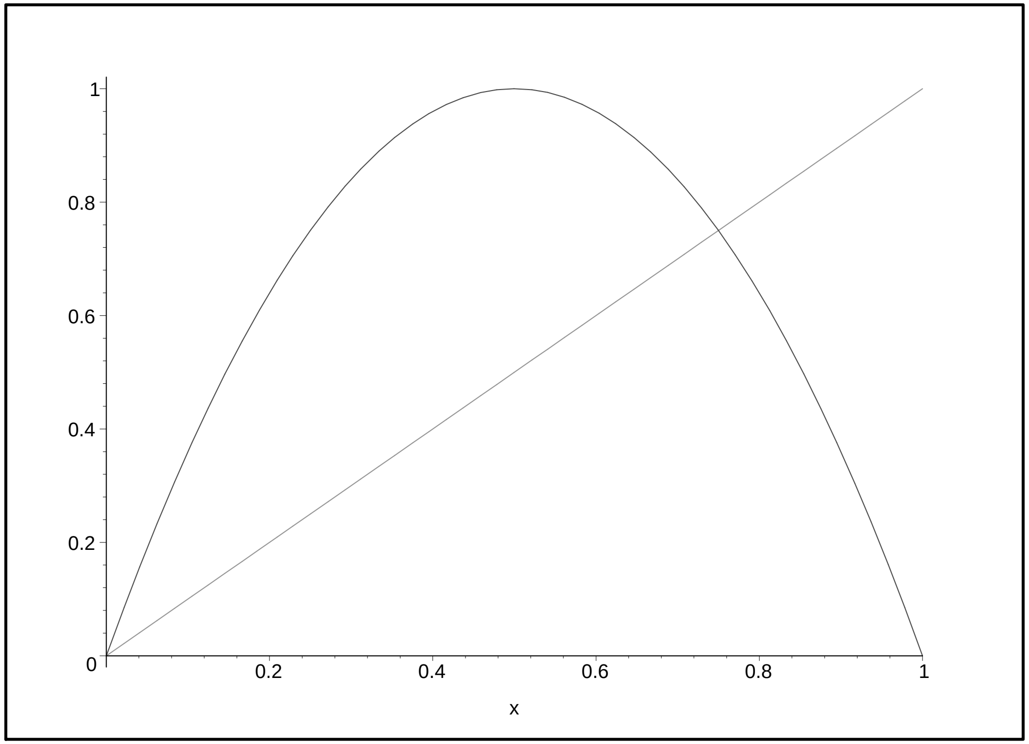

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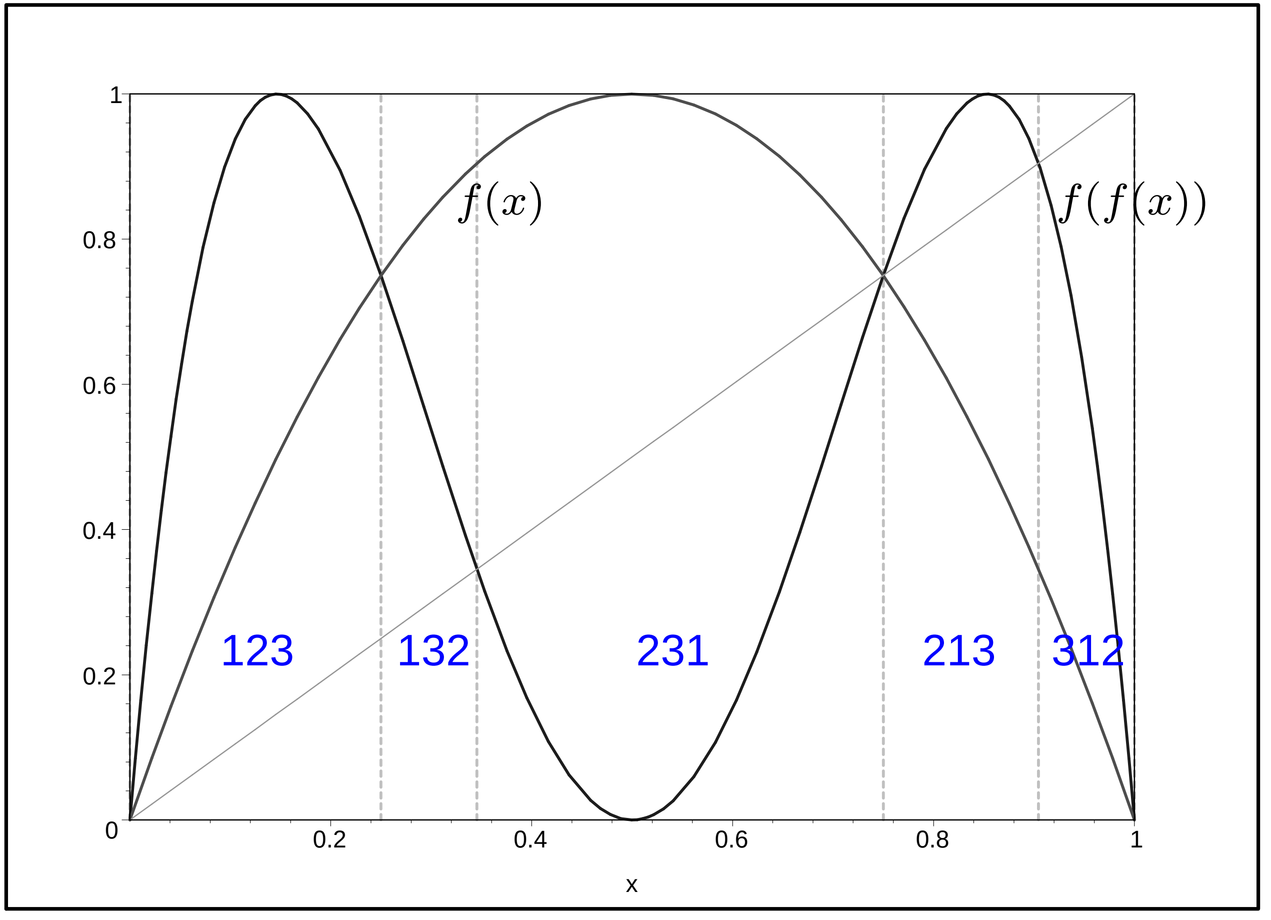
How about the pattern 321?

The pattern 321 does not appear for any x.



We say that 321 is a *forbidden pattern* of f.

Notation

$I \subset \mathbb{R}$ closed interval, $f : I \to I$, $\pi \in \mathcal{S}_n$.

Def. π is *realized* by f if there exists $x \in I$ such that

$$[x, f(x), f(f(x)), \dots, f^{n-1}(x)] \quad \sim \quad [\pi_1, \pi_2, \dots, \pi_n],$$

where $[a_1, \dots, a_n] \sim [b_1, \dots, b_n]$ means that $a_i < a_j$ iff $b_i < b_j$.

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$\mathrm{Allow}_n(f) = \{\pi \in \mathcal{S}_n : \pi$ is realized by $f\}$ $\qquad \mathrm{Forb}_n(f) = \mathcal{S}_n \setminus \mathrm{Allow}_n(f)$

$$\mathrm{Allow}(f) = \bigcup_{n\geq 1} \mathrm{Allow}_n(f) \qquad \mathrm{Forb}(f) = \bigcup_{n\geq 1} \mathrm{Forb}_n(f)$$

Maps have forbidden patterns

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This follows from a result of Bandt-Keller-Pompe that relates allowed patterns of f with its topological entropy.

In fact,

$$|\,\mathrm{Allow}_n(f)| < C^n \ll n! = |\mathcal{S}_n|$$

for some constant C.

Consider a sequence $x_1, x_2, \ldots, x_m$ produced by a black box, with $0 \le x_i \le 1$.

- If the sequence is of the form $x_{i+1} = f(x_i)$, for some piecewise monotone map f, then it must have missing patterns (if m large enough).

  For example, the pattern 321 is missing from

  $$.6416,\ .9198,\ .2951,\ .8320,\ .5590,\ .9861,\ .0550,\ .2078,\ .6584,\ .8996,\ .3612,\ .9230,\ .2844,\ .8141,\ .6054, \ldots .$$

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- On the other hand, if the sequence was generated by m i.i.d. random variables, then the probability that any fixed pattern π is missing goes to 0 exponentially as m grows.

Consecutive patterns in permutations

$\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathcal{S}_n, \quad \pi_1\pi_2\cdots\pi_k \in \mathcal{S}_k$

Definition. σ *contains* π as a consecutive pattern if there exists i such that

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Theorem (E., Noy)**.** *For any* $\pi \in \mathcal{S}_k$ *with* $k \geq 3$ *, there exist constants* $0 < c, d < 1$ *such that*

$$c^n n! \leq |\,\mathrm{Av}_n(\pi)| \leq d^n n! \qquad \text{for all } n \geq k.$$

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$\mathrm{BF}_4(f) = \{\mathbf{1423}, \mathbf{2134}, \mathbf{2143}, \mathbf{3142}, \mathbf{4231}\}$

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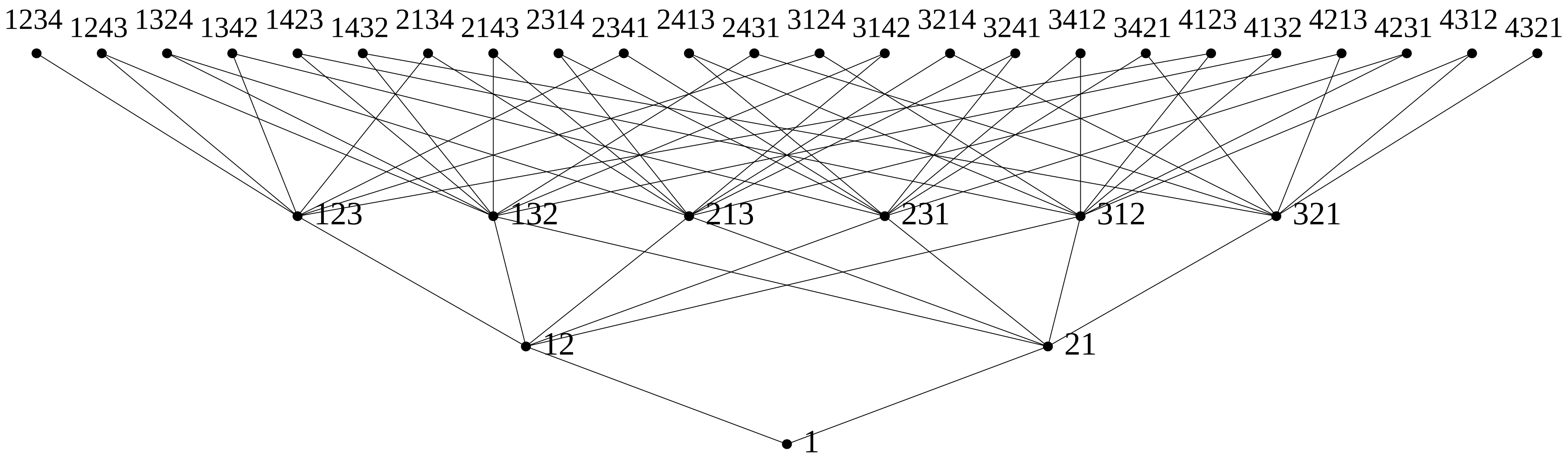
$\mathrm{Forb}_4(f) =$
$\{\mathbf{1423}, 1\underline{432}, \mathbf{2134}, \mathbf{2143}, 2\underline{431}, \mathbf{3142}, \underline{321}4, 3\underline{421}, \underline{421}3, \mathbf{4231}, \underline{431}2, \underline{432}1\}$

Poset of permutations under consec. pattern containment

We can consider the infinite poset of all permutations where

$$\pi \leq \sigma \quad \Leftrightarrow \quad \sigma \text{ contains } \pi \text{ as a consecutive pattern.}$$



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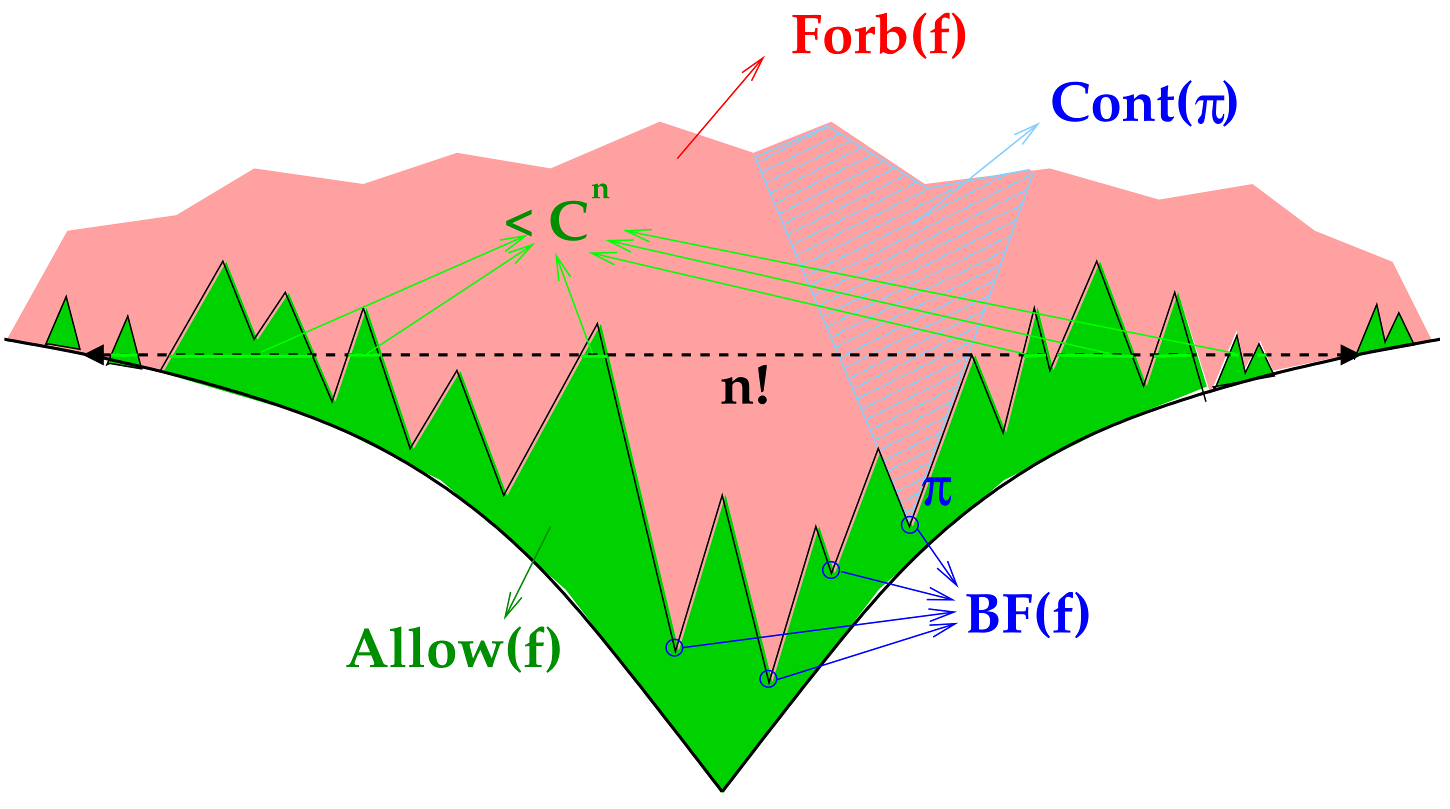

Example

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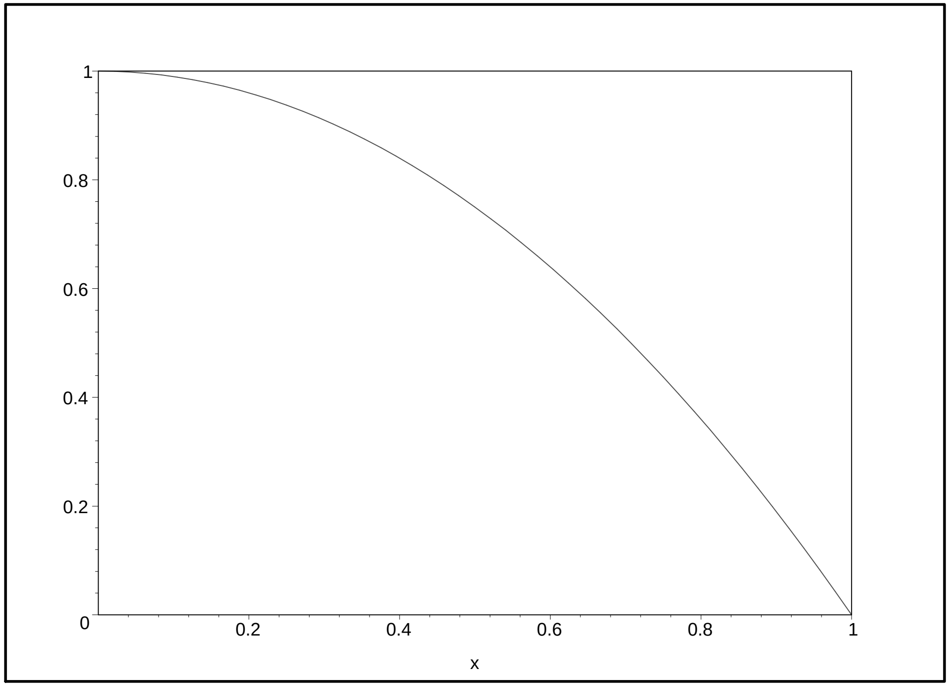

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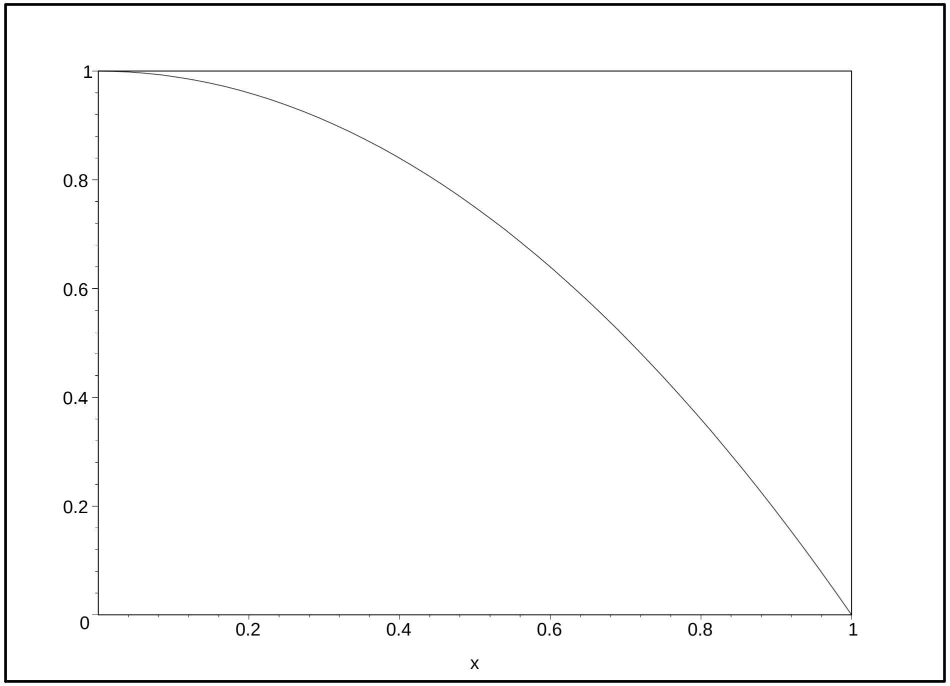

$$\mathrm{BF}(g) = \{\mathbf{123}, \mathbf{132}, \mathbf{312}, \mathbf{321}\}$$

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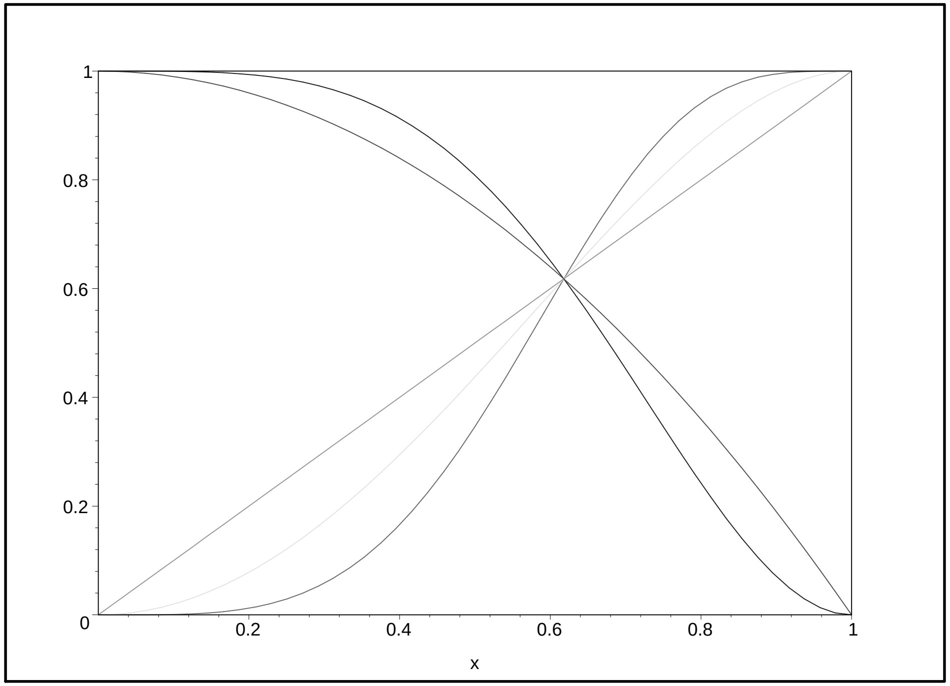

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For example, $h_{10}(0.837435\ldots) = 0.37435\ldots$

One-sided shifts

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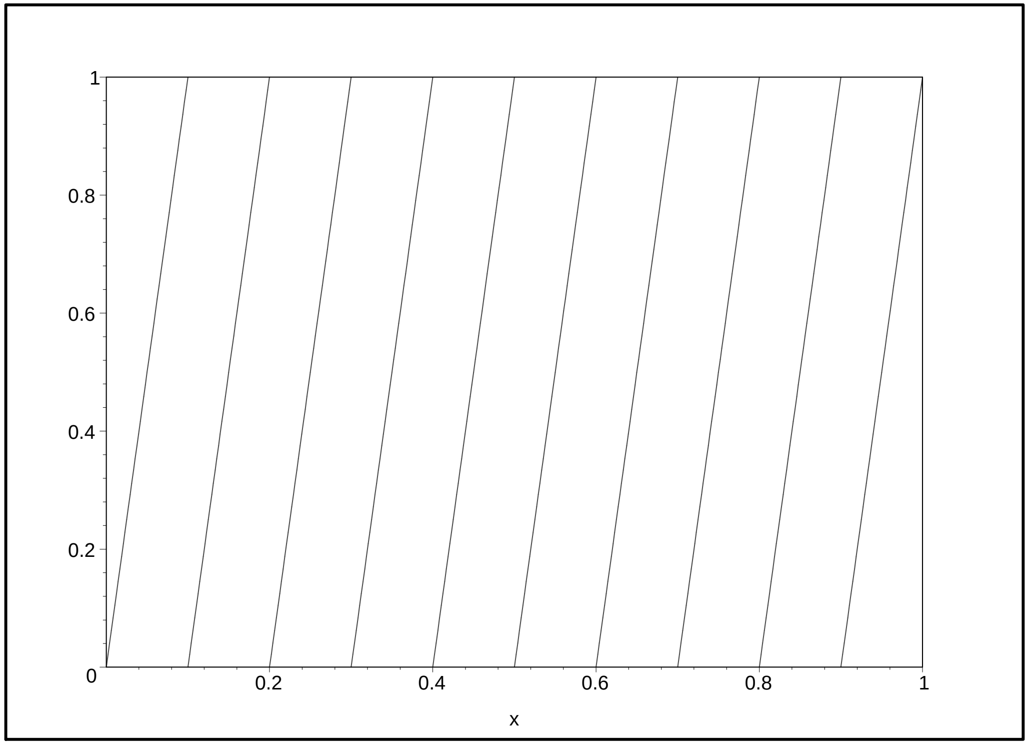
This is a piecewise linear map, so it has forbidden order patterns.



One-sided shift on N symbols

For $N \geq 2$, let $\mathcal{W}_N = \{0, 1, \ldots, N-1\}^{\mathbb{N}}$, and let

$$\begin{array}{cccc} \Sigma_N : & \mathcal{W}_N & \longrightarrow & \mathcal{W}_N \\ & w_1 w_2 w_3 \ldots & \mapsto & w_2 w_3 w_4 \ldots \end{array}$$

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Order patterns are defined using the lexicographic order in $\mathcal{W}_N$.

Example. The word $w = 2102212210 \cdots \in \mathcal{W}_3$ defines the pattern 4217536:

$$\begin{array}{rc} 2102212210\ldots & 4 \\ 102212210\ldots & 2 \\ 02212210\ldots & 1 \\ 2212210\ldots & 7 \\ 212210\ldots & 5 \\ 12210\ldots & 3 \\ 2210\ldots & 6 \end{array}$$

Theorem (Amigó, E., Kennel).

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Example. The smallest forbidden patterns of Σ_4 are

$$\{615243, 324156, 342516, 162534, 453621, 435261\}.$$

The number of permutations realized by a shift

Let $a_{n,N} = |\,\mathrm{Allow}_n(\Sigma_N) \setminus \mathrm{Allow}_n(\Sigma_{N-1})|$,
the # of permutations in $\mathcal{S}_n$ that require N symbols to be realized.

$n \backslash N$	2	3	4	5	6	7
2	2					
3	6					
4	18	6				
5	48	66	6			
6	126	402	186	6		
7	306	2028	2232	468	6	
8	738	8790	19426	10212	1098	6

Let $a_{n,N} = |\,\mathrm{Allow}_n(\Sigma_N) \setminus \mathrm{Allow}_n(\Sigma_{N-1})|$,
the # of permutations in $\mathcal{S}_n$ that require N symbols to be realized.

$n \backslash N$	2	3	4	5	6	7
2	2					
3	6					
4	18	6				
5	48	66	6			
6	126	402	186	6		
7	306	2028	2232	468	6	
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Theorem (E.). *For $n \geq N+2$,*

$$a_{n,N} = \sum_{i=0}^{N-2} (-1)^i \binom{n}{i} \left((N-i-2)(N-i)^{n-2} + \sum_{t=1}^{n-1} \psi_{N-i}(t)(N-i)^{n-t-1} \right),$$

where $\psi_M(k)$ is the number of primitive words of length k over an M-letter alphabet.

The condition of piecewise monotonicity is essential:

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Maps without forbidden patterns

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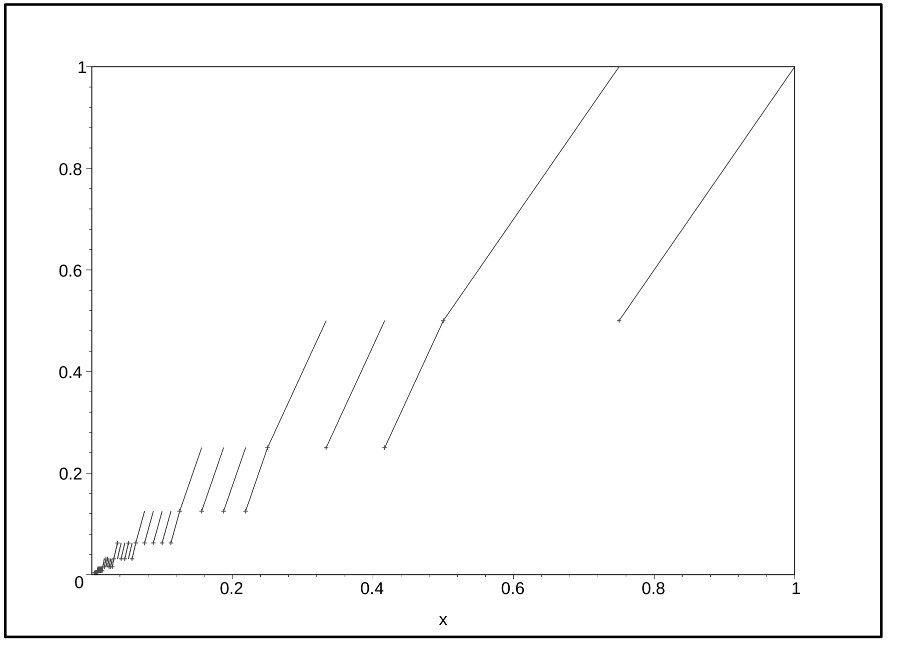
- Define on each I_m a properly scaled version of h_m from I_m to I_m.



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- What else can we say about the structure or the asymptotic growth of $\mathrm{Allow}(f)$ or $\mathrm{Forb}(f)$?