On the link pattern distribution of quarter-turn-symmetric FPL configurations

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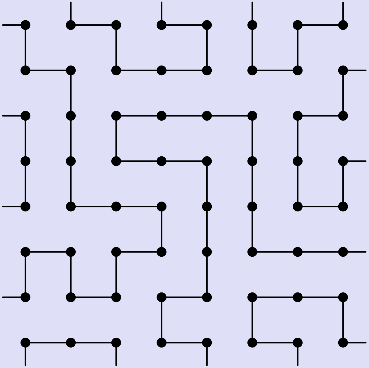
FPSAC 2008 - June 23, 2008

Outline

FPL vs ASM

Fully-Packed Loop configuration:



FPL vs ASM

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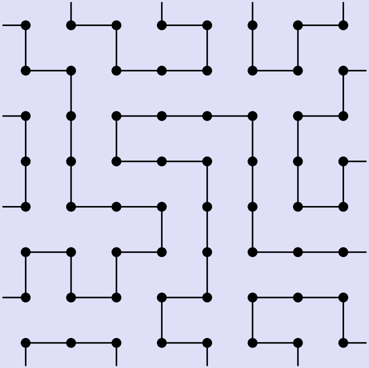


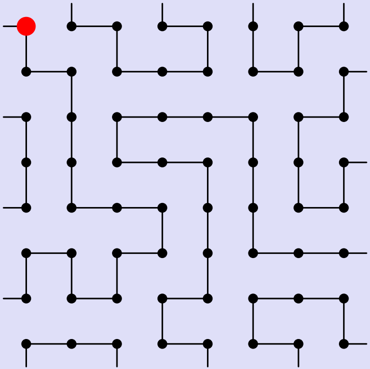
Alternating-Sign Matrix:

$$\begin{matrix}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 \\
1 & -1 & 0 & 1 & 0 & -1 & 1 & 0 \\
0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{matrix}$$

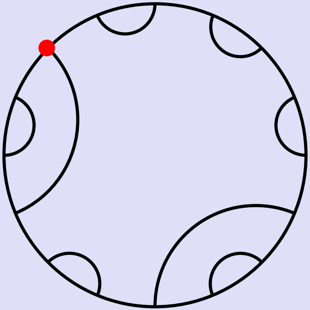
(easy bijection: corner $= 0$, straight line $= \pm 1$)

Link patterns

FPL F



Link pattern C(F)

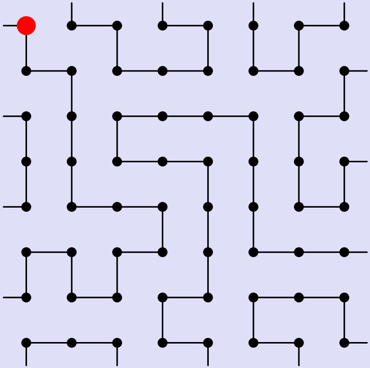


w = aabbabaabbababab
(Catalan)

Each FPL is a collection of “open” and “closed” loops; the open loops determine a perfect matching of the 2n border points, with non-crossing condition: $i < j < \sigma(i) \implies j < \sigma(j) < \sigma(i)$

Link patterns

FPL F



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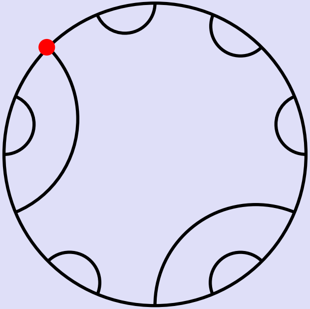

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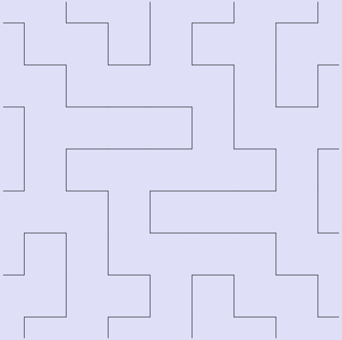
(Catalan)

The link pattern C(F) of an FPL F is this perfect matching, coded as a Dyck word (a for the first endpoint of each edge, b for the second endpoint)

Symmetries

Possible rotational symmetries

Half turn ($2n$, $2n+1$): HTFPL



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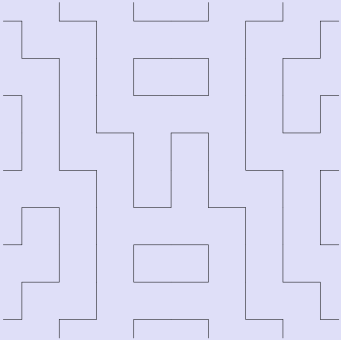

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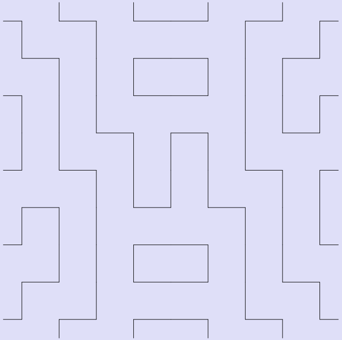
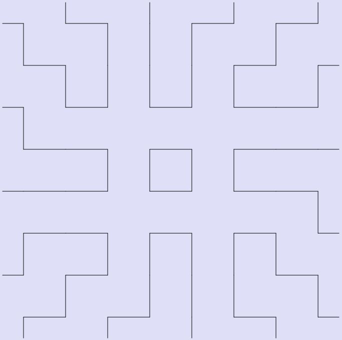


Quarter turn $(4n)$: QTFPL



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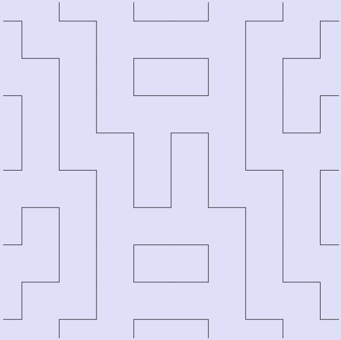

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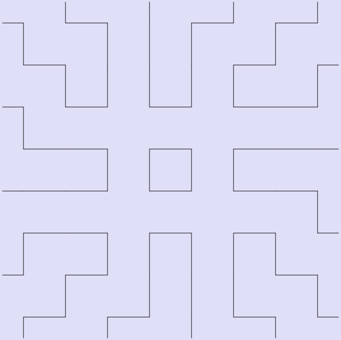
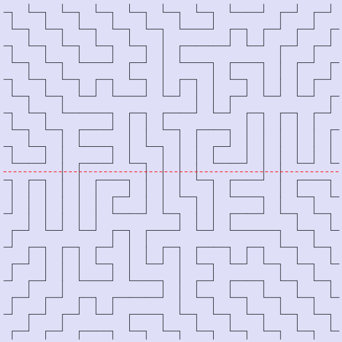


(Quarter-turn invariant ASMs of odd size exist, but do not yield quarter-turn invariant FPLs because the border condition are not quarter-turn invariant for odd size)

Symmetric link patterns

HTFPL (2n)



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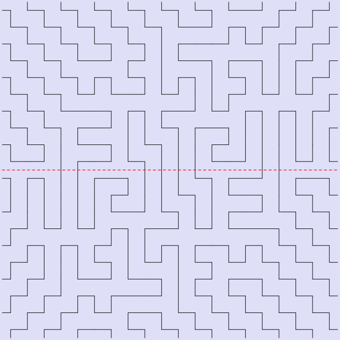
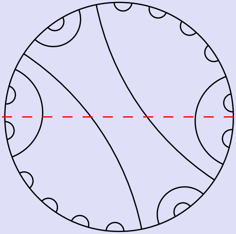


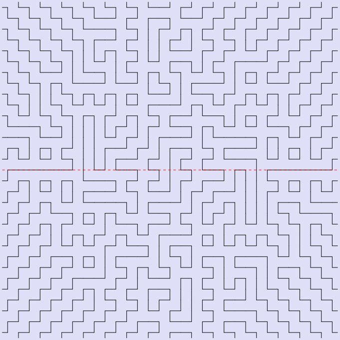
Link pattern



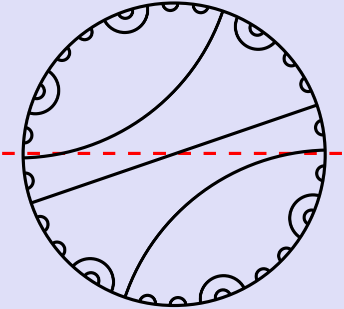
w = abbababababbaabbaaab
(bilateral Dyck)

Symmetric link patterns

HTFPL $(2n + 1)$



Link pattern



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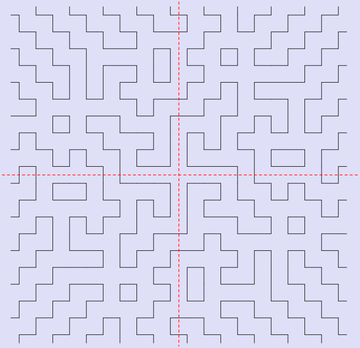
More symmetric link patterns

QTFPL (4n)



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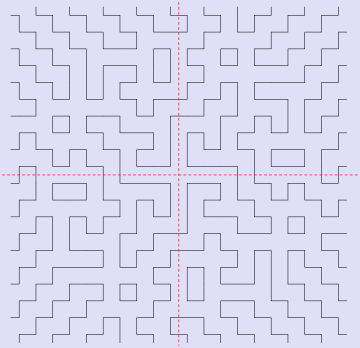
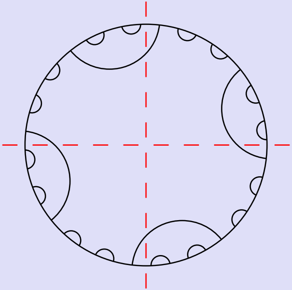


Link pattern



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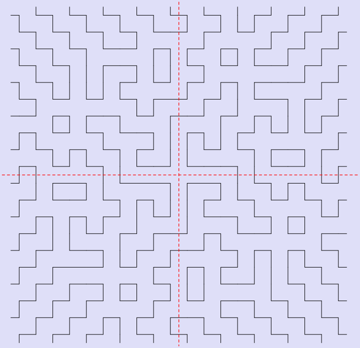

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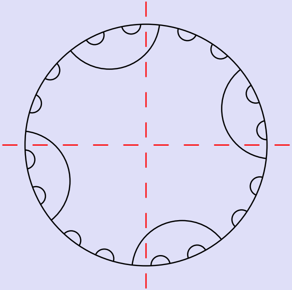

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Link patterns for QTFPLs of size 4n are coded with the same words as those for HTFPLs of size 2n

Known enumerations

Theorem (Zeilberger, Kuperberg)

$$
\begin{aligned}
A(n) &= (-3)^{\binom{n}{2}} \prod_{1\leq i,j\leq n} \frac{3(j-i)+1}{j-i+n} \\
A_{HT}(2n) &= A(n) \times (-3)^{\binom{n}{2}} \prod_{1\leq i,j\leq n} \frac{3(j-i)+2}{j-i+n} \\
A_{QT}(4n) &= A_{HT}(2n) A(n)^2
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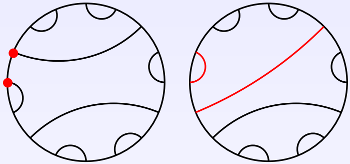
$1, 2, 40, 6860, 9779616\ldots$

Known conjectures on link pattern distribution

Define 2n operators e_i on link patterns of size n:



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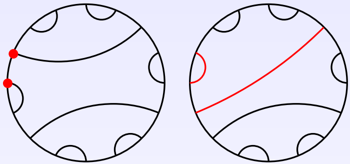

Conjectures (Razumov-Stroganov, de Gier)

$$\sum_{(w',i):e_i(w')=w} A(n,w') = 2nA(n,w)$$

$$\sum_{(w',i):e_i \circ e_{i+n}(w')=w} A_{HT}(n,w') = nA_{HT}(n,w)$$

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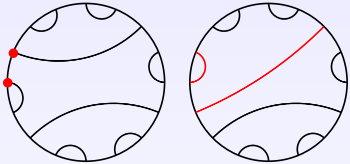

Conjectures (Razumov-Stroganov, de Gier)

Picking a random link pattern with probability proportional to its number of FPLs (respectively, number of HTFPLs), or picking such a random link pattern and then applying to it one of the e_i operators (respectively, symmetrized operators $e_i \circ e_{i+n}$) at random, are equivalent.

Outline

Link patterns for QTFPLs of size 4n

Conjecture (D.)

With link patterns written words of length 8n,

$$\sum_{(w',i):e_i \circ e_{i+2n} \circ e_{i+4n} \circ e_{i+6n}(w')=w} A_{QT}(4n, w') = 2n A_{QT}(4n, w)$$

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Equivalently, up to de Gier's conjecture for even size (and with bilateral Dyck words for patterns),

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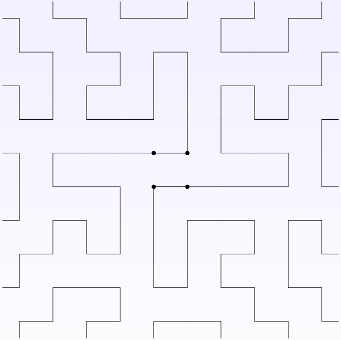
Checked up to $n = 5$ ($A_{QT}(20) = 114,640,611,228$)

Quasi-QTFPLs of size $4n+2$

The "only thing" making it impossible to have quarter-turn symmetric FPLs of size $4n+2$ is the center square; define a quasi-quarter-turn symmetric FPL (qQTFPL) of size $4n+2$ as one having two horizontal edges around the center square, and otherwise QT-symmetric.



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Conjecture (D. 2007)

The number of qQTFPLs of size $4n+2$ is

$$A_{QT}(4n+2) = A_{HT}(2n+1)A(n)A(n+1)$$

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Theorem (Aval, D. 2008)

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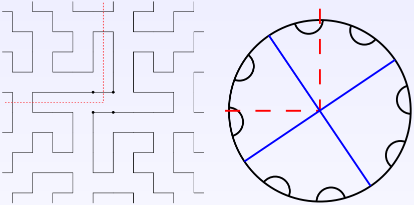
$$A_{QT}(4n + 2) = A_{HT}(2n + 1)A(n)A(n + 1)$$

qQTFPLs have link patterns, too. . .

The link patterns of qQTFPLs can be coded with the same words as those of odd size HTFPLs (any loop touching the center square now coded by c).



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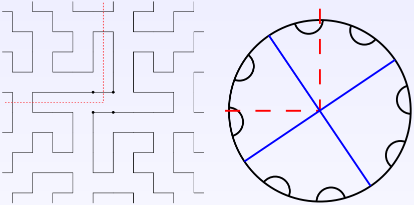

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$$A_{QT}(4n+2, w) = A_{HT}(2n+1, w)A(n)A(n+1).$$

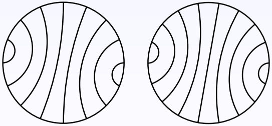
Outline

The rarest link patterns

Note: Wieland's construction proves that the number of (QT,HT-)FPLs with a given pattern is invariant if the pattern is rotated around the circle; for HTFPLs and QTFPLs, this means $A(n, w)$ only depends on w as a cyclic word.

The rarest link pattern for HTFPLs is $w = a^n b^n$ (even size) or $w = a^n b^n c$ (odd size), and one has

$$A_{HT}(2n, a^n b^n) = A_{HT}(2n + 1, a^n b^n c) = 1.$$



Rarest link patterns for (q)QTFPLs

Theorem

The numbers of (quasi-)QTFPLs with the rarest link patterns are

$$\begin{aligned} A_{QT}(4n, a^n b^n) &= A(n)^2 \\ A_{QT}(4n+2, a^n b^n c) &= A(n)A(n+1) \end{aligned}$$

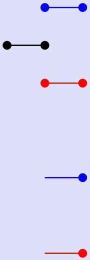
Proof uses the "fixed edges" technique (de Gier; Caselli et al.) and Ciucu's Factorization Theorem.

Fixed edges

Key observation

Some edges appear in all FPL with a given link pattern.

These edges. . .



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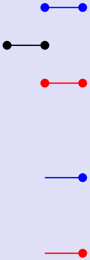
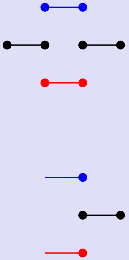
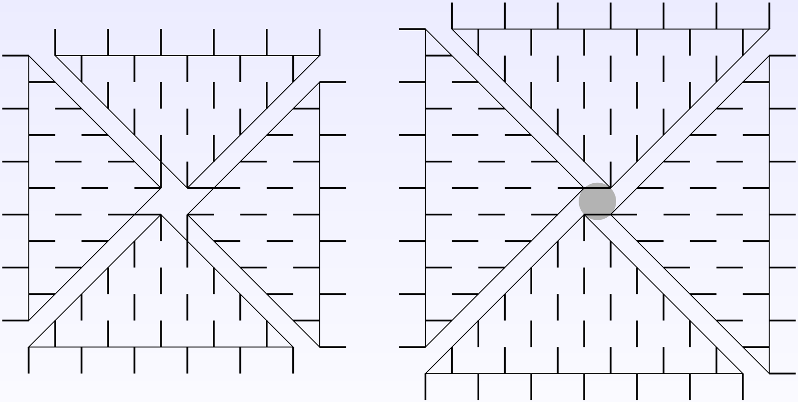


...imply these...

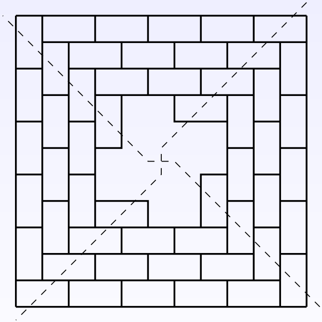


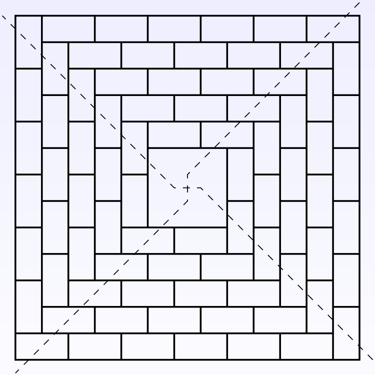
Fixed edges in $\mathcal{A}_{\mathrm{QT}}(4\mathrm{n}, \mathrm{b}^{\mathrm{n}}\mathrm{a}^{\mathrm{n}})$ and $\mathcal{A}_{\mathrm{QT}}(4\mathrm{n}+2; \mathrm{b}^{\mathrm{n}}\mathrm{c}\mathrm{a}^{\mathrm{n}})$



Non fixed edges

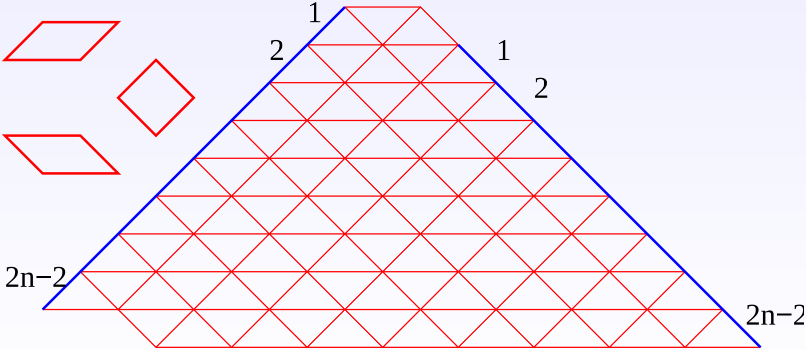
The remaining edges must form a (rotationally symmetric) perfect matching of some graph





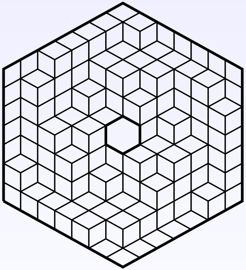
Size 4n

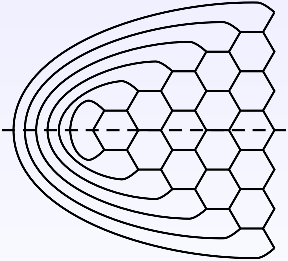
For QTFPLs of size 4n, the quarter-turn invariant matchings are in easy bijection with cyclically symmetric, self-complementary plane partitions of size 2n (cyclically symmetric lozenge tilings of a hexagon), known to be counted by $A(n)^2$



Size $4n + 2$

For size $4n + 2$, the quarter-turn invariant matchings are in easy bijection with "quasi" cyclically symmetric, self-complementary plane partitions of size $2n + 1$ (cyclically symmetric lozenge tilings of a hexagon of odd side, with a hexagonal hole in the center).





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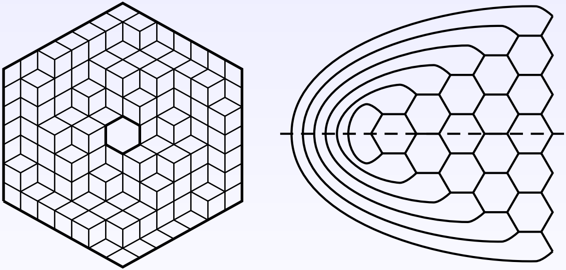

Ciucu's Factorization Theorem lets one express the number of these perfect matchings as a determinant, which Krattenthaler already evaluated to $A(n)A(n + 1)$.

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- qQTFPLs really seem to be the right substitute to nonexistant QTFPLs of size $4n + 2$ (also have a definition in terms of the distributive lattice of ASMs)
- All this calls for bijective proofs; possibly, a unified proof of the Razumov-Stroganov, de Gier, and QTFPL conjectures (ideas welcome!)