The complexity of computing Kronecker coefficients

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Representations of the symmetric group

$$V = \bigoplus_{\nu \vdash n} g_\nu \mathscr{S}_\nu$$

where $g_\nu \geq 0$ denotes the multiplicity of $\mathscr{S}_\nu$ in V (*Schur's lemma*).

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The problem of computing $g_{\lambda,\mu,\nu}$ for given $\lambda, \mu, \nu \vdash n$ is denoted by KronCoeff.

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- $(\mathscr{S}_{\lambda \vdash m} \otimes \mathscr{S}_{\mu \vdash n}) \uparrow_{S_m \times S_n}^{S_{m+n}}$ as an S_{m+n}-representation
  - Littlewood-Richardson coefficients

1. Introduction
   - Motivation
   - Definitions

2. The description from Ballantine and Orellana

3. Proof of the lower bound

Lower bound questions in computational complexity

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- We showed: Computation of Kronecker coefficients is $\#\mathbf{P}$-hard as well
- Positivity of Kronecker coefficients easy to decide? Mulmuley ([Mul07]) conjectures *yes*.

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Definition (#**P**)

The complexity class $\#\mathbf{P}$ consists of the functions $f\colon \{0,1\}^* \to \mathbb{N}$ such that there exists a nondeterministic polynomial-time Turing machine M such that, for all $w \in \{0,1\}^*$,

$$f(w) = \text{ the number of accepting paths of } M \text{ when started with input } w$$

Example

$\mathrm{PERMANENT} : \{\text{undirected bipartite graphs}\} \to \mathbb{N}$,
$G \mapsto |\{\text{perfect matchings in } G\}| \in \#\mathbf{P}$
M chooses nondeterministically a set of edges and checks whether it is a perfect matching.

Definition (Reductions of function problems)

We say that $g\colon \{0,1\}^* \to \mathbb{N}$ *reduces to* $f\colon \{0,1\}^* \to \mathbb{N}$ if the following holds: There are functions $\mathrm{pre} : \{0,1\}^* \to \{0,1\}^*, \mathrm{post} : \mathbb{N} \to \mathbb{N}$, both computable in polynomial time, such that

$$\mathrm{post} \circ f \circ \mathrm{pre} = g.$$

If $\mathrm{post} = \mathrm{id}$, we call the reduction *parsimonious*.

Definition (#**P**-hardness)

f is denoted #**P**-*hard*, if each $g \in \#\mathbf{P}$ reduces to f.

Lemma

Reductions are transitive.

Corollary

Given

- f #**P**-*hard*
- f *reduces to* h

then h *is* #**P**-*hard as well*

Theorem (Main result)

KRONCOEFF *is* #**P**-*hard.*

Definition (Kostka numbers $\mathbf{K}_{\lambda\mu}$)

The *Kostka number* $\mathbf{K}_{\lambda\mu}$ is defined to be number of semistandard Young tableaux of shape λ and type μ.

Definition (The problem KostkaSub)

Given a two-row partition $x = (x_1, x_2) \vdash m$ and $y = (y_1, \ldots, y_\ell)$ with $|y| = m$, compute the Kostka number $\mathbf{K}_{xy}$.

- Narayanan proved that KostkaSub is $\#\mathbf{P}$-hard.

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- Narayanan proved that KostkaSub is $\#\mathbf{P}$-hard.
- Our result: parsimonious reduction from KostkaSub to KronCoeff
- *Purely combinatorial* interpretation of some $g_{\lambda,\mu,\nu}$ from Ballantine and Orellana ([BO07])

- The *reverse reading word* $w^{\leftarrow}$ of a skew tableau T is the sequence of entries in T obtained by reading the entries from right to left and top to bottom, starting with the first row.

Example

		1	2	3
	2	2		
2	3	3		
4				

has shape $(5, 3, 3, 1)/(2, 1)$ and type $(1, 4, 3, 1)$.

$w^{\leftarrow} = (3, 2, 1, 2, 2, 3, 3, 2, 4)$.

- A *lattice permutation* is a sequence $(a_1, a_2, \cdots, a_n)$ such that in any prefix segment $(a_1, a_2, \ldots, a_p)$ the number of i's is at least as large as the number of $(i+1)$'s for all i.

Example

The word (1,1,2,2,3,1,1,3) is a lattice-permutation, because it codes a standard-Tableau:

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Example

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1	2

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3	

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3	4
5	

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1	2	6
3	4	
5		

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Example

The word (1,1,2,2,3,1,1,3) is a lattice-permutation, because it codes a standard-Tableau:

1	2	6	7
3	4		
5			

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1	2	6	7
3	4		
5	8		

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- The concatenation of two lattice permutations is a lattice permutation.

Definition

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ be a partition. A sequence $a = (a_1, a_2, \ldots, a_n)$ is called an *α-lattice permutation* if the concatenation $(1^{\alpha_1}\, 2^{\alpha_2} \cdots n^{\alpha_n} a)$ is a lattice permutation.

- Concatenation:
  Let a be an α-lattice permutation, b be a lattice permutation.
  Then $a\, b$ is an α-lattice permutation.

Definition (Kronecker-Tableaux definition from [BO07])

- λ, α, ν partitions with $\alpha \subseteq \lambda \cap \nu$.
- shape λ/α, type $\nu - \alpha$
- semistandard
- $w^{\leftarrow}$ is an α-lattice permutation
- $\alpha_1 = \alpha_2$ or one of two other technical restrictions

We denote by $k_{\alpha\nu}^{\lambda}$ the number of Kronecker-Tableaux of shape λ/α and type $\nu - \alpha$.

Example

·	·	·	3	3
·	·	·	4	

is a Kronecker-Tableau of shape λ/α and type $\nu - \alpha$ for $\lambda = (5, 4)$, $\nu = (3, 3, 2, 1)$ and $\alpha = (3, 3)$. $w^{\leftarrow} = (3, 3, 4)$ is an α-lattice permutation.

Theorem (Key theorem from [BO07])

Suppose $\mu = (n-p, p), \lambda \vdash n, \nu \vdash n$ *such that* $n \geq 2p$ *and* $\lambda_1 \geq 2p-1$. *Then we have*

$$g_{\lambda,\mu,\nu} = g_{\lambda,(n-p,p),\nu} = \sum_{\substack{\beta \vdash p \\ \beta \subseteq \lambda \cap \nu}} k^{\lambda}_{\beta\nu}.$$

Parsimonious reduction from KostkaSub to KronCoeff:
Given:

- a two-row partition $x = (x_1, x_2) \vdash m$
- $y = (y_1, \ldots, y_\ell)$ with $|y| = m$

we search for

- $n, p \in \mathbb{N}$,
- $\lambda, \nu \vdash n$ with
- $\mathbf{K}_{xy} = g_{\lambda,(n-p,p),\nu}$.

- If we choose λ, ν, n, p correctly, then we have

$$g_{\lambda,(n-p,p),\nu} = \sum_{\substack{\beta \vdash p \\ \beta \subseteq \lambda \cap \nu}} k^{\lambda}_{\beta\nu}.$$

- Assume for a moment that we could choose and fix $\alpha \vdash p, \alpha \in \lambda \cap \nu$ and have only one summand:

$$g_{\lambda,(n-p,p),\nu} = k^{\lambda}_{\alpha\nu}.$$

Then we only have to find $\lambda, \nu \vdash n, \alpha \vdash p$ such that there is a bijection

Young tableaux of shape x, type y

$$\updownarrow$$

Kronecker tableaux of shape λ/α, type $\nu - \alpha$.

Example

$m = 10$, shape $x = (7,3) \vdash m$, type $y = (3,2,2,3)$, $|y| = m$.

1	1	1	2	3	4	4
2	3	4				

- Try

$$\lambda := x, \nu := y, \alpha := ().$$

But for $\alpha = ()$, $w^{\leftarrow}$ is not an α-lattice permutation.

1	1	1	2	3	4	4
2	3	4				

has shape $(x_1 = 7, x_2 = 3) \vdash m$, type $y = (3, 2, 2, 3)$ for m=10.

Lemma

Given any word w of type $y = (y_1, \dots y_\ell)$. Then w is an α-lattice permutation for $\alpha = (\sum_{i>1} y_i, \sum_{i>2} y_i, \dots, y_\ell)$.

Example

Here: $y = (3, 2, 2, 3) \Rightarrow \alpha = (2 + 2 + 3 = 7, 2 + 3 = 5, 3) = (7, 5, 3)$.

·	·	·	·	·	·	·
·	·	·	·	·		
·	·	·				

1	1	1	2	3	4	4
2	3	4				

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·	·	·	·	·	·	·
·	·	·	·	·		
·	·	·				
1						

1	1	1	2	3	4	4
2	3	4				

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Given any word w of type $y = (y_1, \dots y_\ell)$*. Then w is an* α*-lattice permutation for* $\alpha = (\sum_{i>1} y_i, \sum_{i>2} y_i, \dots, y_\ell)$*.*

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·	·	·	·	·	·	·
·	·	·	·	·		
·	·	·				
1	2					

1	1	1	2	3	4	4
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·	·	·	·	·	·	·
·	·	·	·	·		
·	·	·	3			
1	2					

1	1	1	2	3	4	4
2	3	4				

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·	·	·	·	·	·	·
·	·	·	·	·	4	
·	·	·	3			
1	2					

1	1	1	2	3	4	4
2	3	4				

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·	·	·	·	·	·	·	5
·	·	·	·	·	4		
·	·	·	3				
1	2						

1	1	1	2	3	4	4
2	3	4				

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·	·	·	·	·	·	·	5	6
·	·	·	·	·	4			
·	·	·	3					
1	2							

1	1	1	2	3	4	4
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·	·	·	·	·	·	·	5	6	7
·	·	·	·	·	4				
·	·	·	3						
1	2								

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2	3	4				

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·	·	·	·	·	·	·	5	6	7
·	·	·	·	·	4				
·	·	·	3						
1	2	8							

1	1	1	2	3	4	4
2	3	4				

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·	·	·	·	·	·	·	5	6	7
·	·	·	·	·	4				
·	·	·	3	9					
1	2	8							

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Lemma

Given any word w of type $y = (y_1, \dots y_\ell)$. *Then w is an* α*-lattice permutation for* $\alpha = (\sum_{i>1} y_i, \sum_{i>2} y_i, \dots, y_\ell)$.

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·	·	·	·	·	·	·	5	6	7
·	·	·	·	·	4	10			
·	·	·	3	9					
1	2	8							

1	1	1	2	3	4	4
2	3	4				

has shape $(x_1 = 7, x_2 = 3) \vdash m$, type $y = (3, 2, 2, 3)$ for m=10.

- For an aligned embedding, we choose $\alpha = (m, m, 7, 5, 3)$ and shift the type by 2.

·	·	·	·	·	·	·	·	·	·	3	3	3	4	5	6	6
·	·	·	·	·	·	·	·	·	·	4	5	6				
·	·	·	·	·	·	·										
·	·	·	·	·												
·	·	·														

- We have $\lambda = (m + x_1, m + x_2, 7, 5, 3)$ and $\mu = (0, 0, 3, 2, 2, 3) + (m, m, 7, 5, 3) = (10, 10, 10, 7, 5, 3)$.

1	1	1	2	3	4	4
2	3	4				

has shape $(x_1 = 7, x_2 = 3) \vdash m$, type $y = (3, 2, 2, 3)$ for m=10.

- To meet all technical restrictions, we have to add another row:

·	·	·	·	·	·	·	·	·	·	1	1	1	1	1	1	1	1	1	. . .	. . .	. . .	1	1	1
·	·	·	·	·	·	·	·	·	·	4	4	4	5	6	7	7								
·	·	·	·	·	·	·	·	·	·	5	6	7												
·	·	·	·	·	·	·																		
·	·	·	·	·																				
·	·	·																						

- We have $\alpha = (m, m, m, 7, 5, 3)$, $\lambda = (m + M, m + x_1, m + x_2, 7, 5, 3)$ and
  $\mu = (M, 0, 0, 3, 2, 2, 3) + (m, m, m, 7, 5, 3) = (M + m, m, m, 10, 7, 5, 3)$.
- $\mathbf{K}_{xy} = k^{\lambda}_{\alpha\nu}$

1	1	1	2	3	4	4
2	3	4				

has shape $(x_1 = 7, x_2 = 3) \vdash m$, type $y = (3, 2, 2, 3)$ for m=10.

- We have

$$g_{\lambda,(n-p,p),\nu} = \sum_{\substack{\beta \vdash p \\ \beta \subseteq \lambda \cap \nu}} k^{\lambda}_{\beta\nu}.$$

- How can we fix $\alpha \vdash p$ such that only one summand contributes to the sum?

$$g_{\lambda,(n-p,p),\nu} = k^{\lambda}_{\alpha\nu}.$$

- By adjusting λ, μ and choosing the appropriate p.

1	1	1	2	3	4	4
2	3	4				

has shape $(x_1 = 7, x_2 = 3) \vdash m$, type $y = (3, 2, 2, 3)$ for m=10.

·	·	·	·	·	·	·	·	·	·	1	1	1	1	1	1	1	1	...	...	...	1	1	1
·	·	·	·	·	·	·	·	·	·	4	4	4	5	6	7	7							
·	·	·	·	·	·	·	·	·	·	5	6	7											
·	·	·	·	·	·	·	1	1	1														
·	·	·	·	·	1	1	2	2	2														
·	·	·	1	1	2	2	3	3	3														
1	1	1	2	2	3	3	4	4	4														
2	2	2	3	3	4	4	5	5	5														
3	3	3	4	4	5	5	6	6	6														
4	4	4	5	5	6	6	7	7	7														
5	5	5	6	6	7	7																	
6	6	6	7	7																			
7	7	7																					

- For semistandardness, each left column can contain at most 7 entries. $\alpha \vdash p$ has to cover the other 45 boxes. Set $p := 45$ to fix $\alpha \vdash 45$.
  We now have for such $\lambda \vdash n, \nu \vdash n, p$ that $g_{\lambda,(n-p,p),\nu} = \sum_{\substack{\beta \vdash p \\ \beta \subseteq \lambda \cap \nu}} k^{\lambda}_{\beta\nu} = k^{\lambda}_{\alpha\nu}$.

1	1	1	2	3	4	4
2	3	4				

has shape $(x_1 = 7, x_2 = 3) \vdash m$, type $y = (3, 2, 2, 3)$ for m=10.

·	·	·	·	·	·	·	·	·	·	1	1	1	1	1	1	1	1	...	...	...	1	1	1
·	·	·	·	·	·	·	·	·	·	4	4	4	5	6	7	7							
·	·	·	·	·	·	·	·	·	·	5	6	7											
·	·	·	·	·	·	·	1	1	1														
·	·	·	·	·	1	1	2	2	2														
·	·	·	1	1	2	2	3	3	3														
1	1	1	2	2	3	3	4	4	4														
2	2	2	3	3	4	4	5	5	5														
3	3	3	4	4	5	5	6	6	6														
4	4	4	5	5	6	6	7	7	7														
5	5	5	6	6	7	7																	
6	6	6	7	7																			
7	7	7																					

- $w^{\leftarrow}$ is still an α-lattice permutation as it is the concatenation of an α-lattice permutation and a lattice permutation.
  So this is a Kronecker tableau and $\mathbf{K}_{xy} = k^{\lambda}_{\alpha\nu} = g_{\lambda,(n-p,p),\nu}$.

Proposition (Reduction formally)

Let $x = (x_1, x_2) \vdash m$ *and* $y = (y_1, \ldots, y_\ell)$ *with* $|y| = m > 0$ *be given. We define*

$$\alpha := (m, m, m, \sum_{j>1} y_j, \ldots, \sum_{j>\ell-1} y_j)$$

and we set $p := |\alpha|$ *and* $M := 2p - 1 - m$. *Consider*

$$\lambda := (M + m, m + x_1, m + x_2, \underbrace{m, m, \ldots, m}_{\ell \text{ times}}) \alpha$$

$$\nu := (M + m, m, m, m + y_1, m + y_2, \ldots, m + y_{\ell-1}, m + y_\ell) + \alpha.$$

and write $n := |\lambda|$. *Then we have* $\mathbf{K}_{xy} = g_{\lambda,(n-p,p),\nu}$.

- This proves the $\#\mathbf{P}$-hardness of KRONCOEFF.

Thank you.

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