The complexity of computing Kronecker coefficients

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- Introduction
 - Motivation
 - Definitions

2 The description from Ballantine and Orellana

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Representations of the symmetric group

$$V = \bigoplus_{\nu \vdash n} \mathsf{g}_{\nu} \mathscr{S}_{\nu}$$

where $g_{\nu} \geq 0$ denotes the multiplicity of \mathscr{S}_{ν} in V (*Schur's lemma*).

Plethysm problem Describe the decompositions:

• $\mathscr{S}_{\lambda \vdash n} \otimes \mathscr{S}_{\mu \vdash n}$ as an S_n -representation



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The problem of computing $g_{\lambda,\mu,\nu}$ for given $\lambda,\mu,\nu\vdash n$ is denoted by KRONCOEFF.

- $(\mathscr{S}_{\lambda \vdash m} \otimes \mathscr{S}_{\mu \vdash n}) \uparrow_{S_{-} \times S_{-}}^{S_{m+n}}$ as an S_{m+n} -representation
 - Littlewood-Richardson coefficients

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Proof of the lower bound

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- Positivity of Kronecker coefficients easy to decide? Mulmuley ([Mul07]) conjectures *yes*.

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Definition $(\#\mathbf{P})$

The complexity class $\#\mathbf{P}$ consists of the functions $f: \{0,1\}^* \to \mathbb{N}$ such that there exists a nondeterministic polynomial-time Turing machine M such that, for all $w \in \{0,1\}^*$,

f(w) = the number of accepting paths of M when started with input w

Example

PERMANENT : {undirected bipartite graphs} $\rightarrow \mathbb{N}$,

 $G \mapsto |\{\text{perfect matchings in } G\}| \in \#\mathbf{P}$

M chooses nondeterministically a set of edges and checks whether it is a perfect matching.

Definition (Reductions of function problems)

We say that $g: \{0,1\}^* \to \mathbb{N}$ reduces to $f: \{0,1\}^* \to \mathbb{N}$ if the following holds: There are functions pre : $\{0,1\}^* \to \{0,1\}^*$, post : $\mathbb{N} \to \mathbb{N}$, both computable in polynomial time, such that

$$post \circ f \circ pre = g$$
.

If post = id, we call the reduction *parsimonious*.

Definition (#P-hardness)

f is denoted $\#\mathbf{P}$ -hard, if each $g \in \#\mathbf{P}$ reduces to f.

Lemma

Reductions are transitive.

Corollary

Given

- f #P-hard
- f reduces to h

then h is #P-hard as well

Theorem (Main result)

KRONCOEFF is #P-hard.

Definition (Kostka numbers $\mathbf{K}_{\lambda\mu}$)

The Kostka number $\mathbf{K}_{\lambda\mu}$ is defined to be number of semistandard Young tableaux of shape λ and type μ .

Definition (The problem KostkaSub)

Given a two-row partition $x=(x_1,x_2)\vdash m$ and $y=(y_1,\ldots,y_\ell)$ with |y|=m, compute the Kostka number \mathbf{K}_{xy} .

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- Narayanan proved that KostkaSub is #P-hard.
- Our result: parsimonious reduction from KostkaSub to KronCoeff
- Purely combinatorial interpretation of some $g_{\lambda,\mu,\nu}$ from Ballantine and Orellana ([BO07])

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 The reverse reading word w[←] of a skew tableau T is the sequence of entries in T obtained by reading the entries from right to left and top to bottom, starting with the first row.

Example



has shape (5,3,3,1)/(2,1) and type (1,4,3,1).

 $w^{\leftarrow} = (3, 2, 1, 2, 2, 3, 3, 2, 4).$

Example

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1 2 3 4

Example

The word (1,1,2,2,3,1,1,3) is a lattice-permutation, because it codes a standard-Tableau:

1 2

Example

1	2	6
3	4	
5		

Example

1	2	6	7
3	4		
5			

Example

1	2	6	7
3	4		
5	8		

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The word (1,1,2,2,3,1,1,3) is a lattice-permutation, because it codes a standard-Tableau:

1	2	6	7
3	4		
5	8		

 The concatenation of two lattice permutations is a lattice permutation.

Definition

Let $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_\ell)$ be a partition. A sequence $a=(a_1,a_2,\ldots,a_n)$ is called an α -lattice permutation if the concatenation $(1^{\alpha_1} 2^{\alpha_2} \cdots n^{\alpha_n} a)$ is a lattice permutation.

Concatenation:

Let a be an α -lattice permutation, b be a lattice permutation. Then a b is an α -lattice permutation.

Definition (Kronecker-Tableaux definition from [BO07])

- λ, α, ν partitions with $\alpha \subseteq \lambda \cap \nu$.
- shape λ/α , type $\nu-\alpha$
- semistandard
- w^{\leftarrow} is an α -lattice permutation
- $\alpha_1 = \alpha_2$ or one of two other technical restrictions

We denote by $k_{\alpha\nu}^{\lambda}$ the number of Kronecker-Tableaux of shape λ/α and type $\nu-\alpha$.

Example



is a Kronecker-Tableau of shape λ/α and type $\nu-\alpha$ for $\lambda=(5,4)$, $\nu=(3,3,2,1)$ and $\alpha=(3,3)$. $w^{\leftarrow}=(3,3,4)$ is an α -lattice permutation.

Theorem (Key theorem from [BO07])

Suppose $\mu = (n - p, p), \lambda \vdash n, \nu \vdash n$ such that $n \ge 2p$ and $\lambda_1 \ge 2p - 1$. Then we have

$$g_{\lambda,\mu,
u}=g_{\lambda,(n-p,p),
u}=\sum_{egin{array}{c}eta\vdash p\ eta\subseteq\lambda\cap
u\end{array}}k_{eta
u}^{\lambda}.$$

Parsimonious reduction from KOSTKASUB to KRONCOEFF: Given:

- a two-row partition $x = (x_1, x_2) \vdash m$
- $y = (y_1, ..., y_\ell)$ with |y| = m

we search for

- $n, p \in \mathbb{N}$,
- $\lambda, \nu \vdash n$ with
- $K_{xy} = g_{\lambda,(n-p,p),\nu}$.

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• If we choose λ, ν, n, p correctly, then we have

$$g_{\lambda,(n-p,p),
u} = \sum_{egin{subarray}{c}eta \vdash p \ eta \subseteq \lambda \cap
u\end{array}} k_{eta
u}^{\lambda}.$$

• Assume for a moment that we could choose and fix $\alpha \vdash p, \alpha \in \lambda \cap \nu$ and have only one summand:

$$g_{\lambda,(n-p,p),\nu}=k_{\alpha\nu}^{\lambda}.$$

Then we only have to find $\lambda, \nu \vdash n, \alpha \vdash p$ such that there is a bijection Young tableaux of shape x, type y



Kronecker tableaux of shape λ/α , type $\nu-\alpha$.

Example

$$m = 10$$
, shape $x = (7,3) \vdash m$, type $y = (3,2,2,3), |y| = m$.

1	1	1	2	3	4	4
2	3	4				

Try

$$\lambda := x, \nu := y, \alpha := ().$$

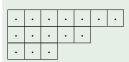
But for $\alpha = ()$, w^{\leftarrow} is not an α -lattice permutation.

Lemma

Given any word w of type $y=(y_1,\ldots y_\ell)$. Then w is an α -lattice permutation for $\alpha=(\sum_{i>1}y_i,\sum_{i>2}y_i,\ldots,y_\ell)$.

Example

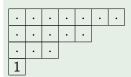
Here:
$$y = (3, 2, 2, 3) \Rightarrow \alpha = (2 + 2 + 3 = 7, 2 + 3 = 5, 3) = (7, 5, 3).$$



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Given any word w of type $y = (y_1, \ldots, y_\ell)$. Then w is an α -lattice permutation for $\alpha = (\sum_{i>1} y_i, \sum_{i>2} y_i, \ldots, y_\ell)$.

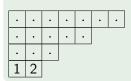
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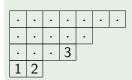
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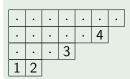
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Example

• For an aligned embedding, we choose $\alpha = (m, m, 7, 5, 3)$ and shift the type by 2.

		•	•		3	3	3	4	5	6	6
					4	5	6				

• We have $\lambda = (m + x_1, m + x_2, 7, 5, 3)$ and $\mu = (0,0,3,2,2,3) + (m,m,7,5,3) = (10,10,10,7,5,3).$

To meet all technical restrictions, we have to add another row:

					1	1	1	1	1	1	1	1	1	 ļ	 . 1	1	1
					4	4	4	5	6	7	7						
					5	6	7										

- We have $\alpha = (m, m, m, 7, 5, 3), \lambda = (m + M, m + x_1, m + x_2, 7, 5, 3)$ and $\mu = (M, 0, 0, 3, 2, 2, 3) + (m, m, m, 7, 5, 3) = (M + m, m, m, 10, 7, 5, 3).$
- $\mathbf{K}_{xy} = k_{\alpha y}^{\lambda}$

We have

$$g_{\lambda,(n-
ho,
ho),
u} = \sum_{egin{subarray}{c}etadashed{}_{eta
u} \ eta^{\lambda}_{eta
u}.$$

• How can we fix $\alpha \vdash p$ such that only one summand contributes to the sum?

$$g_{\lambda,(n-p,p),\nu}=k_{\alpha\nu}^{\lambda}.$$

• By adjusting λ, μ and choosing the appropriate p.

										1	1	1	1	1	1	1	1	
										4	4	4	5	6	7	7		
										5	6	7						
							1	1	1									
					1	1	2	2	2									
			1	1	2	2	3	3	3									
1	1	1	2	2	3	3	4	4	4									
2	2	2	3	3	4	4	5	5	5									
3	3	3	4	4	5	5	6	6	6									
4	4	4	5	5	6	6	7	7	7									
5	5	5	6	6	7	7												
6	6	6	7	7														
7	7	7																

• For semistandardness, each left column can contain at most 7 entries. $\alpha \vdash p$ has to cover the other 45 boxes. Set p:=45 to fix $\alpha \vdash 45$. We now have for such $\lambda \vdash n, \nu \vdash n, p$ that $g_{\lambda,(n-p,p),\nu} = \sum_{\beta \in \lambda \ni \nu} k_{\beta\nu}^{\lambda} = k_{\alpha\nu}^{\lambda}$.

										1	1	Ī
										4	4	Ī
										5	6	Ī
							1	1	1			
					1	1	2	2	2			
			1	1	2	2	3	3	3			
1	1	1	2	2	3	3	4	4	4			
2	2	2	3	3	4	4	5	5	5			
3	3	3	4	4	5	5	6	6	6			
4	4	4	5	5	6	6	7	7	7			
5	5	5	6	6	7	7						
6	6	6	7	7								
7	7	7										

• w^{\leftarrow} is still an α -lattice permutation as it is the concatenation of an α -lattice permutation and a lattice permutation.

6 7 7

5

So this is a Kronecker tableau and $\mathbf{K}_{xy} = k_{\alpha\nu}^{\lambda} = g_{\lambda,(n-p,p),\nu}$.

Proposition (Reduction formally)

Let $x = (x_1, x_2) \vdash m$ and $y = (y_1, \dots, y_\ell)$ with |y| = m > 0 be given. We define

$$\alpha := (m, m, m, \sum_{j>1} y_j, \ldots, \sum_{j>\ell-1} y_j)$$

and we set $p := |\alpha|$ and M := 2p - 1 - m. Consider

$$\lambda := (M + m, m + x_1, m + x_2, \underbrace{m, m, \dots, m}_{\ell \text{ times}}) \alpha$$

$$\nu := (M + m, m, m, m + y_1, m + y_2, \dots, m + y_{\ell-1}, m + y_{\ell}) + \alpha.$$

and write $n := |\lambda|$. Then we have $K_{xy} = g_{\lambda,(n-p,p),\nu}$.

• This proves the #P-hardness of KronCoeff.

Thank you.



Cristina M. Ballantine and Rosa C. Orellana.

A combinatorial interpretation for the coefficients in the Kronecker product $s_{(n-p,p)} * s_{\lambda}$.

Sém. Lothar. Combin., 54A:Art. B54Af, 29 pp. (electronic), 2005/07.



Allen Knutson and Terence Tao.

The honeycomb model of $GL_n(\mathbf{C})$ tensor products. I. Proof of the saturation conjecture.

J. Amer. Math. Soc., 12(4):1055-1090, 1999.



Ketan D. Mulmuley and Milind Sohoni.

Geometric complexity theory. I. An approach to the P vs. NP and related problems.

SIAM J. Comput., 31(2):496-526 (electronic), 2001.

Ketan D. Mulmuley and Milind Sohoni.

Geometric complexity theory III: On deciding positivity of Littlewood-Richardson coefficients. cs.ArXive preprint cs.CC/0501076, 2005.

Ketan D. Mulmuley and Milind Sohoni.

Geometric complexity theory II: Towards explicit obstructions for embeddings among class varieties.

cs.ArXive preprint cs.CC/0612134. To appear in SIAM J. Comput., 2006.



Ketan D. Mulmuley.

Geometric complexity theory VI: The flip via saturated and positive integer programming in representation theory and algebraic geometry,. Technical Report TR-2007-04, Computer Science Department, The University of Chicago, 2007.



Hariharan Narayanan.

On the complexity of computing Kostka numbers and Littlewood-Richardson coefficients.

J. Algebraic Combin., 24(3):347-354, 2006.



Volker Strassen.

Rank and optimal computation of generic tensors.

Lin. Alg. Appl., 52:645-685, 1983.