Staircase Macdonald polynomials and the q-discriminant

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Formal Power Series and Algebraic Combinatorics 2008

Introduction

Let $\mathbb{X} = \{x_1, \ldots, x_n\}$.

q-discriminant:

$$\mathcal{D}_1(\mathbb{X}, q) := \prod_{i \neq j} (qx_i - x_j).$$

Example: $n = 4$

$$\mathcal{D}_1(\mathbb{X}, q) = \underbrace{\begin{matrix} (qx_1 - x_4) \\ (qx_1 - x_3)(qx_2 - x_4) \\ (qx_1 - x_2)(qx_2 - x_3)(qx_3 - x_4) \end{matrix}}_{i<j} \cdot \underbrace{\begin{matrix} (qx_4 - x_1) \\ (qx_4 - x_2)(qx_3 - x_1) \\ (qx_4 - x_3)(qx_3 - x_2)(qx_2 - x_1) \end{matrix}}_{i>j}$$

Introduction

We define the "polarized powers" of the q-discriminant by:

$$\mathcal{D}_k(\mathbb{X}, q) := \prod_{i=1}^{k} \mathcal{D}_1(\mathbb{X}, q^{2l-1})$$

Example: for $k = 3$

$$\mathcal{D}_3(\mathbb{X}, q) = \mathcal{D}_1(\mathbb{X}, q^1).\mathcal{D}_1(\mathbb{X}, q^3).\mathcal{D}_1(\mathbb{X}, q^5)$$

Aim 1

$$\mathcal{D}_k(\mathbb{X}, q^{-1/2}) = cst.P_{2k\rho}(\mathbb{X}, q, q^{-\frac{2k-1}{2}})$$

is a Macdonald polynomial indexed by the staircase partition
$2k\rho = [2k(n-1), \ldots, 2k, 0]$.

Example: for $k = 2$ and $n = 4$,

$$\prod_{i \neq j}(q^{-1/2}x_i - x_j)\prod_{i \neq j}(q^{-3/2}x_i - x_j) = q^{-12}P_{[12,8,4,0]}(\mathbb{X}, q, q^{-3/2})$$

is a product of q-discriminants.

Aim 2

$$\mathcal{D}_k(\mathbb{X}, q) = \sum_\lambda c_\lambda(q) S_\lambda$$

We give a caracterisation of $\{\lambda | c_\lambda(q) \neq 0\}$.

Generalisation of a result of King, Toumazet and Wybourne ($k = 1$) (2004).

When $q = 1$, $\mathcal{D}_k(\mathbb{X}, 1)$ is exactly an even power of the vanderMonde.

$$\mathcal{D}_k(\mathbb{X}, 1) = \prod_{i \leq j} (x_i - x_j)^{2k}$$

Used to describe the fractionnal quantum Hall effect (*Laughlin's wave functions, Coulomb gases and expansions of the discriminant*, Di Francesco, Gaudinn, Itzykson, Lesage, 1994).

Plan

1. Macdonald operators and functions
2. The “polarized powers” of the q-discriminant and the Macdonald polynomials
3. The “polarized powers” of the q-discriminant and the Schur functions

Plan

1. Macdonald operators and functions
   - Macdonald polynomials
   - Macdonald operator
2. The “polarized powers” of the q-discriminant and the Macdonald polynomials
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Macdonald polynomials

(q, t)-deformation of the usual scalar product :

$$< p_\lambda | p_\mu >_{q,t} = \delta_{\lambda\mu} z_\lambda \prod_{i=1}^{l(\lambda)} \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}}$$

The family of Macdonald polynomials is the unique basis of the symmetric functions which:

- are orthogonal for the (q, t)-deformed scalar product,
- verify
$$P_\lambda(\mathbb{X}; q, t) = m_\lambda(\mathbb{X}) + \sum_{\mu < \lambda} u_{\mu,\lambda} m_\mu(\mathbb{X}).$$

Macdonald operator $\mathcal{M}_1$

Resultant: $R(\mathbb{X}, \mathbb{Y}) = \prod_{\substack{x\in\mathbb{X}\\ y\in\mathbb{Y}}} (x - y)$.

Macdonald operator:

$$\mathcal{M}_1.f(\mathbb{X}) = (f(\mathbb{X} - (1-q)x_1)R(tx_1; \mathbb{X} - x_1)) \quad \partial_1 \ldots \partial_{n-1}$$

where ∂_i are the divided differences defined by

$$f(x_1, \ldots, x_n)\partial_i := \frac{f(x_1, \ldots, x_i, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_{i+1}, x_i, \ldots, x_n)}{x_i - x_{i+1}}$$

Example: for $n = 2$ we have

$$\mathcal{M}_1.f(x_1, x_2) = (f(q.x_1, x_2)(t.x_1 - x_2)) \quad \partial_1.$$

Macdonald polynomials are the eigenfunctions of $\mathcal{M}_1$

Eigenspaces of the Macdonal operator $\mathcal{M}_1$: dimension 1 for q and t generic.

Eigenfunctions :

$$\mathcal{M}_1.P_\lambda(\mathbb{X}, q, t) = [[\lambda]]_{q,t}.P_\lambda(\mathbb{X}, q, t)$$

Eigenvalues :

$$[[\lambda]]_{q,t} := q^{\lambda_1} t^{n-1} + q^{\lambda_2} t^{n-2} + \cdots + q^{\lambda_n}.$$

Plan

The $\mathcal{D}_k$ are Macdonald polynomials

Theorem

$$\mathcal{D}_k(\mathbb{X}, q^{-1/2}) = (-q^{-1/2})^{k^2 n(n-1)/2}.P_{2k\rho}(\mathbb{X}; q, q^{-\frac{2k-1}{2}})$$

Example: for $k = 2$ and $n = 4$ we have:

$$\prod_{i \neq j}(q^{-1/2}x_i - x_j)\prod_{i \neq j}(q^{-3/2}x_i - x_j) = q^{-12}P_{[12,8,4,0]}(\mathbb{X}, q, q^{-3/2})$$

Main steps of the proof:

- $\mathcal{D}_k$ is an eigenfunction. We calculate the associated eigenvalue.
- It belongs to an eigenspace of dimention 1.
- We compute the constant.

$\mathcal{D}_k$ is an eigenfunction

$\mathcal{D}_2(x_1, x_2, x_3; q^{-1/2})$ is an eigenfunction of $\mathcal{M}_1$.

By definition of $\mathcal{M}_1$, we have

$$\mathcal{M}_1.\mathcal{D}_2(x_1, x_2, x_3, q^{-1/2}) = \\ \left(\mathcal{D}_2(qx_1, x_2, x_3, q^{-1/2}).R(q^{-\frac{2k-1}{2}}x_1, \{x_2, x_3\})\right)\partial_1\partial_2.$$

$$\mathcal{D}_2(qx_1, x_2, x_3, q^{-1/2})R(q^{-\frac{2k-1}{2}}x_1, \{x_2, x_3\}) =$$

$$\mathcal{D}_2(qx_1, x_2, x_3, q^{-1/2})R(q^{-\frac{2k-1}{2}}x_1, \{x_2, x_3\}) =$$

$$\frac{(q^{-3/2}x_3 - qx_1)}{(q^{-3/2}x_2 - qx_1).(q^{-3/2}x_3 - x_2)}$$

$$\frac{(q^{-1/2}x_3 - qx_1)}{(q^{-1/2}x_2 - qx_1).(q^{-1/2}x_3 - x_2)} \qquad \frac{(q^{1/2}x_1 - x_3)}{(q^{1/2}x_1 - x_2).(q^{-1/2}x_2 - x_3)}$$

$$\frac{(q^{-1/2}x_1 - x_3)}{(q^{-1/2}x_1 - x_2).(q^{-3/2}x_2 - x_3)} \qquad \frac{(q^{-3/2}x_1 - x_3)}{(q^{-3/2}x_1 - x_2)}$$

$$\mathcal{D}_2(qx_1, x_2, x_3, q^{-1/2})R(q^{-\frac{2k-1}{2}}x_1, \{x_2, x_3\}) =$$

$$\frac{(q^{-3/2}x_3 - qx_1)}{(q^{-3/2}x_2 - qx_1).(q^{-3/2}x_3 - x_2)}$$

$$\frac{(q^{-1/2}x_3 - qx_1)}{(q^{-1/2}x_2 - qx_1).(q^{-1/2}x_3 - x_2)} \qquad \frac{(q^{1/2}x_1 - x_3)}{(q^{1/2}x_1 - x_2).(q^{-1/2}x_2 - x_3)}$$

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$$\mathcal{D}_2(qx_1, x_2, x_3, q^{-1/2})R(q^{-\frac{2k-1}{2}}x_1, \{x_2, x_3\}) =$$

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$$\frac{(q^{-1/2}x_3 - qx_1)}{(q^{-1/2}x_2 - qx_1).(q^{-1/2}x_3 - x_2)} \qquad \frac{(q^{1/2}x_1 - x_3)}{(q^{1/2}x_1 - x_2).(q^{-1/2}x_2 - x_3)}$$

$$\frac{(q^{-1/2}x_1 - x_3)}{(q^{-1/2}x_1 - x_2).(q^{-3/2}x_2 - x_3)} \qquad \frac{(q^{-3/2}x_1 - x_3)}{(q^{-3/2}x_1 - x_2)}.$$

$$\mathcal{D}_2(qx_1, x_2, x_3, q^{-1/2})R(q^{-\frac{2k-1}{2}}x_1, \{x_2, x_3\}) =$$

$$\begin{tikzcd}
 & \begin{matrix}(q^{-3/2}x_3 - qx_1)\\ (q^{-3/2}x_2 - qx_1).\color{red}{(q^{-3/2}x_3 - x_2)}\end{matrix} \arrow[dl] & \\
\begin{matrix}(q^{-1/2}x_3 - qx_1)\\ (q^{-1/2}x_2 - qx_1).\color{red}{(q^{-1/2}x_3 - x_2)}\end{matrix} \arrow[rr] & & \begin{matrix}(q^{1/2}x_1 - x_3)\\ (q^{1/2}x_1 - x_2).\color{red}{(q^{-1/2}x_2 - x_3)}\end{matrix} \arrow[dll] \\
\begin{matrix}(q^{-1/2}x_1 - x_3)\\ (q^{-1/2}x_1 - x_2).\color{red}{(q^{-3/2}x_2 - x_3)}\end{matrix} \arrow[rr] & & \begin{matrix}(q^{-3/2}x_1 - x_3)\\ (q^{-3/2}x_1 - x_2).\end{matrix}
\end{tikzcd}$$

$$\mathcal{D}_2(qx_1, x_2, x_3, q^{-1/2})R(q^{-\frac{2k-1}{2}}x_1, \{x_2, x_3\}) =$$

$$\frac{(q^{-3/2}x_3 - qx_1)}{(q^{-3/2}x_2 - qx_1)}$$

$$\frac{(q^{-1/2}x_3 - qx_1)}{(q^{-1/2}x_2 - qx_1).(q^{-3/2}x_3 - x_2)} \qquad \frac{(q^{1/2}x_1 - x_3)}{(q^{1/2}x_1 - x_2).(q^{-1/2}x_3 - x_2)}$$

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$$\mathcal{D}_2(qx_1, x_2, x_3, q^{-1/2})R(q^{-\frac{2k-1}{2}}x_1, \{x_2, x_3\}) =$$

$$\begin{tikzcd}
 & \begin{array}{c}(q^{-3/2}x_3 - qx_1)\\ (q^{-3/2}x_2 - qx_1)\end{array} & \\
\begin{array}{c}(q.q^{-3/2}x_3 - qx_1)\\ (q.q^{-3/2}x_2 - qx_1).(q^{-3/2}x_3 - x_2)\end{array} \arrow[ur, "q^2"] & & \begin{array}{c}(q^{1/2}x_1 - q^{1/2}.q^{-1/2}x_3)\\ (q^{1/2}x_1 - q^{1/2}.q^{-1/2}x_2).(q^{-1/2}x_3 - x_2)\end{array} \arrow[ul, "q"']
\end{tikzcd}$$

$$\begin{array}{c}(q^{-1/2}x_1 - x_3)\\ (q^{-1/2}x_1 - x_2).(q^{-1/2}x_2 - x_3)\end{array} \qquad \begin{array}{c}(q^{-3/2}x_1 - x_3)\\ (q^{-3/2}x_1 - x_2).(q^{-3/2}x_2 - x_3)\end{array}$$

$$\mathcal{D}_2(qx_1, x_2, x_3, q^{-1/2})R(q^{-\frac{2k-1}{2}}x_1, \{x_2, x_3\}) =$$

$$\frac{(q^{5/2}x_1 - x_3)}{(q^{5/2}x_1 - x_2)}$$

$$\frac{(q^{-3/2}x_3 - x_1)}{(q^{-3/2}x_2 - x_1).(q^{-3/2}x_3 - x_2)} \qquad \frac{(q^{-1/2}x_3 - x_1)}{(q^{-1/2}x_2 - x_1).(q^{-1/2}x_3 - x_2)}$$

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$$\mathcal{D}_2(qx_1, x_2, x_3, q^{-1/2})R(q^{-\frac{2k-1}{2}}x_1, \{x_2, x_3\}) = R(q^{\frac{2k-1}{2}}x_1, \{x_2, x_3\})$$

$$\frac{(q^{-3/2}x_3 - x_1)}{(q^{-3/2}x_2 - x_1).(q^{-3/2}x_3 - x_2)} \qquad \frac{(q^{-1/2}x_3 - x_1)}{(q^{-1/2}x_2 - x_1).(q^{-1/2}x_3 - x_2)}$$

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$$\mathcal{D}_2(qx_1, x_2, x_3, q^{-1/2})R(q^{-\frac{2k-1}{2}}x_1, \{x_2, x_3\}) = R(q^{\frac{2k-1}{2}}x_1, \{x_2, x_3\})\mathcal{D}_2(x_1, x_2, x_3, q^{-1/2})$$

$$\mathcal{D}_2(qx_1, x_2, x_3, q^{-1/2})R(q^{-\frac{2k-1}{2}}x_1, \{x_2, x_3\}) = R(q^{\frac{2k-1}{2}}x_1, \{x_2, x_3\})\mathcal{D}_2(x_1, x_2, x_3, q^{-1/2})$$

$\mathcal{D}_2(x_1, x_2, x_3; q^{-1/2})$ is an eigenvector of $\mathcal{M}_1$

We have just shawn

$$\mathcal{D}_k(qx_1, x_2, \ldots, x_n, q^{-1/2})R(q^{-\frac{2k-1}{2}}x_1, \mathbb{X}) = R(q^{\frac{2k+1}{2}}x_1, \mathbb{X})\mathcal{D}_k(\mathbb{X}, q^{-1/2}).$$

We deduce from $\mathcal{M}_1$ definition that:

$$\mathcal{M}_1.\mathcal{D}_k(\mathbb{X}, q^{-1/2}) = R(q^{\frac{2k+1}{2}}x_1, \mathbb{X})\mathcal{D}_k(\mathbb{X}, q^{-1/2})\partial_1 \ldots \partial_k.$$

Since $\mathcal{D}_k(\mathbb{X}, q)$ is symmetric, we deduce that

$$\mathcal{M}_1.\mathcal{D}_k(\mathbb{X}, q^{-1/2}) = \underbrace{R(q^{\frac{2k+1}{2}}x_1, \mathbb{X})\partial_1 \ldots \partial_{n-1}}_{\text{eigenvalue}}.\mathcal{D}_k(\mathbb{X}, q^{-1/2}).$$

End of the proof

The eigenvalue associated to $\mathcal{D}_k(\mathbb{X})$ is:

$$R(q^{\frac{2k+1}{2}}x_1,\mathbb{X})\partial_1\ldots\partial_{n-1} = [[2k\rho]]_{q,q^{\frac{1-2k}{2}}}.$$

Under the specialisation $t \longrightarrow q^{\frac{1-2k}{2}}$, the Macdonald polynomial $P_{2k\rho}(\mathbb{X},q,q^{\frac{1-2k}{2}})$ belongs to an eigenspace of $\mathcal{M}_1$ of dimension 1.

$$[[\lambda]]_{q,q^{\frac{1-2k}{2}}} \neq [[2k\rho]]_{q,q^{\frac{1-2k}{2}}} \text{ for all } \lambda \neq 2k\rho.$$

We deduce that

$$D_k(\mathbb{X},q^{-1/2}) = cst.P_{2k\rho}(\mathbb{X},q,q^{(1-2k)/2}).$$

We get the constant *cst* by computing the coefficients of the dominant monomial of $D_k(\mathbb{X},q^{-1/2})$.

Plan

1. Macdonald operators and functions

2. The "polarized powers" of the q-discriminant and the Macdonald polynomials

3. The "polarized powers" of the q-discriminant and the Schur functions
    - (n, m)-admissible partitions
    - Enumerate the (n, m)-admissible partitions
    - "Polarized powers" of the q-discriminant on the Schur basis

$\mathcal{D}_k(\mathbb{X}, q)$ in the Schur basis

$$\mathcal{D}_k(\mathbb{X}, q) = \sum_{\lambda} c_\lambda S_\lambda(\mathbb{X})$$

We want to characterise λ when $c_\lambda(q) \neq 0$.

We generalize a result by Toumazet, King and Wybourne ($k = 1$) (2004).

(n, m)-admissible partitions

We recall that $\rho = [n-1, n-2, \dots, 1, 0]$.
The set of (n, m)-admissible partitions can be obtained by the induction:

$$A_{n,1} = \{[\lambda_1, \dots, \lambda_n] | \lambda \leq \rho\}$$
$$A_{n,m} = \{[\lambda_1 + \sigma(1) - 1, \dots, \lambda_n + \sigma(n) - 1] | \sigma \in S_n \text{ and } \lambda \in A_{n,m-1}\}$$

Example : $n = 3,\ m = 2$

$$\begin{array}{c} A_{3,1} = \{[2,1,0],[1,1,1]\} \\ \Big\downarrow{\scriptstyle 210,201,120,102,021,012} \\ A_{3,2} = \{[4,2,0],[4,1,1],[3,3,0],[3,2,1],[2,2,2]\} \end{array}$$

The (n, m)-admissible partitions are all the partitions with n parts which are lower or equal than $m\rho$ with respect to the dominance order.

Enumerate the (n, m)-admissible partitions

Let $\tau_1 = [-1, +1, 0]$ and $\tau_2 = [0, -1, +1]$.

$$
\begin{tikzcd}[column sep=tiny]
 & & & & \boxed{[420]} \arrow[dl, "\tau_1"'] \arrow[dr, "\tau_2"] & & & & \\
 & & & \boxed{[330]} \arrow[dl] \arrow[dr] & & \boxed{[411]} \arrow[dl] \arrow[dr] & & & \\
 & & [240] \arrow[dl] \arrow[dr] & & \boxed{[321]} \arrow[dl] \arrow[dr] & & [402] \arrow[dl] \arrow[dr] & & \\
 & [150] \arrow[dl] \arrow[dr] & & [231] \arrow[dl] \arrow[dr] & & [312] \arrow[dl] \arrow[dr] & & [4\ {-1}\ 3] \arrow[dl] \arrow[dr] & \\
 [060] & & [141] & & \boxed{[222]} & & [303] & & [4\ {-2}\ 4]
\end{tikzcd}
$$

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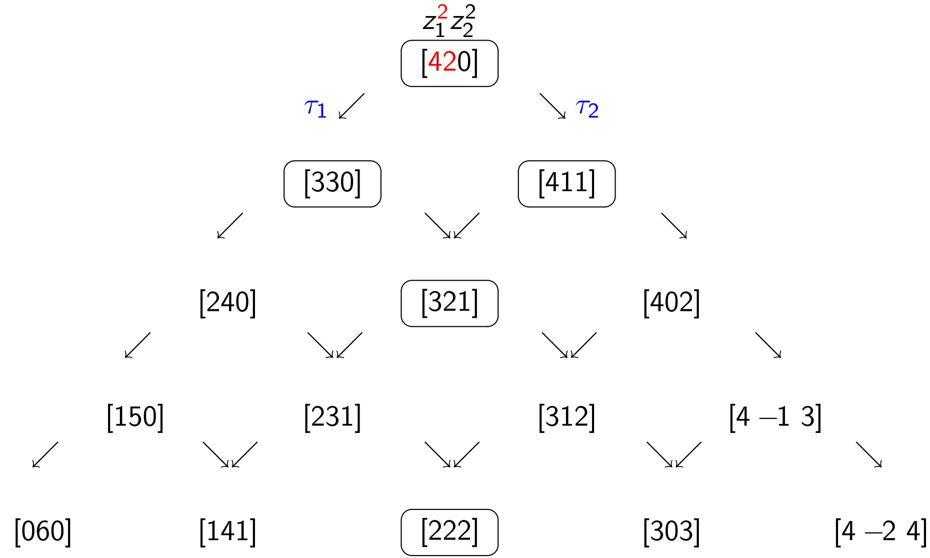
Let $\tau_1 = [-1, +1, 0]$ and $\tau_2 = [0, -1, +1]$.



Enumerate the (n, m)-admissible partitions

Let $\tau_1 = [-1, +1, 0]$ and $\tau_2 = [0, -1, +1]$.

$$
\begin{tikzcd}[column sep=small]
 & & & & \overset{z_1^{2} z_2^{2}}{\boxed{[420]}} \arrow[dl, "{.z_1^{-2} z_2 \quad \tau_1}"'] \arrow[dr, "{\tau_2 \quad .z_1 z_2^{-2}}"] & & & & \\
 & & & \boxed{[330]} \arrow[dl] \arrow[dr] & & \boxed{[411]} \arrow[dl] \arrow[dr] & & & \\
 & & {[240]} \arrow[dl] \arrow[dr] & & \boxed{[321]} \arrow[dl] \arrow[dr] & & {[402]} \arrow[dl] \arrow[dr] & & \\
 & {[150]} \arrow[dl] \arrow[dr] & & {[231]} \arrow[dl] \arrow[dr] & & {[312]} \arrow[dl] \arrow[dr] & & {[4\ {-1}\ 3]} \arrow[dl] \arrow[dr] & \\
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$$

Enumerate the (n, m)-admissible partitions

Let $\tau_1 = [-1, +1, 0]$ and $\tau_2 = [0, -1, +1]$.

$$
\begin{tikzcd}[column sep=tiny]
 & & & & \overset{z_1^{\color{red}2} z_2^2}{\boxed{[{\color{red}42}0]}} \arrow[dl, swap, ".z_1^{-2}z_2 \quad \tau_1"] \arrow[dr, "\tau_2 \quad .z_1 z_2^{-2}"] & & & & \\
 & & & \overset{z_1^0 z_2^3}{\boxed{[330]}} \arrow[dl] \arrow[dr] & & \overset{z_1^3 z_0^0}{\boxed{[411]}} \arrow[dl] \arrow[dr] & & & \\
 & & [240] \arrow[dl] \arrow[dr] & & \boxed{[321]} \arrow[dl] \arrow[dr] & & [402] \arrow[dl] \arrow[dr] & & \\
 & [150] \arrow[dl] \arrow[dr] & & [231] \arrow[dl] \arrow[dr] & & [312] \arrow[dl] \arrow[dr] & & [4\ {-1}\ 3] \arrow[dl] \arrow[dr] & \\
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$$

Enumerate the (n, m)-admissible partitions

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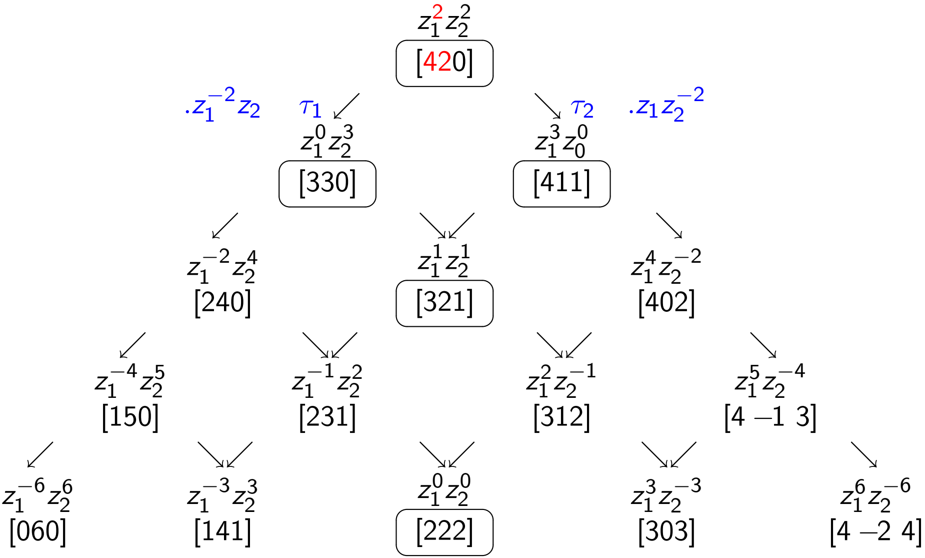

Generating series

Let Ω_{z_1,z_2} be the MacMahon Omega operator defined by

$$\Omega_{z_1,z_2} \sum_{n_1,n_2 \in \mathbb{Z}} a_{n_1,n_2} z_1^{n_1} z_2^{n_2} = \sum_{n_1,n_2 \in \mathbb{N}} a_{n_1,n_2} z_1^{n_1} z_2^{n_2}$$

The generating function of the $(3,2)$-admissible partitions is

$$\Omega_{z_1,z_2} \frac{z_1^2 z_2^2}{(1 - \frac{z_2}{z_1^2} q)(1 - \frac{z_1}{z_2^2} q)} = z^{22} + (z^{03} + z^{30})q + z^{11} q^2 + z^{00} q^4$$

"Polarized powers" of the q-discriminant on the Schur basis

We want to prove

$$\mathcal{D}_k(\mathbb{X};q) = \sum_{\lambda} c_\lambda(q) S_\lambda(\mathbb{X})$$

$c_\lambda(q) \neq 0$ if and only if λ is $(n, 2k)$-admissible.

Sketch of the proof

We just have to prove that $c_\lambda(-1) \neq 0$.

$$\mathcal{D}_k(\mathbb{X}, -1) = \prod_{i \neq j} (x_i + x_j)^k = (S_\rho(\mathbb{X}))^{2k}$$

Proof by induction:

- For $k = 1$, King, Toumazet and Wybourne have proved that:
$$S_\rho^2 = \sum_{\lambda \text{ is } (n,2)\text{-admissible}} c_\lambda^{n,2} S_\lambda \qquad \text{with} \qquad c_\lambda^{n,2} > 0$$
- For $k = m$,
$$S_\rho^m = S_\rho^{m-1} S_\rho = \sum_\lambda c_\lambda^{n,m-1} (S_\lambda S_\rho) \qquad \text{with} \qquad c_\lambda^{n,m-1} > 0.$$
$$S_\rho^m = \sum_{\mu \leq m\rho} \bullet S_\mu = \sum_{\mu \text{ is } (n,m)\text{-admissible}} \bullet S_\mu$$
$$\mu \leq m\rho \quad \Rightarrow \quad \exists \sigma | \mu - \rho.\sigma \text{ is } (n, m-1)\text{-admissible.}$$
$$\mu - \lambda \text{ is a permutation of } \rho \quad \Rightarrow \quad < S_\mu | S_\lambda S_\rho > \geq \ 1.$$

$$c_\lambda^{n,2k} \neq 0 \text{ iff the coefficient } \lambda \text{ is } (n, 2k)\text{-admissible.}$$

Conclusion

We have seen that:

$$\mathcal{D}_k(\mathbb{X}, q^{-1/2}) = cst.P_{2k\rho}(\mathbb{X}, q, q^{-\frac{2k-1}{2}}),$$

and we have characterized all the set $\{\lambda | c_\lambda(q) \neq 0\}$ in

$$\mathcal{D}_k(\mathbb{X}, q) = \sum_\lambda c_\lambda(q) S_\lambda.$$

- Can we find an algebraic and combinatoric interpretation of $c_\lambda(q)$ in

$$\mathcal{D}_k(\mathbb{X}, q) = \sum_\lambda c_\lambda(q) S_\lambda?$$

- There exist other products being Macdonald polynomials:

$$\prod_{l=1}^{k-1} \prod_{i \neq j} (x_i - q^l x_j) = \bullet\, P_{((k-1)(n-1))^n}(\frac{1-q}{1-q^k}\mathbb{X}, q, q^k).$$

- What are its properties on the Schur basis?