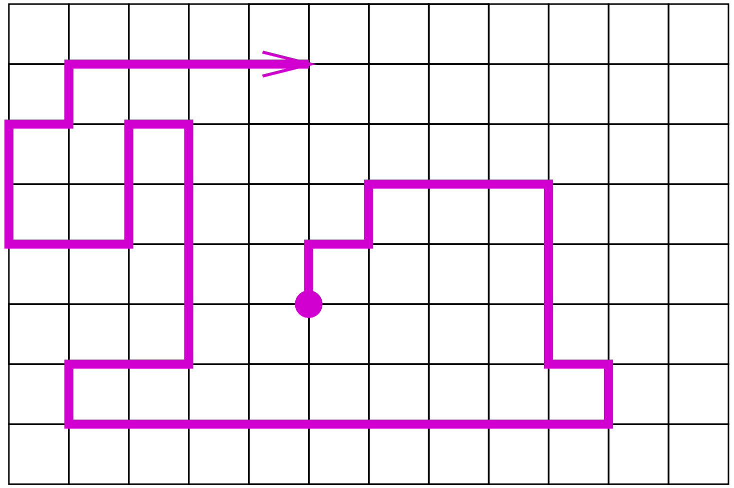
Prudent self-avoiding walks

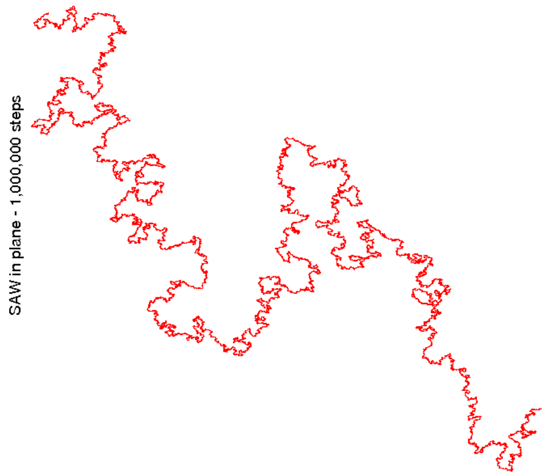
Mireille Bousquet-Mélou, CNRS, Bordeaux, France

http://www.labri.fr/∼bousquet

+ ArXiv 2008

Self-avoiding walks (SAW)





Conjectures ($d = 2$)

• The number of n-step SAW is equivalent to $(\kappa)\,\mu^n\, n^{11/32}$ for n large.

• The endpoint lies on average at distance $n^{3/4}$ from the starting point.

• The scaling limit of SAW is $\mathrm{SLE}_{8/3}$ (proved under an assumption of conformal invariance [Lawler, Schramm, Werner 02]).

This is too hard!

... for exact enumeration

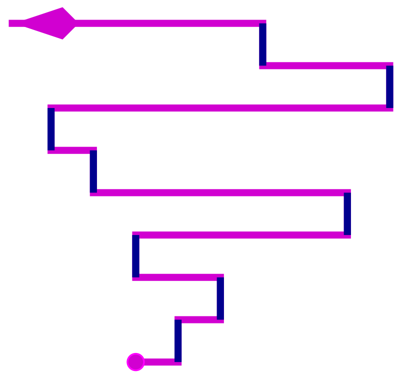
$\Rightarrow$ **Study of toy models**, that should be as general as possible, but still tractable

- develop new techniques in exact enumeration

- solve better and better approximations of real SAW

A toy model: Partially directed walks

Steps N, W, E



"Markovian with memory 1"

Generating function:

$$\sum_n a(n)t^n = \frac{1+t}{1-2t-t^2} \quad \Rightarrow \quad a(n) \sim (1+\sqrt{2})^n \sim (2.41...)^n$$

$$\mathbb{E}(|X_n|) \sim \sqrt{n}, \quad \mathbb{E}(Y_n) \sim n$$

A hierarchy of formal power series

• The formal power series $A(t)$ is rational if it can be written

$$A(t) = \frac{P(t)}{Q(t)}$$

where $P(t)$ and $Q(t)$ are polynomials in t.

• The formal power series $A(t)$ is algebraic (over $\mathbb{Q}(t)$) if it satisfies a (non-trivial) polynomial equation:

$$P(t, A(t)) = 0.$$

• The formal power series $A(t)$ is D-finite if it satisfies a (non-trivial) linear differential equation with polynomial coefficients:

$$P_0(t)A^{(k)}(t) + P_1(t)A^{(k-1)}(t) + \cdots + P_k(t)A(t) = 0.$$

Rat $\subset$ Alg $\subset$ D-finite

I. Prudent self-avoiding walks:

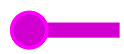
Definition

Self-directed walks [Turban-Debierre 86], Exterior walks **[Préa 97]**,
Outwardly directed SAW [Santra-Seitz-Klein 01]

Prudent walks **[Duchi 05], [Dethridge, Guttmann, Jensen 07], [mbm 08]**

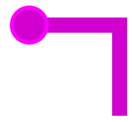
Prudent self-avoiding walks

Every new step can be repeated indefinitely without creating intersections.



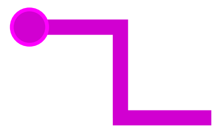
Prudent self-avoiding walks

Every new step can be repeated indefinitely without creating intersections.



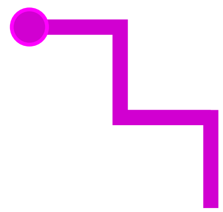
Prudent self-avoiding walks

Every new step can be repeated indefinitely without creating intersections.



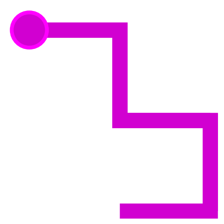
Prudent self-avoiding walks

Every new step can be repeated indefinitely without creating intersections.



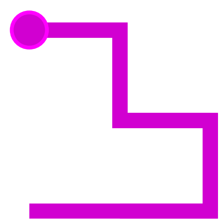
Prudent self-avoiding walks

Every new step can be repeated indefinitely without creating intersections.



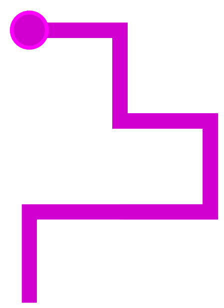
Prudent self-avoiding walks

Every new step can be repeated indefinitely without creating intersections.



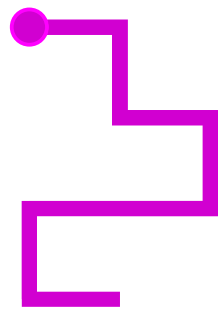
Prudent self-avoiding walks

Every new step can be repeated indefinitely without creating intersections.



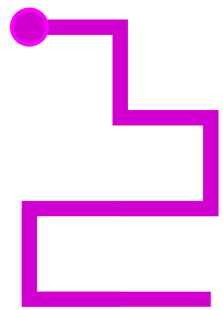
Prudent self-avoiding walks

Every new step can be repeated indefinitely without creating intersections.



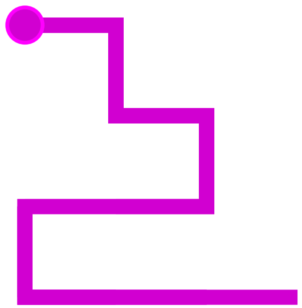
Prudent self-avoiding walks

Every new step can be repeated indefinitely without creating intersections.



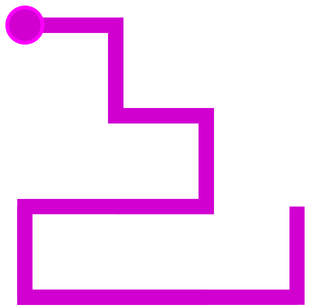
Prudent self-avoiding walks

Every new step can be repeated indefinitely without creating intersections.



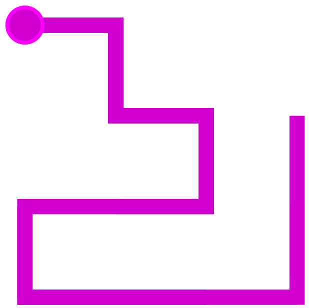
Prudent self-avoiding walks

Every new step can be repeated indefinitely without creating intersections.



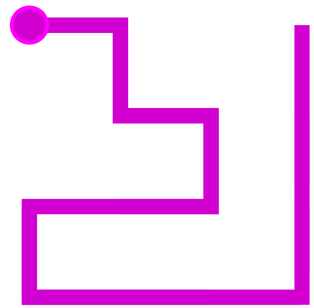
Prudent self-avoiding walks

Every new step can be repeated indefinitely without creating intersections.



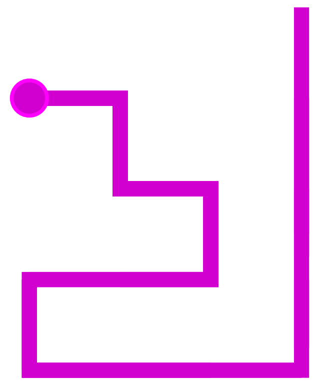
Prudent self-avoiding walks

Every new step can be repeated indefinitely without creating intersections.



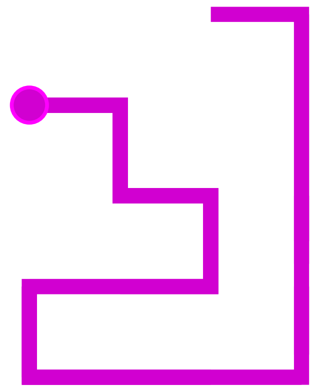
Prudent self-avoiding walks

Every new step can be repeated indefinitely without creating intersections.



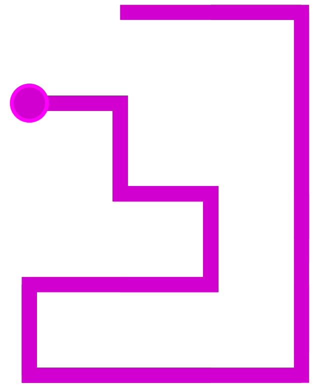
Prudent self-avoiding walks

Every new step can be repeated indefinitely without creating intersections.



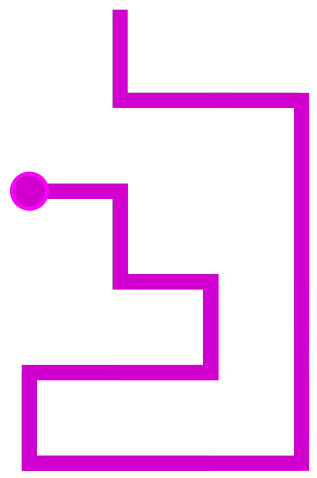
Prudent self-avoiding walks

Every new step can be repeated indefinitely without creating intersections.



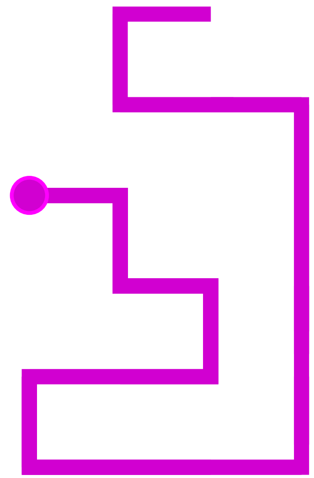
Prudent self-avoiding walks

Every new step can be repeated indefinitely without creating intersections.



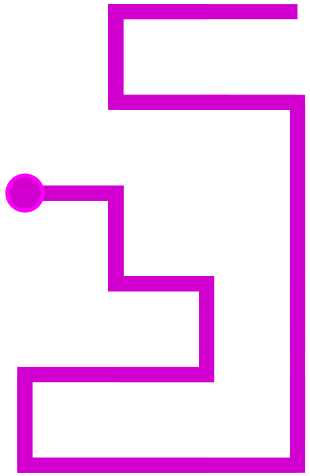
Prudent self-avoiding walks

Every new step can be repeated indefinitely without creating intersections.



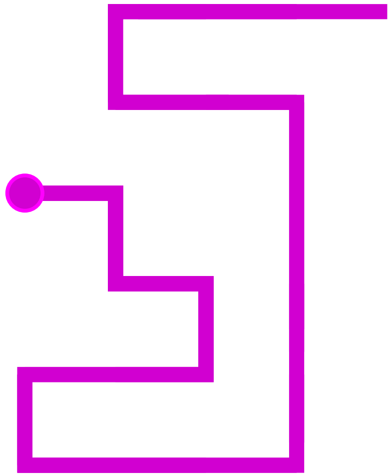
Prudent self-avoiding walks

Every new step can be repeated indefinitely without creating intersections.



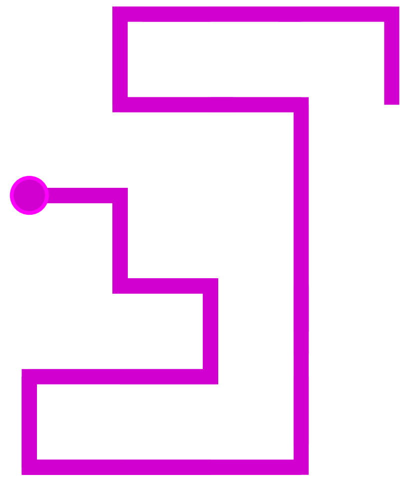
Prudent self-avoiding walks

Every new step can be repeated indefinitely without creating intersections.



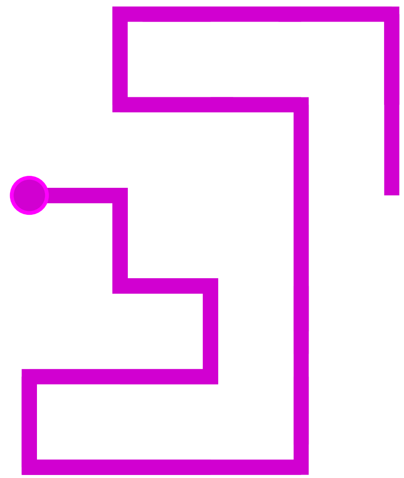
Prudent self-avoiding walks

Every new step can be repeated indefinitely without creating intersections.



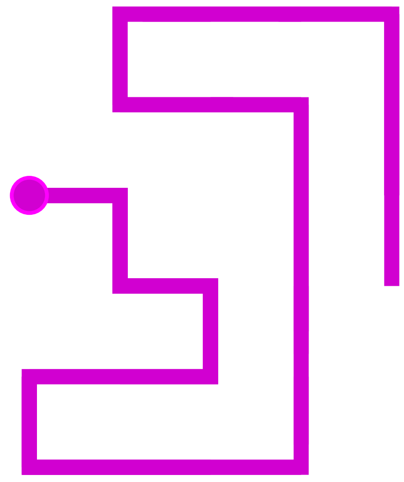
Prudent self-avoiding walks

Every new step can be repeated indefinitely without creating intersections.



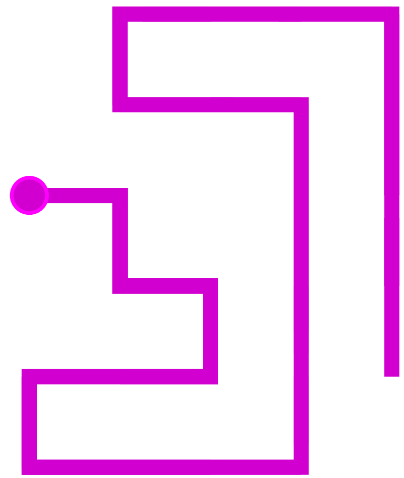
Prudent self-avoiding walks

Every new step can be repeated indefinitely without creating intersections.



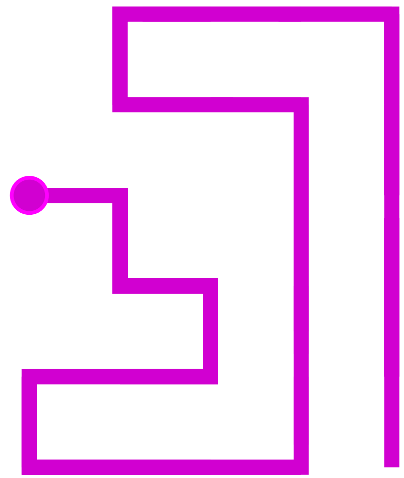
Prudent self-avoiding walks

Every new step can be repeated indefinitely without creating intersections.



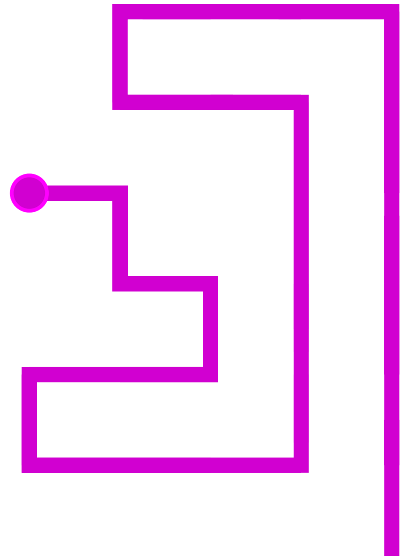
Prudent self-avoiding walks

Every new step can be repeated indefinitely without creating intersections.



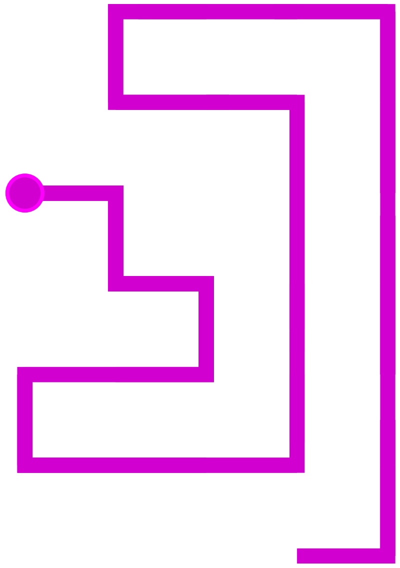
Prudent self-avoiding walks

Every new step can be repeated indefinitely without creating intersections.



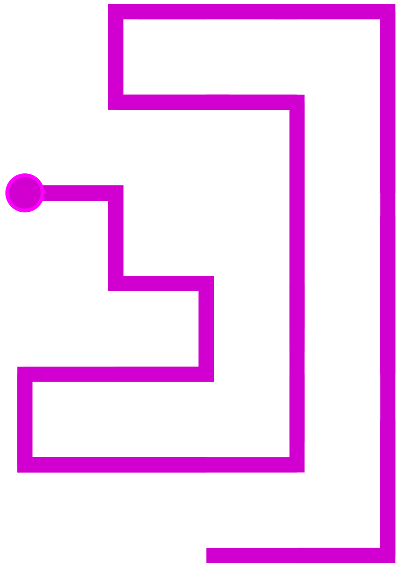
Prudent self-avoiding walks

Every new step can be repeated indefinitely without creating intersections.



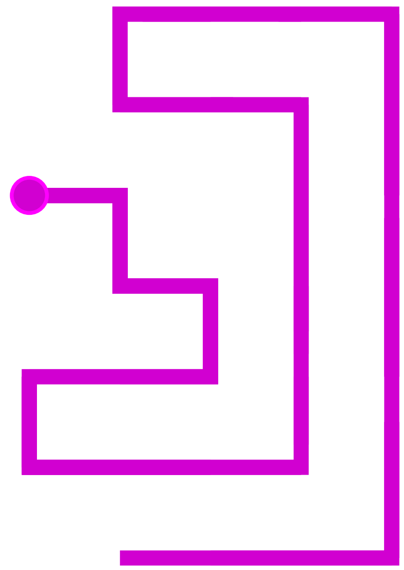
Prudent self-avoiding walks

Every new step can be repeated indefinitely without creating intersections.



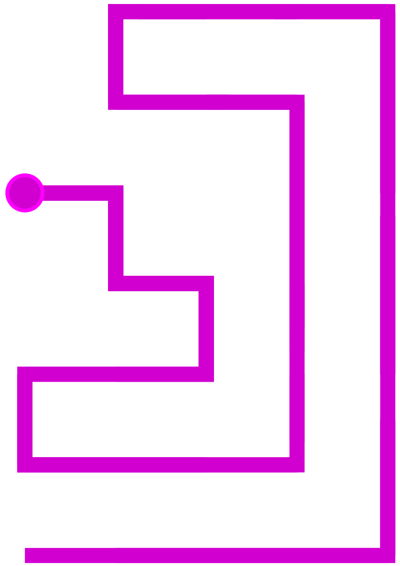
Prudent self-avoiding walks

Every new step can be repeated indefinitely without creating intersections.



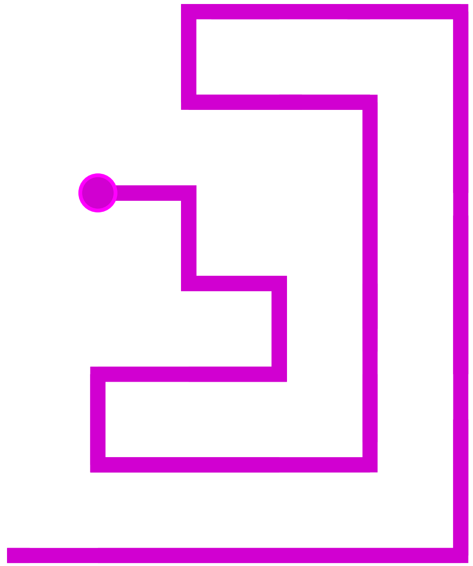
Prudent self-avoiding walks

Every new step can be repeated indefinitely without creating intersections.



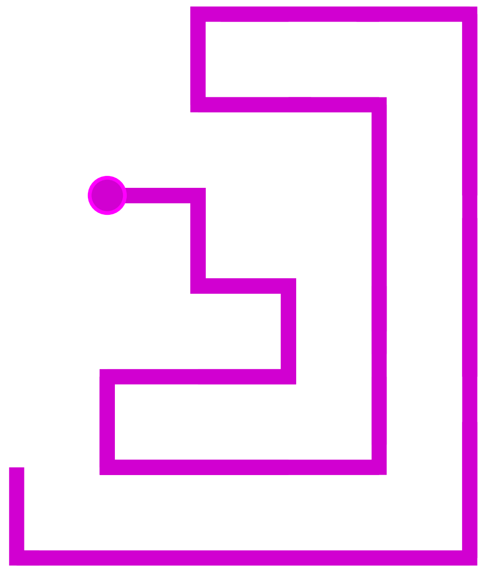
Prudent self-avoiding walks

Every new step can be repeated indefinitely without creating intersections.



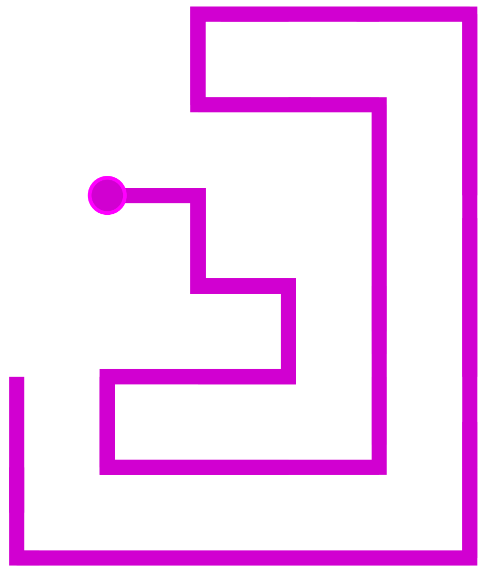
Prudent self-avoiding walks

Every new step can be repeated indefinitely without creating intersections.



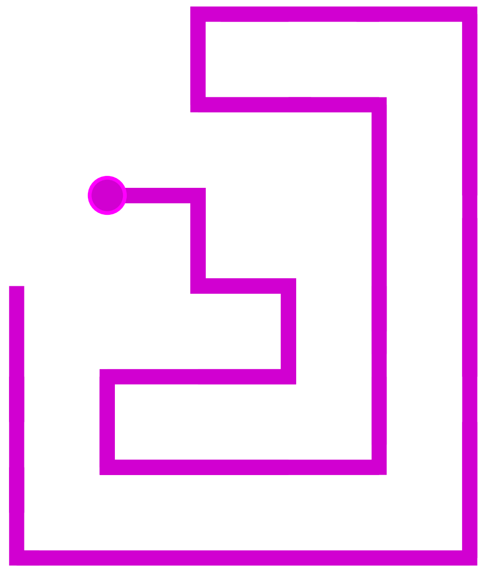
Prudent self-avoiding walks

Every new step can be repeated indefinitely without creating intersections.



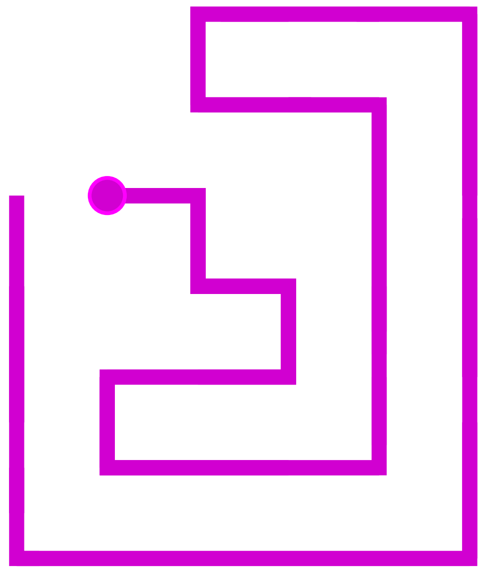
Prudent self-avoiding walks

Every new step can be repeated indefinitely without creating intersections.



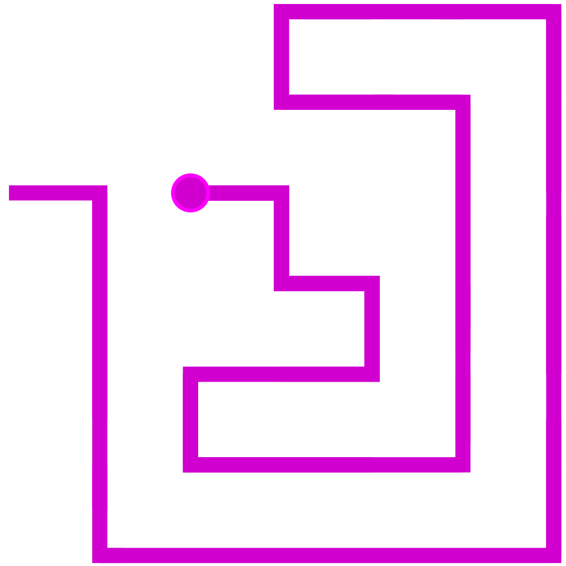
Prudent self-avoiding walks

Every new step can be repeated indefinitely without creating intersections.



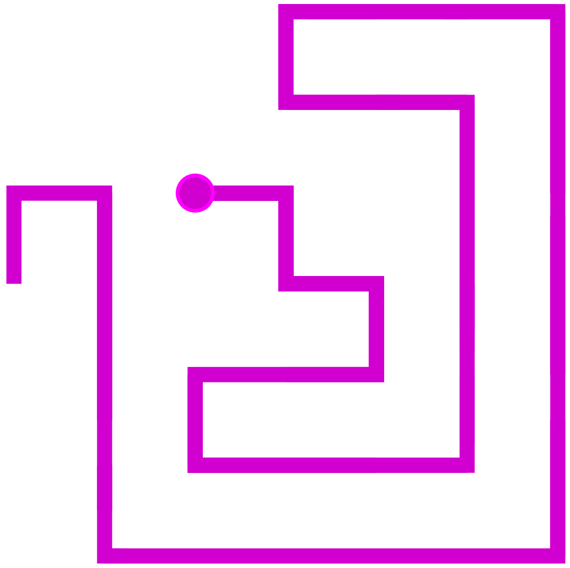
Prudent self-avoiding walks

Every new step can be repeated indefinitely without creating intersections.



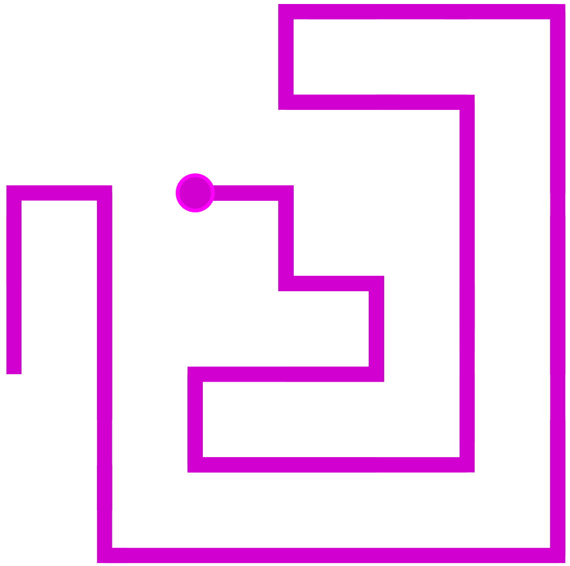
Prudent self-avoiding walks

Every new step can be repeated indefinitely without creating intersections.



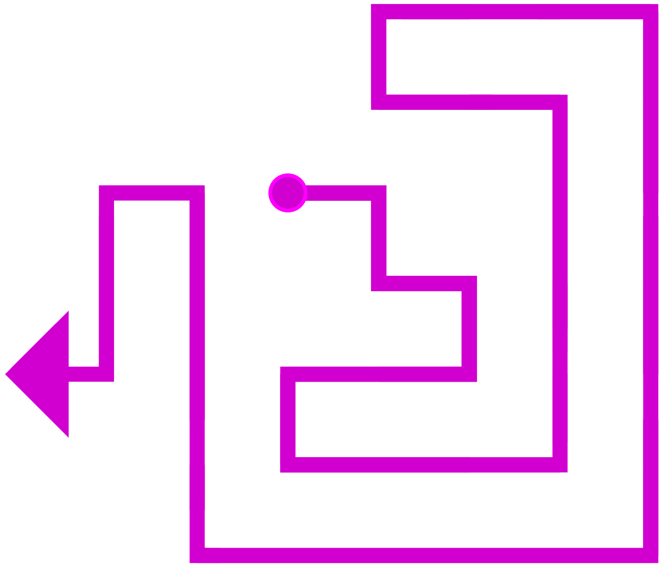
Prudent self-avoiding walks

Every new step can be repeated indefinitely without creating intersections.

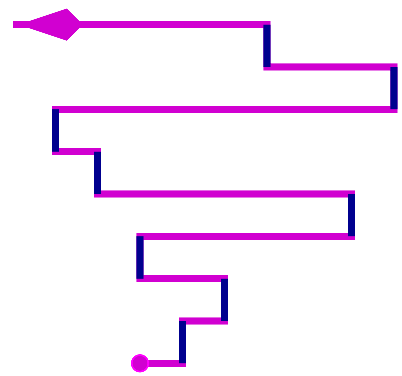


Prudent self-avoiding walks

Every new step can be repeated indefinitely without creating intersections.



Remark: Partially directed walks are prudent



II. Enumeration and properties of prudent walks: Some tools

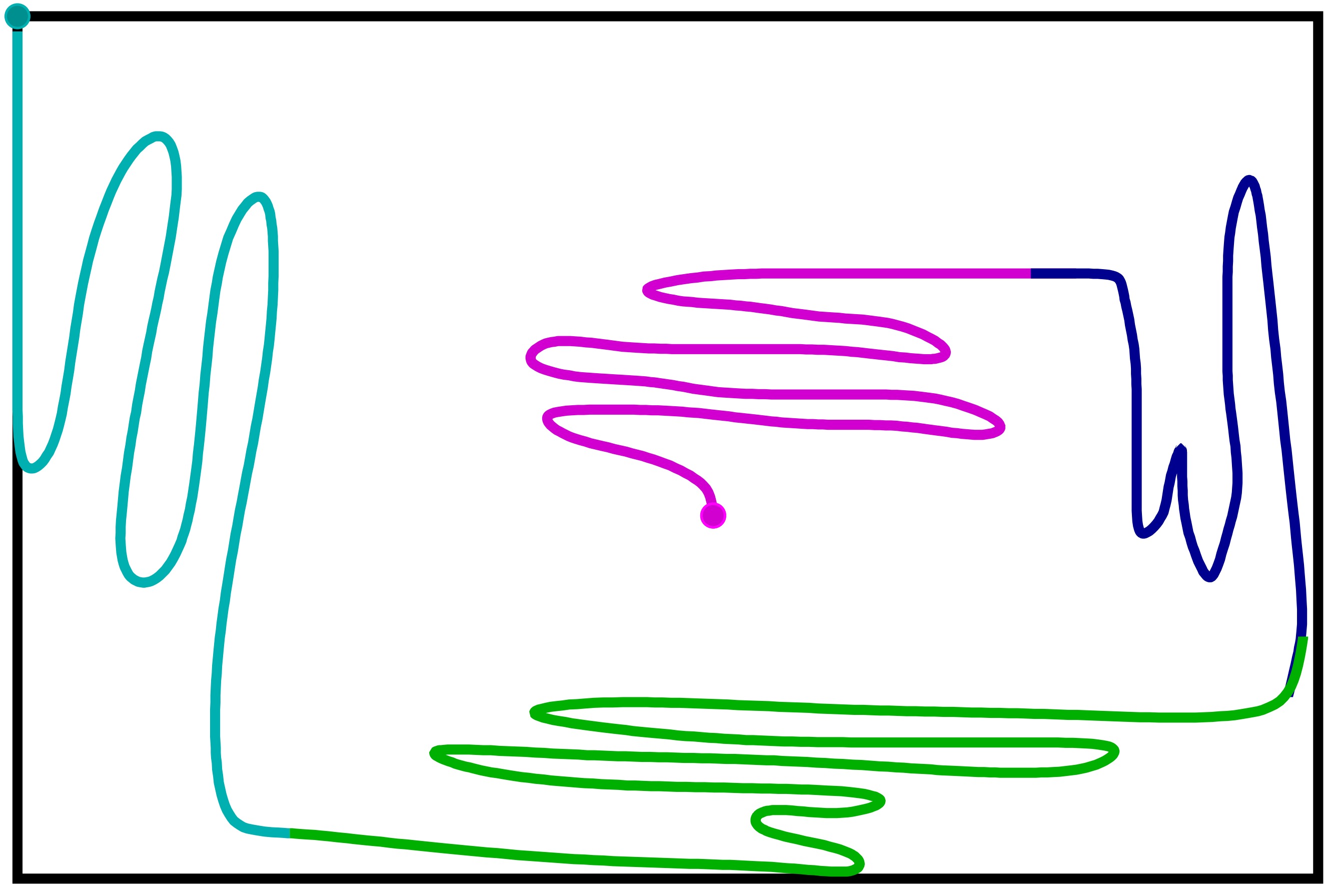
- Recursive description of the walks $\Rightarrow$ Functional equations with "catalytic" variables

- Solving equations: The kernel method

- Singularity analysis $\Rightarrow$ Nature of the series and asymptotics

Some properties of prudent walks

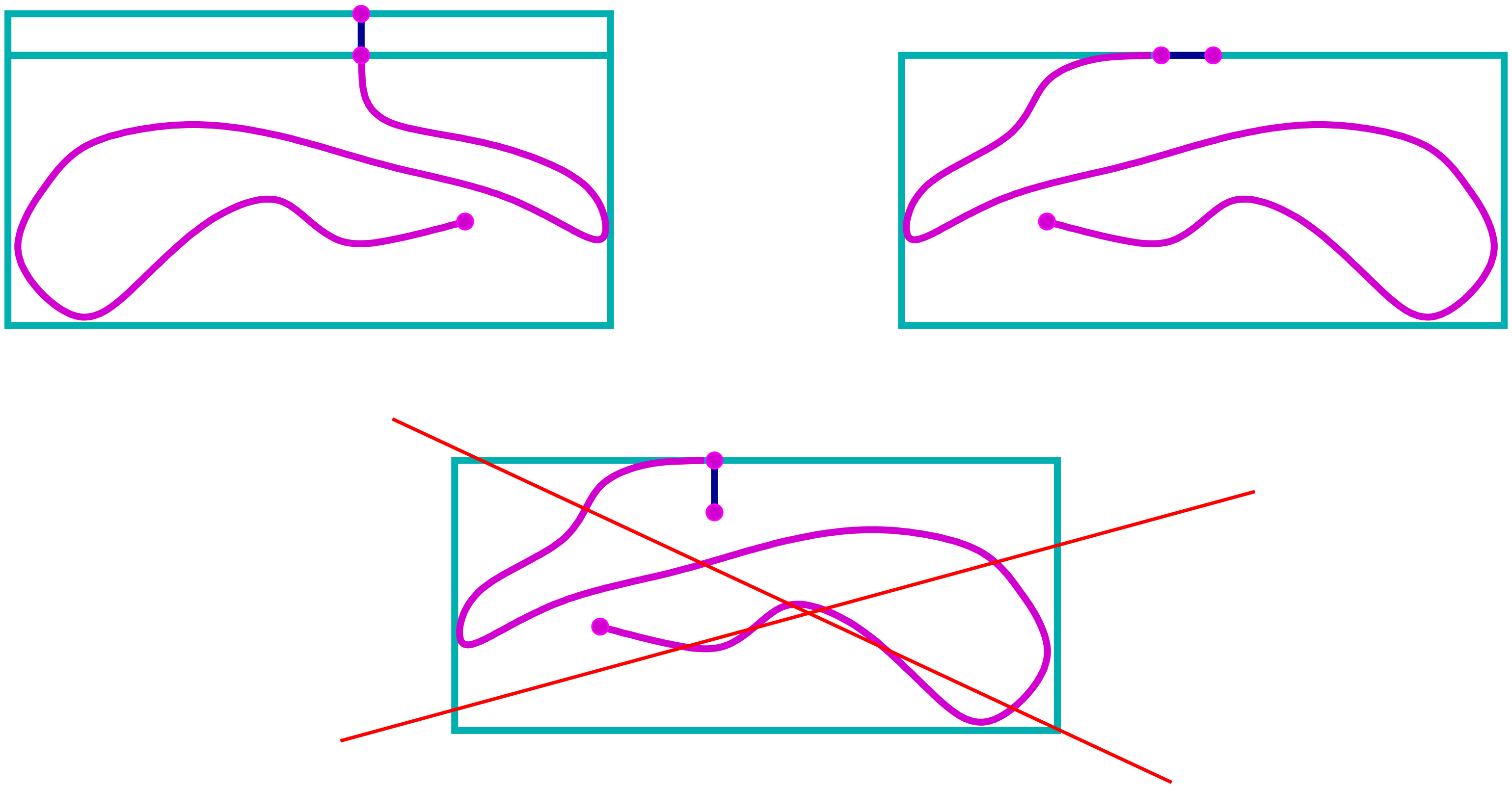
Some properties of prudent walks



The box of a prudent walk

Some properties of prudent walks

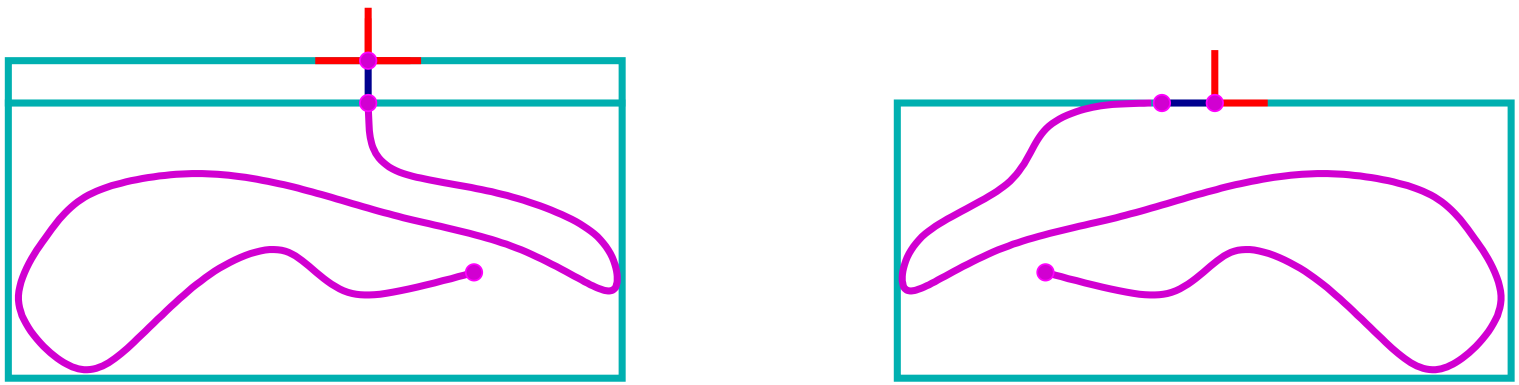
The endpoint of a prudent walk is always on the border of the box.



Each new step either inflates the box or walks (prudently) along the border.

Some properties of prudent walks

Each new step either inflates the box or walks (prudently) along the border.



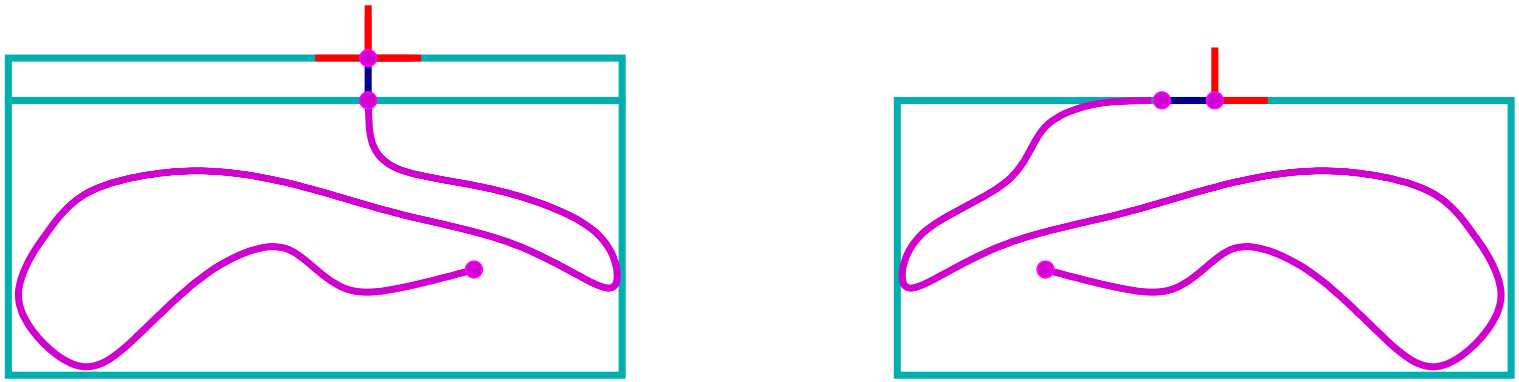
- After an inflating step, 3 possible extensions

- Otherwise, only 2.

⇒ Count prudent walks by looking for inflating steps

Some properties of prudent walks

Each new step either inflates the box or walks (prudently) along the border.



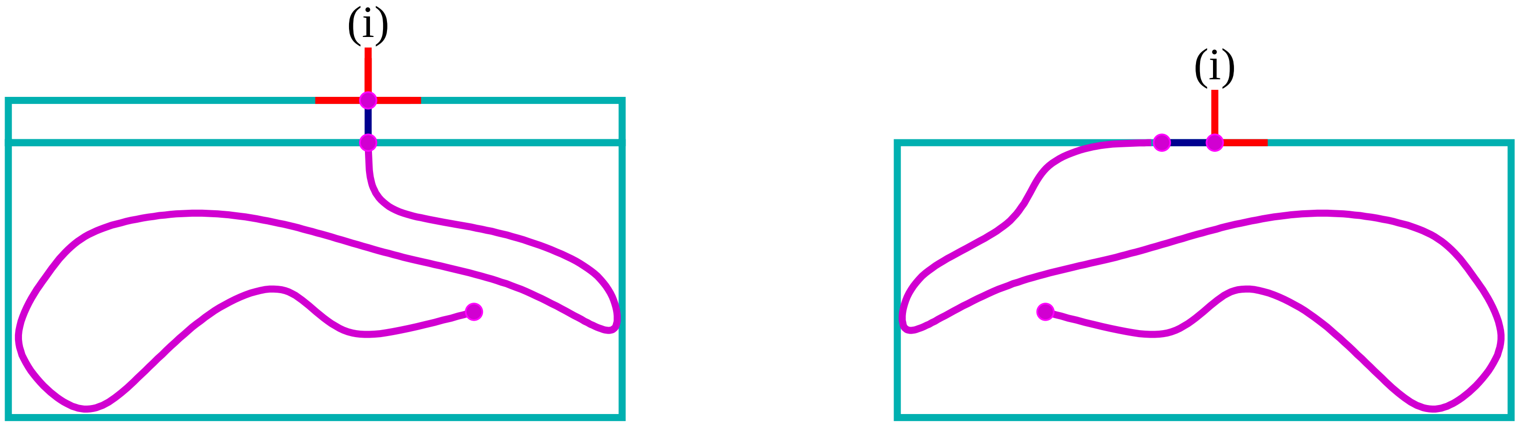
- After an inflating step, 3 possible extensions
- Otherwise, only 2.

⇒ Count prudent walks by looking for inflating steps

When do we create an inflating step?

Some properties of prudent walks

Each new step either inflates the box or walks (prudently) along the border.



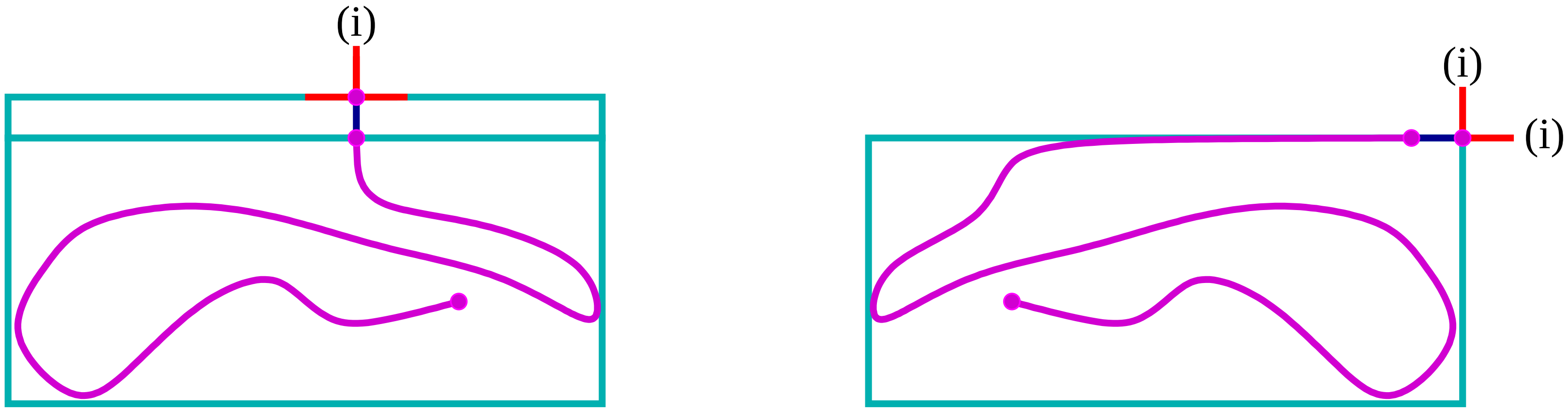
- After an inflating step, 3 possible extensions
- Otherwise, only 2.

⇒ Count prudent walks by looking for inflating steps

When do we create an inflating step?

Some properties of prudent walks

Each new step either inflates the box or walks (prudently) along the border.



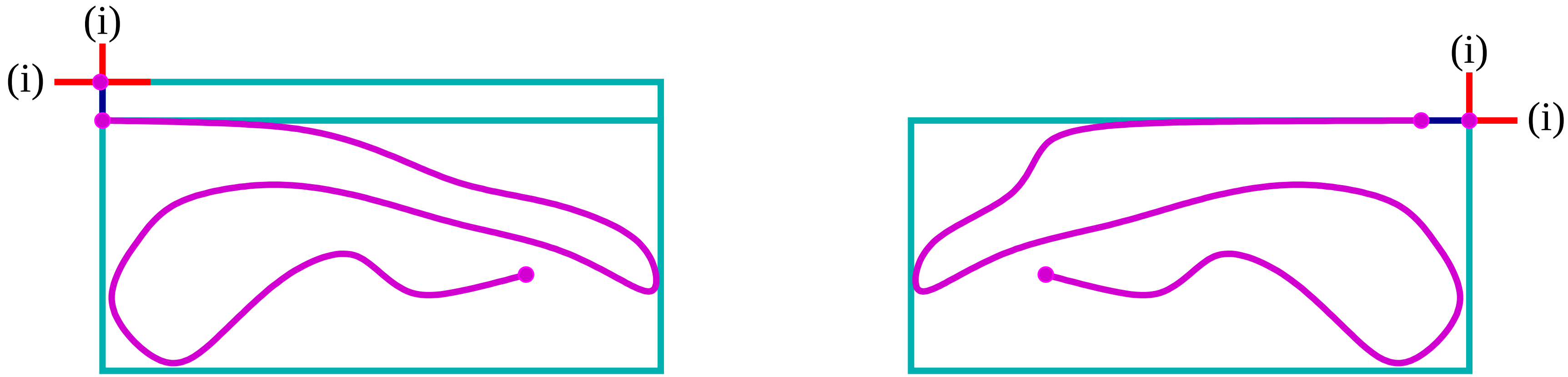
- After an inflating step, 3 possible extensions
- Otherwise, only 2.

$\Rightarrow$ Count prudent walks by looking for inflating steps

When do we create an inflating step?

Some properties of prudent walks

Each new step either inflates the box or walks (prudently) along the border.

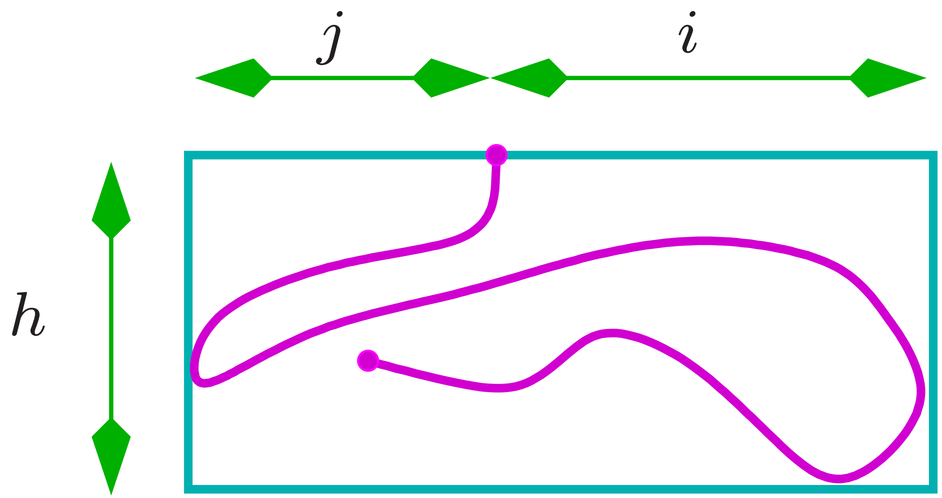


- After an inflating step, 3 possible extensions
- Otherwise, only 2.

⇒ Count prudent walks by looking for inflating steps

When do we create an inflating step?

More parameters



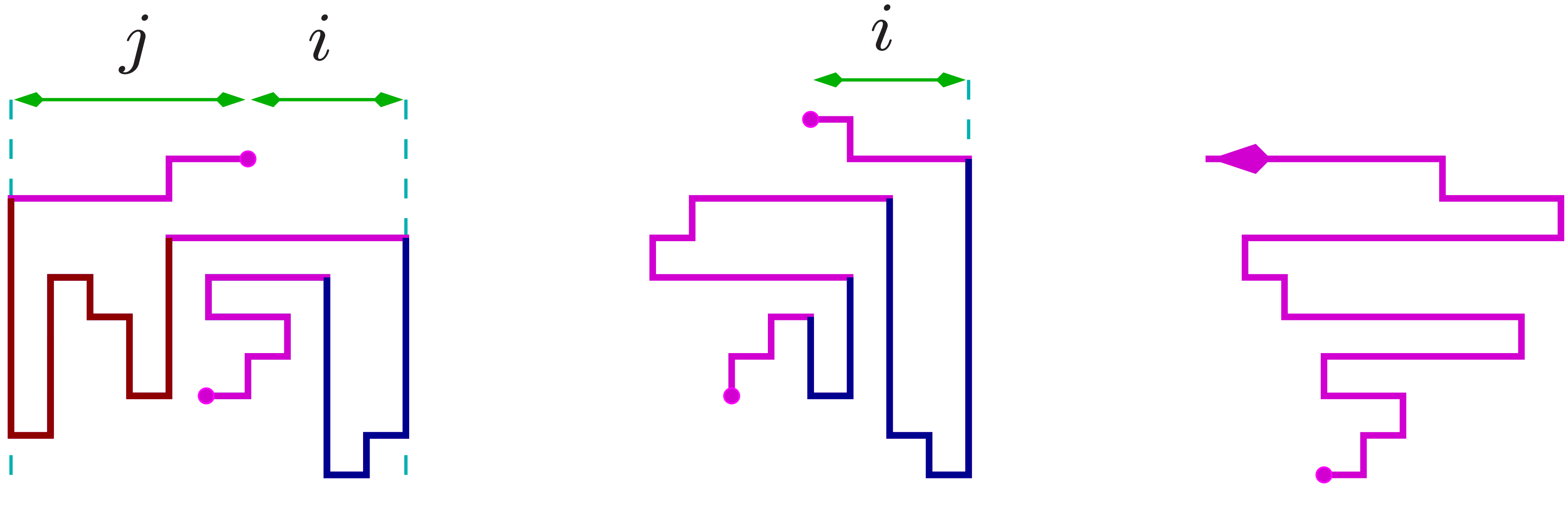
If one knows:

- the direction of the last step,
- whether it is inflating or not,
- the distances i, j and h,

then one can decide which steps can be appended to the walk, and the new values of these parameters.

$\Rightarrow$ Count prudent walks by looking for inflating steps, keeping track of the distances i, j, h

Simpler families of prudent walks [Préa 97]



3-sided 2-sided 1-sided

- The endpoint of a 3-sided walk lies always on the top, right or left side of the box
- The endpoint of a 2-sided walk lies always on the top or right side of the box
- The endpoint of a 1-sided walk lies always on the top side of the box (= partially directed!)

Functional equations for two-sided prudent walks:
The more general the class, the more additional variables

(Walks ending on the top of the box)

1. Two-sided walks : one catalytic variable

$$\left(1-\frac{tu(1-t^2)}{(1-tu)(u-t)}\right)T(t;u)=\frac{1}{1-tu}+t\,\frac{u-2t}{u-t}\,T(t;t).$$

2. Three-sided walks : two catalytic variables

$$\left(1-\frac{uvt(1-t^2)}{(u-tv)(v-tu)}\right)T(t;u,v)=1+\cdots-\frac{t^2v}{u-tv}T(t;tv,v)-\frac{t^2u}{v-tu}T(t;u,tu)$$

3. General prudent walks : three catalytic variables

$$\left(1-\frac{uvwt(1-t^2)}{(u-tv)(v-tu)}\right)T(t;u,v,w)=1+\mathcal{G}(w,u)+\mathcal{G}(w,v)-tv\frac{\mathcal{G}(v,w)}{u-tv}-tu\frac{\mathcal{G}(u,w)}{v-tu}$$

with $\mathcal{G}(u,v)=tvT(t;u,tu,v)$.

Two-sided walks: the kernel method

$$\Big((1-tu)(u-t)-tu(1-t^2)\Big)T(t;u)=u-t+t(u-2t)(1-tu)T(t;t).$$

- If $u=U(t)$ cancels $(1-tu)(u-t)-tu(1-t^2)$, then

$$0=U(t)-t+t(U(t)-2t)(1-tU(t))T(t;t),$$

that is,

$$T(t;t)=\frac{t-U(t)}{t(U(t)-2t)(1-tU(t))}$$

- We know such a series $U(t)$:

$$U(t)=\frac{1-t+t^2+t^3-\sqrt{(1-t^4)(1-2t-t^2)}}{2t}$$

[Knuth 72], [mbm-Petkovšek 2000]

Two-sided walks

- The length generating function of 2-sided walks is

$$P(t) = \frac{1}{1-2t-2t^2+2t^3}\left(1+t-t^3+t(1-t)\sqrt{\frac{1-t^4}{1-2t-t^2}}\right)$$

[Duchi 05]

- Dominant singularity: a simple pole for $1-2t-2t^2+2t^3=0$, that is, $t_c = 0.40303...$. Asymptotically,

$$p(n) \sim \kappa \, (2.48...)^n$$

Compare with 2.41... for partially directed walks.

Three-sided walks: two catalytic variables

- Functional equation for $T(t; u, v) \equiv T(u, v)$:

$$K(u, v)T(u, v) = A(u, v) + B(u, v)\,\Phi(u) + B(v, u)\,\Phi(v)$$

for polynomials $K(u, v), A(u, v), B(u, v)$, with $\Phi(u) = T(u, tu)$.

Three-sided walks: two catalytic variables

- Functional equation for $T(t; u, v) \equiv T(u, v)$:

$$K(u, v)T(u, v) = A(u, v) + B(u, v)\,\Phi(u) + B(v, u)\,\Phi(v)$$

for polynomials $K(u, v), A(u, v), B(u, v)$, with $\Phi(u) = T(u, tu)$.

- Cancellation of the kernel: $K(u, V(u)) = 0$ for a series $V(u) \equiv V(t; u)$

$$\Phi(u) = -\frac{A(u, V(u))}{B(u, V(u))} - \frac{B(V(u), u)}{B(u, V(u))}\,\Phi(V(u))$$

Three-sided walks: two catalytic variables

- Functional equation for $T(t;u,v) \equiv T(u,v)$:

$$K(u,v)T(u,v) = A(u,v) + B(u,v)\,\Phi(u) + B(v,u)\,\Phi(v)$$

for polynomials $K(u,v), A(u,v), B(u,v)$, with $\Phi(u) = T(u,tu)$.

- Cancellation of the kernel: $K(u,V(u)) = 0$ for a series $V(u) \equiv V(t;u)$

$$\Phi(u) = -\frac{A(u,V(u))}{B(u,V(u))} - \frac{B(V(u),u)}{B(u,V(u))}\,\Phi(V(u))$$

- If it is possible to iterate (...), denote $V^{(k)} = V(V(V(\cdots(u))))$ (k iterations):

$$\Phi(u) = \sum_{k\geq 0} (-1)^{k-1} \frac{B(V^{(1)},u)B(V^{(2)},V^{(1)})\cdots B(V^{(k)},V^{(k-1)})A(V^{(k)},V^{(k+1)})}{B(u,V^{(1)})B(V^{(1)},V^{(2)})\cdots B(V^{(k-1)},V^{(k)})B(V^{(k)},V^{(k+1)})}$$

Sum of algebraic series — iteration of algebraic functions

Three-sided prudent walks

- Let

$$U(w) = \frac{1 - tw + t^2 + t^3 w - \sqrt{(1-t^2)(1+t-tw+t^2w)(1-t-tw-t^2w)}}{2t},$$

and

$$q = U(1) = \frac{1 - t + t^2 + t^3 - \sqrt{(1-t^4)(1-2t-t^2)}}{2t}.$$

- The length generating function of three-sided prudent walks is:

$$P(t;1) = \frac{1}{1-2t-t^2}\left(\frac{1+3t+tq(1-3t-2t^2)}{1-tq} + 2t^2 q\, T(t;1,t)\right)$$

where

$$T(t;1,t) = \sum_{k\geq 0} (-1)^k \frac{\prod_{i=0}^{k-1}\left(\frac{t}{1-tq} - U(q^{i+1})\right)}{\prod_{i=0}^{k}\left(\frac{tq}{q-t} - U(q^i)\right)}\left(1 + \frac{U(q^k)-t}{t(1-tU(q^k))} + \frac{U(q^{k+1})-t}{t(1-tU(q^{k+1}))}\right)$$

Three-sided prudent walks

The generating function of three-sided prudent walks is:

$$P(t;1) = \frac{1}{1-2t-t^2}\left(\frac{1+3t+tq(1-3t-2t^2)}{1-tq} + 2t^2q\; T(t;1,t)\right)$$

$$T(t;1,t) = \sum_{k\geq 0}(-1)^k \frac{\prod_{i=0}^{k-1}\left(\frac{t}{1-tq} - U(q^{i+1})\right)}{\prod_{i=0}^{k}\left(\frac{tq}{q-t} - U(q^i)\right)}\left(1 + \frac{U(q^k)-t}{t(1-tU(q^k))} + \frac{U(q^{k+1})-t}{t(1-tU(q^{k+1}))}\right)$$

• The series $P(t;1)$ has infinitely many poles, satisfying $\frac{tq}{q-t} = U(q^i)$ for some $i \geq 0$. Hence it is neither algebraic, nor even D-finite.

• Dominant singularity: (again) a simple pole for $1-2t-2t^2+2t^3=0$. Asymptotically,

$$p(n) \sim \kappa\,(2.48...)^n$$

General prudent walks: three catalytic variables

$$\left(1 - \frac{uvwt(1-t^2)}{(u-tv)(v-tu)}\right) T(u,v,w) = 1 + \mathcal{G}(w,u) + \mathcal{G}(w,v) - tv\frac{\mathcal{G}(v,w)}{u-tv} - tu\frac{\mathcal{G}(u,w)}{v-tu}$$

with $\mathcal{G}(u,v) = tvT(u,tu,v)$.

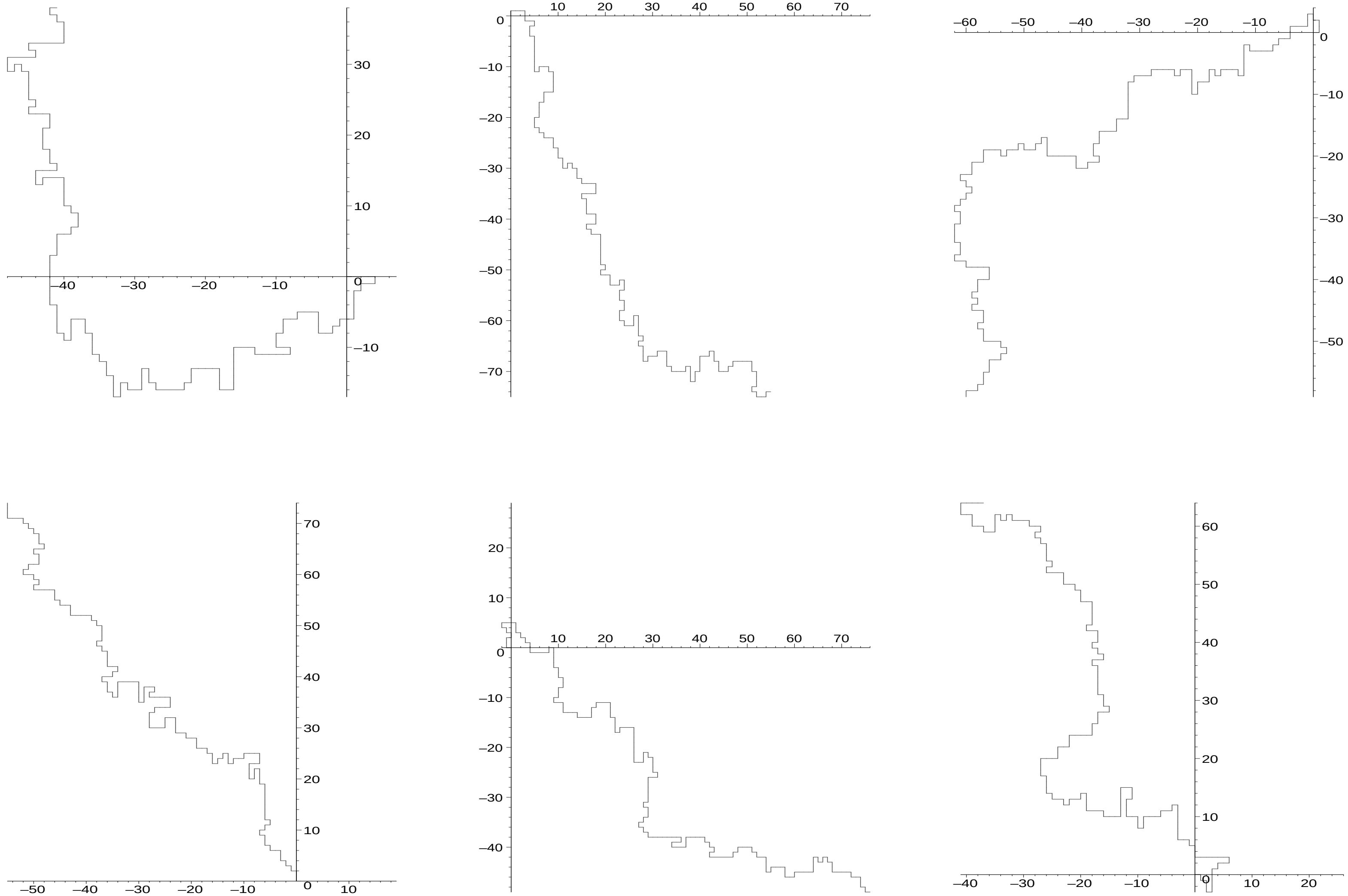
?

Summary of the results

	Nature of the g.f.	Asympt. growth	End-to-end distance
1-sided (part. dir)	Rat.	$(2.41...)^n$	n
2-sided	Alg. [Duchi 05]	$(2.48...)^n$	n
3-sided	not D-finite	$(2.48...)^n$	n
4-sided (general)	not D-finite	$(2.48...)^n$	n
square lattice SAW	?	$(2.63...)^n \, n^{11/32}$	$n^{3/4}$

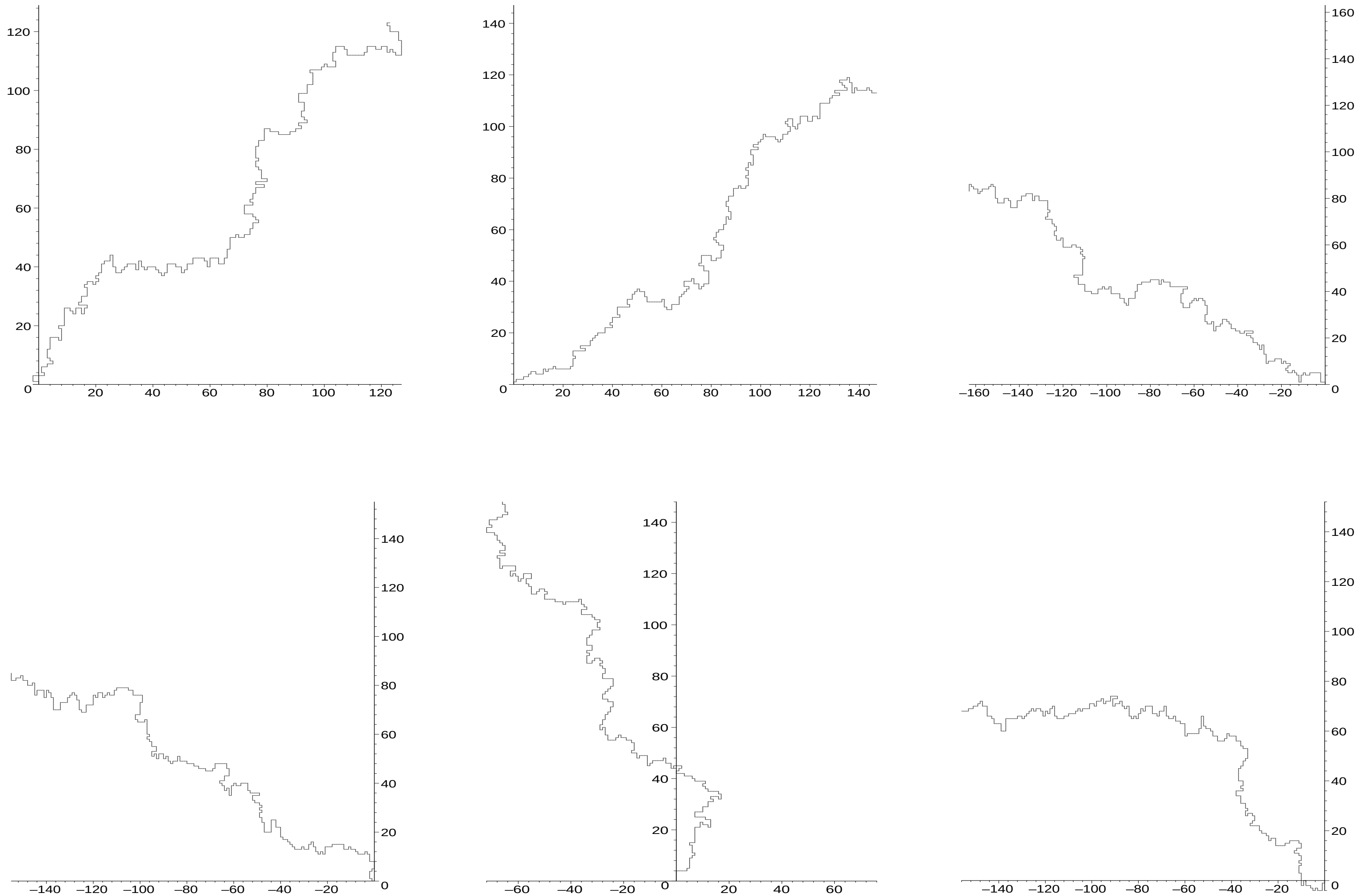
Random prudent walks

195 steps (sic)



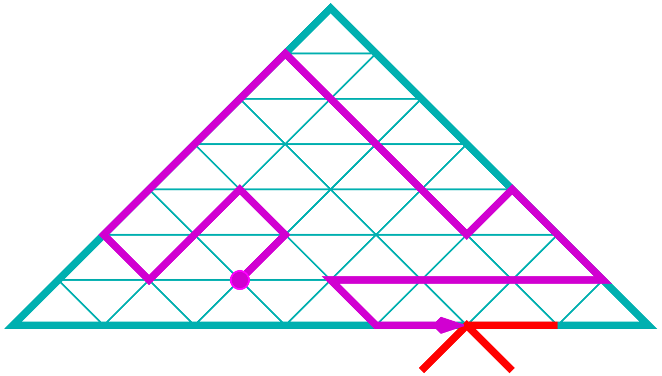
Three-sided prudent walks

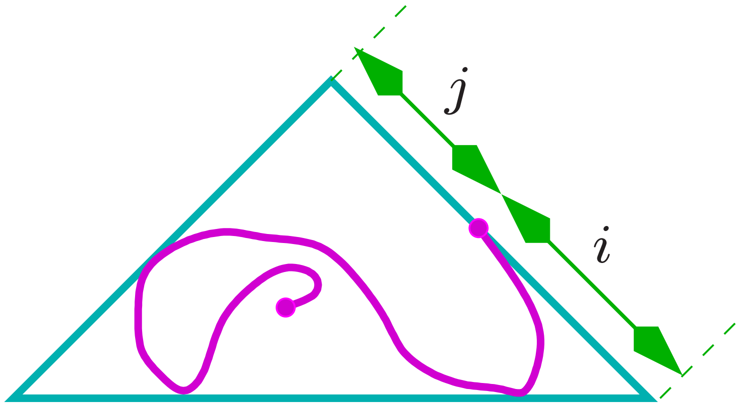
400 steps



An isotropic model with only two additional parameters: triangular prudent walks

Definition: Every new step either inflates the box or walks (prudently) along the border





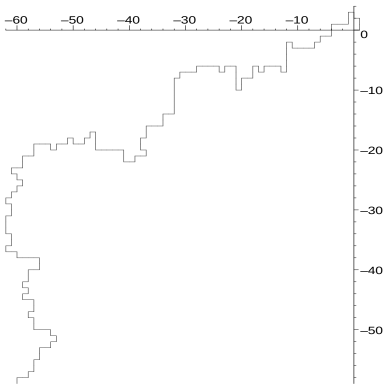
- Explicit form of the generating function (infinite sum of quadratic series)

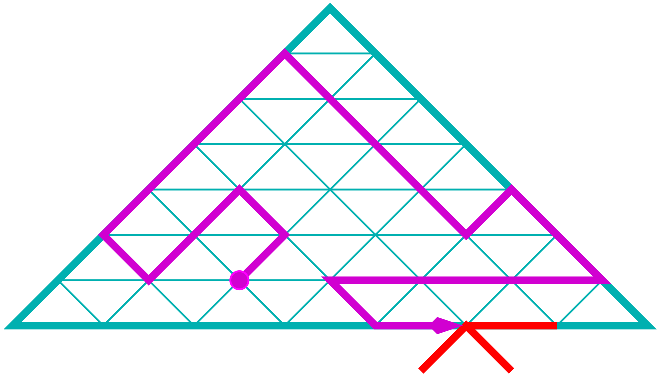
- Infinitely many singularities, accumulating on a portion of the unit circle ⇒ non-D-finite

Some questions

• General prudent walks on the square lattice: growth constant? Exact enumeration?

• More efficient procedures for random generation (maximal length 200 for general prudent walks...)





• Limit processes?

• The number of triangular prudent walks whose box has size k is

$$2^{k-1}(k+1)(k+2)!$$

Combinatorial explanation?

Triangular prudent walks

The length generating function of triangular prudent walks is

$$P(t;1) = \frac{6t(1+t)}{1-3t-2t^2}\Big(1 + t\,(1+2t)\,R(t;1,t)\Big)$$

with

$$R(t;1,t) = (1+Y)(1+tY)\sum_{k\geq 0} \frac{t^{\binom{k+1}{2}}\left(Y(1-2t^2)\right)^k}{(Y(1-2t^2);t)_{k+1}}\left(\frac{Yt^2}{1-2t^2};t\right)_k$$

and

$$Y = \frac{1-2t-t^2-\sqrt{(1-t)(1-3t-t^2-t^3)}}{2t^2}$$

Notation:

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}).$$

- The series $P(t;1)$ is neither algebraic, nor even D-finite (infinitely many poles at $Yt^k(1-2t^2)=0$)