A variation on tableau switching and a Pak-Vallejo's conjecture

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Overview

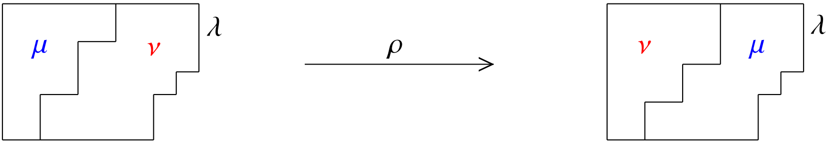
1. Fundamental symmetry map and Pak-Vallejo's conjecture
2. Interlacing phenomenon and GT patterns
3. Decreasing chain sliding/Reverse Schensted row insertion
4. The bijection ρ_3
5. Benkart-Sottile-Stroomer tableau switching and interlacing phenomenon

1. Fundamental symmetry map and Pak-Vallejo's conjecture

Definition (PV04)

The fundamental symmetry is a bijection

$$\rho : LR_n[\lambda/\mu, \nu] \longrightarrow LR_n[\lambda/\nu, \mu].$$



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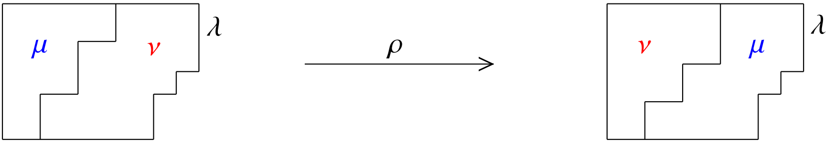

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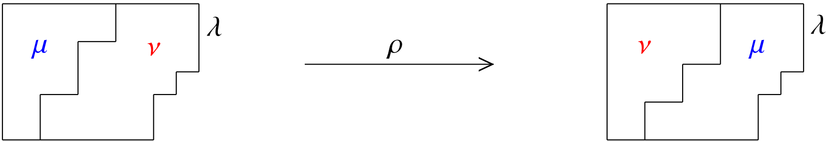

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- ρ_2^{-1}

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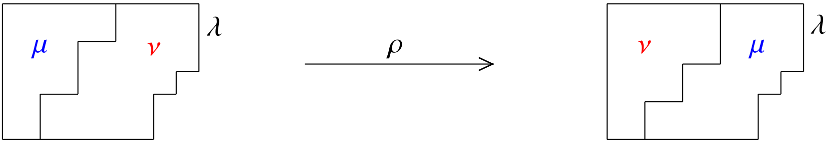

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- ρ_3 [A. 98;00]

Conjecture [PV04] The fundamental symmetries ρ_1, ρ_2, ρ_2^{-1}, ρ_3 are identical involutions.

[DK05] Danilov, Koshevoy: ρ_1 and $\rho_2 = \rho_2^{-1}$ are identical involutions.

2. Interlacing phenomenon

2.1 Invariant factors of a product of matrices over a *pid* with one prime p

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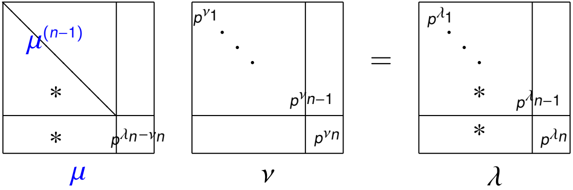

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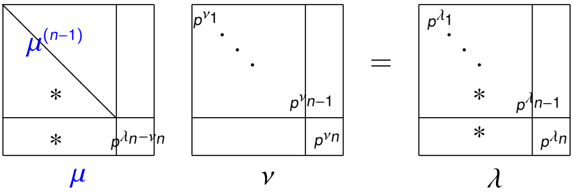

$$(\mu^{(n-1)}, \nu_{[n-1]}, \lambda_{[n-1]}) \longrightarrow T' \in LR(\mu^{(n-1)}, \nu_{[n-1]}, \lambda_{[n-1]}).$$

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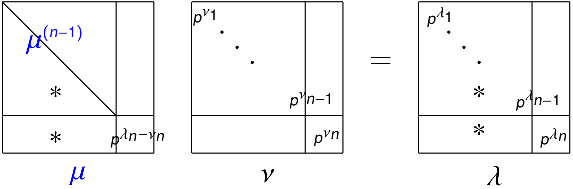

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$$\begin{array}{ccccccccc} & \mu_1^{(n-1)} & & \mu_2^{(n-1)} & & \cdots & & \mu_{n-1}^{(n-1)} & \\ \mu_1 & & \mu_2 & & \mu_3 & \cdots & \mu_{n-1} & & \mu_n \end{array}$$

$$
\begin{array}{ccccccccccc}
 & & & & \mu_1^{(1)} & & & & & & \\
 & & & \mu_1^{(2)} & & \mu_2^{(2)} & & & & & \\
 & & \mu_1^{(3)} & & \mu_2^{(3)} & & \mu_3^{(3)} & & & & \\
 & & \cdots & & \cdots & & \cdots & & & & \\
 & \mu_1^{(n-1)} & & \mu_2^{(n-1)} & & \mu_3^{(n-1)} & \cdots & \mu_{n-1}^{(n-1)} & & & \\
\mu_1^{(n)} & & \mu_2^{(n)} & & \mu_3^{(n)} & \cdots & \mu_{n-1}^{(n)} & & \mu_n^{(n)}. & &
\end{array}
$$

GT pattern $G = [\mu^{(1)}, \ldots, \mu^{(n-1)}, \mu^n = \mu]$ of base μ and weight $\lambda - \nu$,

$$\sum_{j=1}^{i} (\mu_j^{(i)} - \mu_j^{(i-1)}) = \lambda_i - \nu_i, \; i = 1, \ldots, n.$$

Transposition

$$B^t A^t = C^t \longrightarrow (\nu, \mu, \lambda) \longrightarrow \mathrm{t}(T) \in LR_n(\lambda/\nu, \mu)$$

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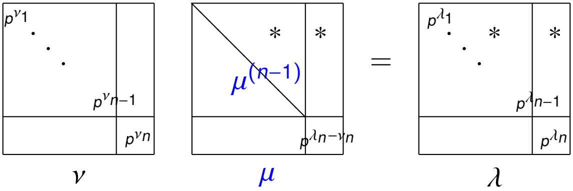

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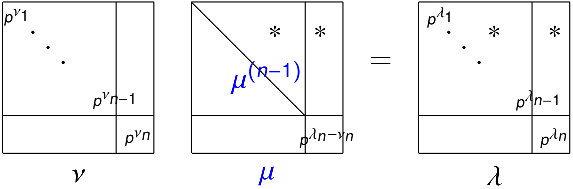

Question: Does $G = [\mu^{(1)}, \mu^{(2)}, \cdots, \mu^{(n-1)}, \mu]$ define $\mathrm{t}(T)$?

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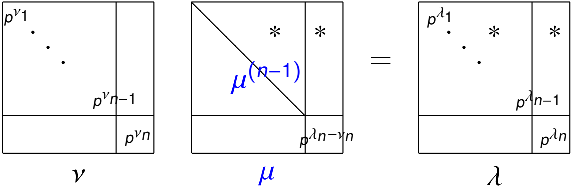

Question: Does $G = [\mu^{(1)}, \mu^{(2)}, \cdots, \mu^{(n-1)}, \mu]$ define $\mathrm{t}(T)$?

- Yes, if $\mathrm{t}(T') \in LR(\lambda_{[n-1]}/\nu_{[n-1]}, \mu^{(n-1)})$ can be obtained by suppression of the last row of $\mathrm{t}(T)$.

LR tableaux and GT patterns

- There is a bijection between $LR_n[\lambda/\nu, \mu]$ and GT patterns $G = [\mu^{(1)}, \cdots, \mu^{(n-1)}, \mu^{(n)}]$ of base μ and weight $\lambda - \nu$,

$$
\begin{array}{ccccccccccc}
 & & & & & \mu_1^{(1)} & & & & & \\
 & & & & \mu_1^{(2)} & & \mu_2^{(2)} & & & & \\
 & & & \mu_1^{(3)} & & \mu_2^{(3)} & & \mu_3^{(3)} & & & \\
 & & & \cdots & & \cdots & & \cdots & & & \\
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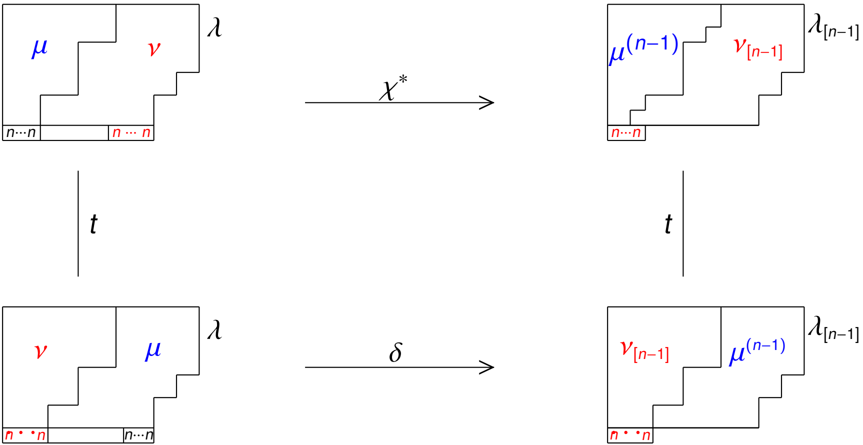
\end{array}
$$

such that

$$\nu_{i-1} - \nu_i \geq \sum_{j=1}^{r} (\mu_j^{(i)} - \mu_j^{(i-1)}) - \sum_{j=1}^{r-1} (\mu_j^{(i)} - \mu_j^{(i-1)}), \quad 1 \leq r \leq i-1, \quad 2 \leq i \leq n.$$

Combinatorial scheme of LR tableaux



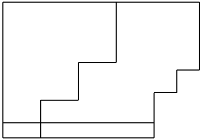
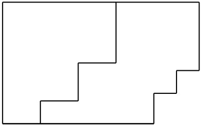
$$\chi^* = t \circ \delta \circ t$$

$$\nu_{[n-1]} = (\nu_1, \cdots, \nu_{n-1}), \qquad \lambda_{[n-1]} = (\lambda_1, \cdots, \lambda_{n-1})$$

2.1. Young tableau shape interlacing

Theorem

$T \in ST_n(\lambda/\mu, m)$

$T =$  λ $\equiv P \xrightarrow{\delta} \tilde{T} =$  $\lambda_{[n-1]}$ $\equiv \tilde{P}$

P the rectification of T with $shape(P) = \sigma = (\sigma_1, \ldots, \sigma_{n-1}, \sigma_n)$
$\tilde{P}$ the rectification of $\tilde{T}$ with $shape(\tilde{P}) = \tilde{\sigma} = (\tilde{\sigma}_1, \ldots, \tilde{\sigma}_{n-1})$

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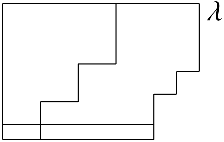
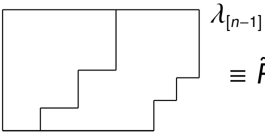
$T =$ 

$\equiv P \xrightarrow{\delta} \tilde{T} =$ 

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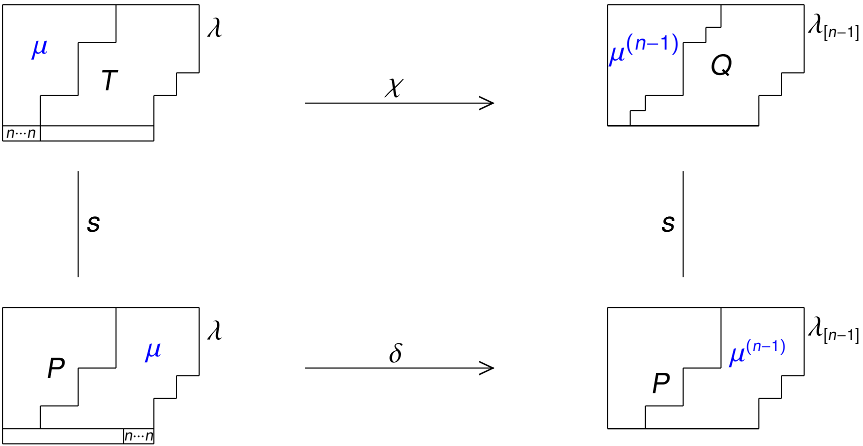
Then

$$\begin{array}{ccccccccccc} & \tilde{\sigma}_1 & & \tilde{\sigma}_2 & & \ldots & & \tilde{\sigma}_{n-1} & \\ \sigma_1 & & \sigma_2 & & \sigma_3 & \ldots & \sigma_{n-1} & & \sigma_n \end{array}$$

Young tableau combinatorial scheme

- $T \equiv P$, P with $n-1$ rows



$$\chi = s \circ \delta \circ s$$

$$\nu_{[n-1]} = (\nu_1, \cdots, \nu_{n-1}), \qquad \lambda_{[n-1]} = (\lambda_1, \cdots, \lambda_{n-1})$$

- If (P, R) is the switching of $(Y(\mu), T)$, the last row of the GT pattern defining the LR tableau R can be obtained by some *sliding up* operations in the last row of T.

3. Decreasing chain sliding/Reverse Schensted row insertion

- If $T \equiv P$ where P has $n - 1$ rows, the strictly decreasing chains starting in the bottom row do not reach the top row of T.

$T =$

•	•	•	•	•	•	•	2
•	•	•	•	•	2	2	
1	2	2	3	3	4		
2	3	4	6	7			
4	4	6	7				
5	6	7					

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•	•	•	•	•	2	2	
1	2	2	3	3	4		
2	3	4	6	7			
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5	6	7					

•	•	•	•	•	•	2	2
•	•	•	•	•	2	3	
1	2	2	3	4	4		
2	3	6	6	7			
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5	6						

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$$
T = \begin{array}{cccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & 2 \\
\bullet & \bullet & \bullet & \bullet & \bullet & 2 & 2 & \\
1 & 2 & 2 & 3 & 3 & 4 & & \\
2 & 3 & 4 & 6 & 7 & & & \\
4 & 4 & 6 & 7 & & & & \\
5 & 6 & 7 & & & & &
\end{array}
\qquad
\begin{array}{cccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & 2 & 2 \\
\bullet & \bullet & \bullet & \bullet & \bullet & 2 & 3 & \\
1 & 2 & 2 & 3 & 4 & 4 & & \\
2 & 3 & 6 & 6 & 7 & & & \\
4 & 4 & 7 & 7 & & & & \\
5 & 6 & & & & & &
\end{array}
$$

$$
T' = \begin{array}{cccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & 2 & 2 \\
\bullet & \bullet & \bullet & 1 & 2 & 2 & 3 & \\
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- $T \equiv T'$, $\begin{array}{ccccc} & 6 & & 3 & \\ 7 & & 5 & & 0 \end{array}$

Proposition $T \in ST_n(\lambda/\mu, m)$, $T \equiv P$, P of normal shape with $n-1$ rows. If T' is obtained by reverse Schensted row insertion in the last row of T, $T' \in ST_{n-1}(\lambda_{[n-1]}/\mu', m)$ with $T' \equiv T$ and the inner shape μ' of T' interlaces with the inner shape μ of T.

Lemma
$w \equiv Y(\mu)$. Then $shuffle(n \dots 21, w) \equiv Y(\mu + (1, \dots, 1))$

Corollary
$T \in LR(\lambda/\mu, \nu)$, then T can be rectified by reverse Schensted row insertion.

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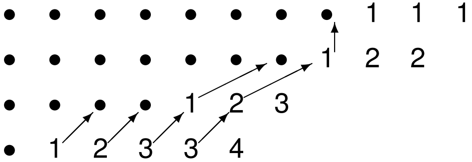

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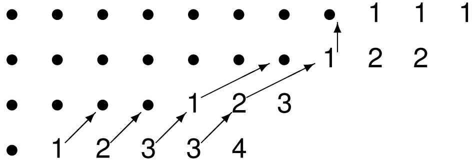

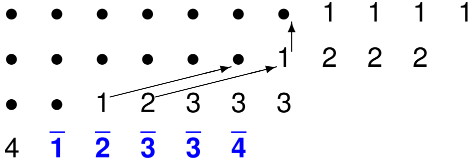

•	•	•	•	•	•	1	1	1	1	1
•	•	•	•	•	1	2	2	2	2	
3	3	3	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{3}$				
4	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{3}$	$\bar{4}$					

•	•	•	•	•	•	1	1	1	1	1
•	•	•	•	•	1	2	2	2	2	
3	3	3	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{3}$				
4	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{3}$	$\bar{4}$					

•	•	•	•	•	1	1	1	1	1	1
2	2	2	2	$\bar{1}$	$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{2}$	$\bar{2}$	
3	3	3	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{3}$				
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$$\begin{array}{ccccccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & 1 & 1 & 1 & 1 & 1 \\
\bullet & \bullet & \bullet & \bullet & \bullet & 1 & 2 & 2 & 2 & 2 & \\
3 & 3 & 3 & \bar{1} & \bar{2} & \bar{3} & \bar{3} & & & & \\
4 & \bar{1} & \bar{2} & \bar{3} & \bar{3} & \bar{4} & & & & &
\end{array}$$

$$\begin{array}{ccccccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & \bar{1} & \bar{2} & \bar{2} & \bar{2} & \bar{2} & \bar{2} & \\
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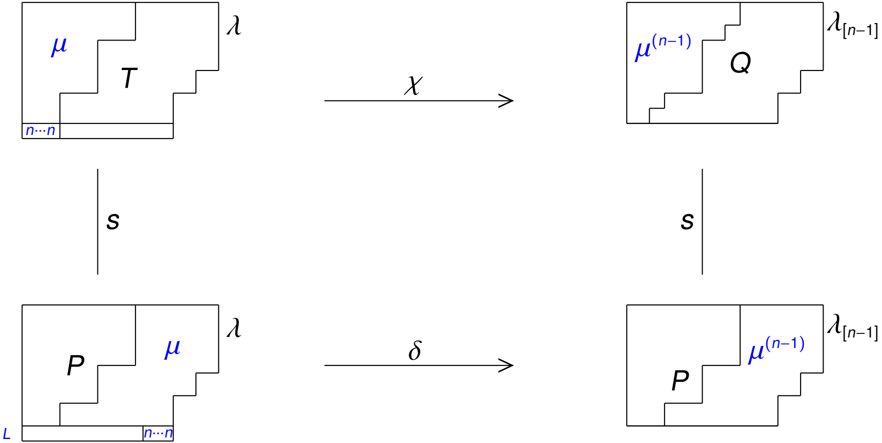
4 & \bar{1} & \bar{2} & \bar{3} & \bar{3} & \bar{4} & & & & &
\end{array}$$

$$\begin{array}{ccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & \bar{1} & \bar{1} & \bar{1} & \bar{1} & \bar{1} \\
2 & 2 & 2 & 2 & \bar{1} & \bar{2} & \bar{2} & \bar{2} & \bar{2} & \bar{2} & \\
3 & 3 & 3 & \bar{1} & \bar{2} & \bar{3} & \bar{3} & & & & \\
4 & \bar{1} & \bar{2} & \bar{3} & \bar{3} & \bar{4} & & & & &
\end{array} = \rho_3(T)$$

- Can $(Y(\mu^{(n-1)}), Q)$ be obtained by reverse Schensted row insertion from the bottom row of T?



$$\chi = s \circ \delta \circ s$$

$$\nu_{[n-1]} = (\nu_1, \cdots, \nu_{n-1}), \qquad \lambda_{[n-1]} = (\lambda_1, \cdots, \lambda_{n-1})$$

5. Benkart-Sottile-Stroomer tableau switching and interlacing phenomenon

Theorem

Let $T \in St_n(\lambda/\mu, m)$ and $(Y(\mu), T) \overset{s}{\leftrightarrow} (P, R)$ where P has $n-1$ rows and $R \in LR(\lambda/\nu, \mu)$.

If $R = R^{(n-1)} \cup [L, n^{\mu_n}]$ and $(P, R^{(n-1)}) \overset{s}{\leftrightarrow} (Y(\mu^{(n-1)}), Q)$, then

- Q is obtained by reverse Schensted row insertion in the last row of T.
- $L = 1^{r_1} \cdots (n-1)^{r_{n-1}}$ such that $\mu - \mu^{(n-1)} = (r_1, \cdots, r_{n-1}, \mu_n)$.

Corollary

$T \in LR(\lambda/\mu, \nu)$, $(Y(\mu), T) \overset{s}{\leftrightarrow} (Y(\nu), R)$

- The GT pattern defining R can be obtained by successive reverse Schensted row insertion operations starting in the bottom row of T.
- $\rho_3(T) = \rho_1(T) = R$.

Proof by induction on $|L|$

- $|L|$=1

$$(Y(\mu), T) = \begin{array}{lllllllll} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & \spadesuit & \theta_1 \\ 3 & 3 & 3 & 3 & 3 & \diamondsuit_1 & \diamondsuit_2 & \theta_2 & \\ 4 & x & y & z & w & v & \theta_3 & & \\ \theta_4 & & & & & & & & \end{array} \xrightarrow{s} (P, R{=}R^{(n-1)} \cup L)$$

$$\theta_4 > \theta_3 > \theta_2 > \theta_1$$

$$(P, R^{(n-1)}) \xrightarrow{s} (Y(\mu'), Q) = \begin{array}{lllllllll} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \theta_1 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & \spadesuit & \theta_2 \\ 3 & 3 & 3 & 3 & 3 & \diamondsuit_1 & \diamondsuit_2 & \theta_3 & \\ 4 & x & y & z & w & v & \theta_4 & & \end{array}$$

1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	♠	θ_1
3	3	3	3	3	$\diamondsuit_1$	$\diamondsuit_2$	θ_2	
4	x	y	z	w	v	θ_3		
θ_4								

1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	$\spadesuit$	
3	3	3	3	3	$\Diamond_1$	$\Diamond_2$		
4	x	y	z	w	v			

1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	♠	
3	3	3	3	3	$\diamondsuit_1$	$\diamondsuit_2$		
4	x	y	z	w	v			

1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	♠	
$\diamondsuit_1$	3	$\diamondsuit_2$	3	3	3	3		
x	y	z	w	v	4			

1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	$\spadesuit$	
3	3	3	3	3	$\diamond_1$	$\diamond_2$		
4	x	y	z	w	v			

1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	$\spadesuit$	
$\diamond_1$	3	$\diamond_2$	3	3	3	3		
x	y	z	w	v	4			

1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	$\spadesuit$	
$\diamond_1$	y	$\diamond_2$	w	v	3	3		
x	z	3	3	3	4			

1	1	1	1	1	1	1	1	1
2	2	♠	2	2	2	2	2	
$\diamondsuit_1$	y	$\diamondsuit_2$	w	v	3	3		
x	z	3	3	3	4			

1	1	1	1	1	1	1	1	1
2	2	$\spadesuit$	2	2	2	2	2	
$\diamondsuit_1$	y	$\diamondsuit_2$	w	v	3	3		
x	z	3	3	3	4			

1	1	1	1	1	1	1	1	1
$\diamondsuit_1$	y	$\spadesuit$	w	v	2	2	2	
x	$\diamondsuit_2$	2	2	2	3	3		
z	2	3	3	3	4			

1	1	1	1	1	1	1	1	1
2	2	$\spadesuit$	2	2	2	2	2	
$\diamondsuit_1$	y	$\diamondsuit_2$	w	v	3	3		
x	z	3	3	3	4			

1	1	1	1	1	1	1	1	1
$\diamondsuit_1$	y	$\spadesuit$	w	v	2	2	2	
x	$\diamondsuit_2$	2	2	2	3	3		
z	2	3	3	3	4			

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1	1	1	1	1	1	1	1	1
$\diamondsuit_1$	y	$\spadesuit$	w	v	2	2	2	θ_1
x	$\diamondsuit_2$	2	2	2	3	3	θ_2	
z	2	3	3	3	4	θ_3		
θ_4								

1	1	1	1	1	1	1	1	1
$\diamond_1$	y	♠	w	v	θ_3	2	2	θ_1
x	$\diamond_2$	2	2	2	2	3	θ_2	
z	2	3	3	3	3	4		
θ_4								

1	1	1	1	1	1	1	1	1
$\diamond_1$	y	♠	w	v	θ_3	2	2	θ_1
x	$\diamond_2$	2	2	2	2	3	θ_2	
z	2	3	3	3	3	4		
θ_4								

1	1	1	1	1	1	1	1	1
$\diamond_1$	y	♠	θ_2	v	θ_3	2	2	θ_1
x	$\diamond_2$	w	2	2	2	3	3	
z	2	2	3	3	3	4		
θ_4								

1	1	1	1	1	1	1	1	1
$\diamond_1$	y	$\spadesuit$	w	v	θ_3	2	2	θ_1
x	$\diamond_2$	2	2	2	2	3	θ_2	
z	2	3	3	3	3	4		
θ_4								

1	1	1	1	1	1	1	1	1
$\diamond_1$	y	$\spadesuit$	θ_2	v	θ_3	2	2	θ_1
x	$\diamond_2$	w	2	2	2	3	3	
z	2	2	3	3	3	4		
θ_4								

1	1	1	1	1	1	1	1	1
$\diamond_1$	y	$\spadesuit$	θ_1	v	θ_3	2	2	2
x	$\diamond_2$	θ_2	2	2	2	3	3	
z	w	2	3	3	3	4		
θ_4								

- $$(P,R) = \begin{array}{lllllllll} \diamond_1 & y & \spadesuit & \theta_1 & v & \theta_3 & 1 & 1 & 1 \\ x & \diamond_2 & \theta_2 & 1 & 1 & 1 & 2 & 2 & 2 \\ z & w & 1 & 2 & 2 & 2 & 3 & 3 & \\ \theta_4 & 1 & 2 & 3 & 3 & 3 & 4 & & \\ 1 & & & & & & & & \end{array}$$

  $$\theta_4 > \theta_3 \geq v \geq w > \theta_2 > \theta_1$$

- $$(P,R^{(n-1)}) = \begin{array}{lllllllll} \diamond_1 & y & \spadesuit & \theta_1 & v & \theta_3 & 1 & 1 & 1 \\ x & \diamond_2 & \theta_2 & 1 & 1 & 1 & 2 & 2 & 2 \\ z & w & 1 & 2 & 2 & 2 & 3 & 3 & \\ \theta_4 & 1 & 2 & 3 & 3 & 3 & 4 & & \end{array}$$

$\theta_4 > \theta_3 \geq v \geq w > \theta_2 > \theta_1$

$\diamond_1$	y	♠	θ_1	v	θ_3	1	1	1
x	$\diamond_2$	θ_2	1	1	1	2	2	2
z	w	1	2	2	2	3	3	
1	2	3	3	3	4	θ_4		

$\theta_4 > \theta_3 \geq v \geq w > \theta_2 > \theta_1$

$\diamond_1$	y	♠	θ_1	v	θ_3	1	1	1
x	$\diamond_2$	θ_2	1	1	1	2	2	2
z	w	1	2	2	2	3	3	
1	2	3	3	3	4	θ_4		

$\diamond_1$	y	♠	θ_1	v	1	1	1	1
x	$\diamond_2$	θ_2	1	1	2	2	2	2
z	w	1	2	2	3	3	θ_3	
1	2	3	3	3	4	θ_4		

$\theta_4 > \theta_3 \geq v \geq w > \theta_2 > \theta_1$

$\diamond_1$	y	♠	θ_1	v	θ_3	1	1	1
x	$\diamond_2$	θ_2	1	1	1	2	2	2
z	w	1	2	2	2	3	3	
1	2	3	3	3	4	θ_4		

$\diamond_1$	y	♠	θ_1	v	1	1	1	1
x	$\diamond_2$	θ_2	1	1	2	2	2	2
z	w	1	2	2	3	3	θ_3	
1	2	3	3	3	4	θ_4		

$\diamond_1$	y	♠	θ_1	1	1	1	1	1
x	$\diamond_2$	θ_2	1	2	2	2	2	2
z	1	2	w	v	3	3	θ_3	
1	2	3	3	3	4	θ_4		

$\theta_4 > \theta_3 \geq v \geq w > \theta_2 > \theta_1$

$\diamond_1$	y	$\spadesuit$	θ_1	v	θ_3	1	1	1
x	$\diamond_2$	θ_2	1	1	1	2	2	2
z	w	1	2	2	2	3	3	
1	2	3	3	3	4	θ_4		

$\diamond_1$	y	$\spadesuit$	θ_1	v	1	1	1	1
x	$\diamond_2$	θ_2	1	1	2	2	2	2
z	w	1	2	2	3	3	θ_3	
1	2	3	3	3	4	θ_4		

$\diamond_1$	y	$\spadesuit$	θ_1	1	1	1	1	1
x	$\diamond_2$	θ_2	1	2	2	2	2	2
z	1	2	w	v	3	3	θ_3	
1	2	3	3	3	4	θ_4		

$\diamond_1$	y	$\spadesuit$	1	1	1	1	1	θ_1
x	$\diamond_2$	1	2	2	2	2	2	θ_2
z	1	2	w	v	3	3	θ_3	
1	2	3	3	3	4	θ_4		

- $v \geq w \geq z > \diamondsuit_2 > \spadesuit$

$$
\begin{array}{ccccccccc}
\diamondsuit_1 & y & \spadesuit & 1 & 1 & 1 & 1 & 1 & \theta_1 \\
x & \diamondsuit_2 & 1 & 2 & 2 & 2 & 2 & 2 & \theta_2 \\
z & 1 & 2 & w & v & 3 & 3 & \theta_3 & \\
1 & 2 & 3 & 3 & 3 & 4 & \theta_4 & &
\end{array}
$$

- $v \geq w \geq z > \diamond_2 > \spadesuit$

$$\begin{array}{ccccccccc} \diamond_1 & y & \spadesuit & 1 & 1 & 1 & 1 & 1 & \theta_1 \\ x & \diamond_2 & 1 & 2 & 2 & 2 & 2 & 2 & \theta_2 \\ z & 1 & 2 & w & v & 3 & 3 & \theta_3 & \\ 1 & 2 & 3 & 3 & 3 & 4 & \theta_4 & & \end{array}$$

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$$\begin{array}{cccccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \theta_1 \\ \diamond_1 & y & \spadesuit & w & v & 2 & 2 & 2 & \theta_2 & \\ x & \diamond_2 & 2 & 2 & 2 & 3 & 3 & \theta_3 & & \\ z & 2 & 3 & 3 & 3 & 4 & \theta_4 & & & \end{array}$$

- $v \geq w \geq z > \diamondsuit_2 > \spadesuit$

$$\begin{array}{ccccccccc}
\diamondsuit_1 & y & \spadesuit & 1 & 1 & 1 & 1 & 1 & \theta_1 \\
x & \diamondsuit_2 & 1 & 2 & 2 & 2 & 2 & 2 & \theta_2 \\
z & 1 & 2 & w & v & 3 & 3 & \theta_3 & \\
1 & 2 & 3 & 3 & 3 & 4 & \theta_4 & &
\end{array}$$

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$$\begin{array}{cccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \theta_1 \\
\diamondsuit_1 & y & \spadesuit & w & v & 2 & 2 & 2 & \theta_2 & \\
x & \diamondsuit_2 & 2 & 2 & 2 & 3 & 3 & \theta_3 & & \\
z & 2 & 3 & 3 & 3 & 4 & \theta_4 & & &
\end{array}$$

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$$\begin{array}{ccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \theta_1 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & \spadesuit & \theta_2 \\
3 & 3 & 3 & 3 & 3 & \diamondsuit_1 & \diamondsuit_2 & \theta_3 & \\
4 & x & y & z & w & v & \theta_4 & &
\end{array}$$

Conjecture: ρ_1, ρ_2 and ρ_3 coincide with the involution defined by $AB = C$.