Growth Rates of Permutation Classes

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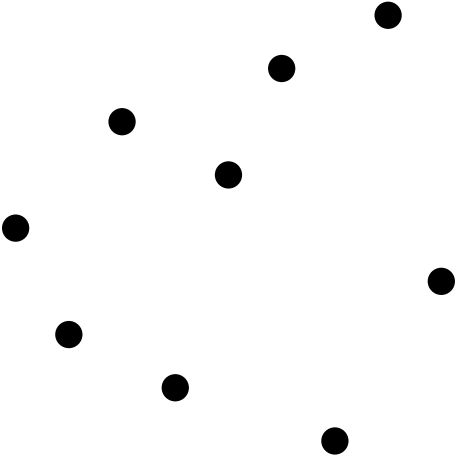
Permutation and Subpermutation



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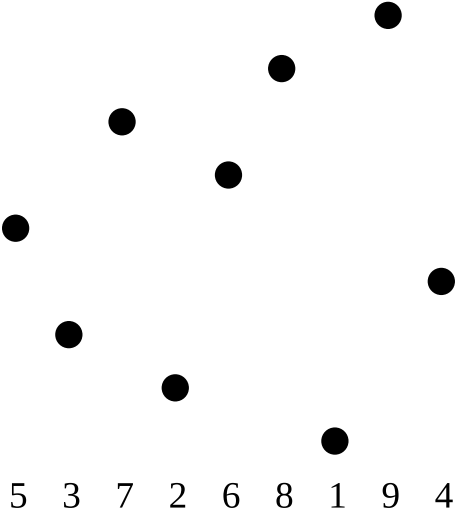

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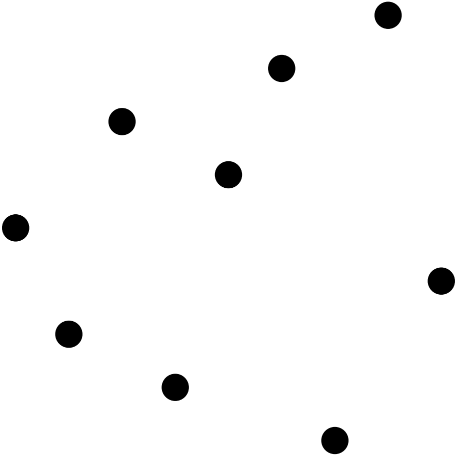
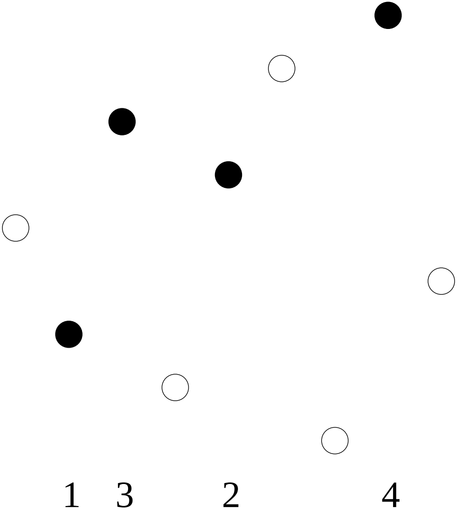


5 **3** **7** 2 **6** 8 1 **9** 4

Permutation and Subpermutation



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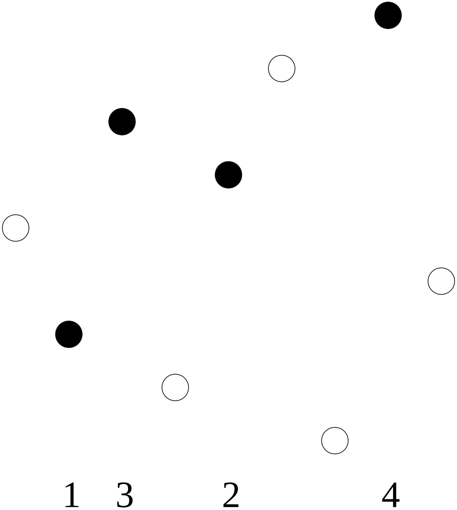

$1324 \preceq 537268194$

Definition

If we construe permutations as sequences, then the *involvement order* on permutations is defined by:

$$\alpha \preceq \beta \quad \text{iff} \quad \left\{ \begin{array}{l} \beta \text{ contains a subsequence} \\ \text{whose terms are in the same} \\ \text{relative order as those of } \alpha \end{array} \right\}$$

A *permutation class*, $\mathcal{C}$ is a down-closed set for $\preceq$. Its *basis* X consists of the $\preceq$-minimal permutations *not in* $\mathcal{C}$, and then:

$$\mathcal{C} = \mathrm{Av}(X) \stackrel{\text{def}}{=} \{\beta : \forall \alpha \in X \ \alpha \not\preceq \beta\}.$$

Av(21)

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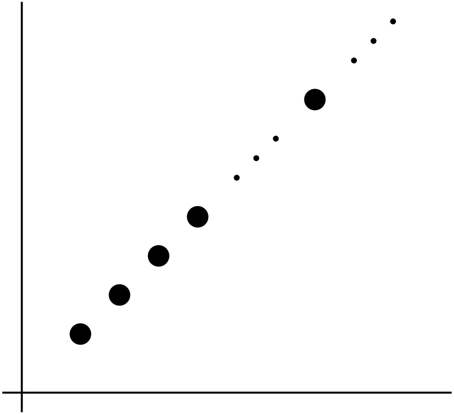

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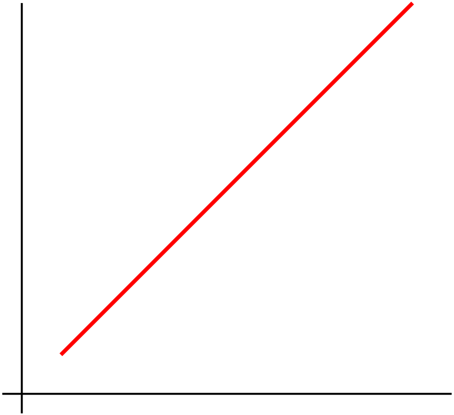

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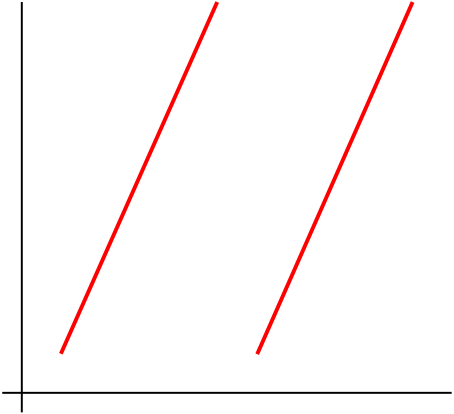
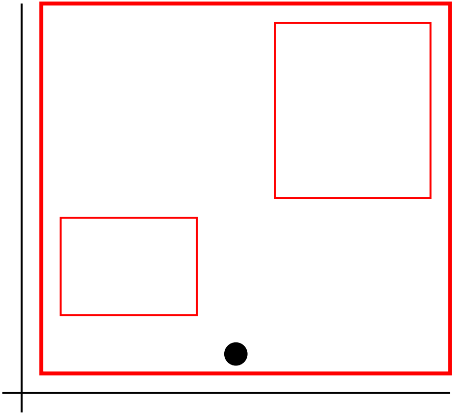
Av(321, 2143, 3142)



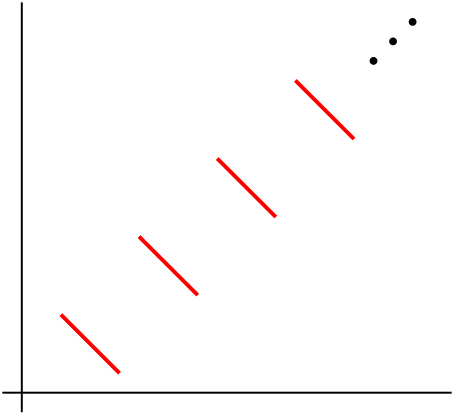
$\mathrm{Av}(321, 2143, 3142) = \{\alpha\beta : \text{both } \alpha \text{ and } \beta \text{ are increasing}\}$

Av(312)



$\mathrm{Av}(312) = \{\alpha\, 1\, \beta \;:\; \alpha < \beta$ both avoiding 312$\}$

$\mathrm{Av}(312, 231)$



$\mathrm{Av}(312, 231) = \{\alpha_1 \alpha_2 \cdots \alpha_k \ : \ \alpha_i \text{ decreasing layers}\}$

Prehistory I

Theorem (Erdős-Szekeres)

Every permutation of length larger than $(n-1)(k-1)$ *contains either the pattern* $123\ldots n$ *or the pattern* $k(k-1)\ldots 321$*.*

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In particular, a permutation class is finite if and only if its basis contains both an increasing and a decreasing permutation.

Prehistory II

Theorem (Knuth)

The permutations that can be generated by a single stack operating on the input sequence $1, 2, \ldots, n$ *are precisely the ones that avoid the pattern 312. There is a one to one correspondence between such permutations and allowable operation sequences of the stack.*

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The structural decomposition of these permutations illustrated earlier also translates naturally into an equation for their generating function C:

$$C = 1 + CtC.$$

Observations

- Av(321) has the same enumeration sequence as Av(312).
- By symmetries, all Av(π) with $|\pi| = 3$ have the same enumeration
- In *principal classes* with a single basis element of length four, other such enumerative coincidences exist. Up to symmetry there should be seven such classes, in fact there are only three.
- All these classes have an exponential bound to their size.

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Every proper subclass of the class of all permutations has size $O(c^n)$ for some constant $c > 0$ (c can depend on the class of course!)

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$$c \leq 15^{2k^4\binom{k^2}{k}}$$
- Arratia had conjectured $c \leq (k-1)^2$, the actual value for the class Av($123 \cdots k$), but this was refuted for $\pi = 4231$ by MA et al.

What now?

For every proper class $\mathcal{C}$ we may define the *(upper) growth rate*:

$$\mathrm{gr}(\mathcal{C}) = \limsup_{n\to\infty} |\mathcal{C} \cap \mathcal{S}_n|^{1/n}.$$

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- What values can occur?
- Is the limit superior always a limit?
- How do the growth rates of subclasses of $\mathcal{C}$ compare with the growth rate of $\mathcal{C}$ itself?
- Can we compute or estimate the growth rates of “interesting” classes?

$\mathrm{gr}(\mathcal{C}) \leq 2$

These were classified by Kaiser and Klazar (EJC **9**, 2003):

Theorem

If $\mathrm{gr}(\mathcal{C}) < 2$ *then:*

- $\mathcal{C}$ *contains no "infinite alternation"*
- $\mathrm{gr}(C)$ *satisfies a polynomial:*

$$1 - x - x^2 - \cdots - x^k = 0$$

  for some $k \geq 1$*.*

- *More specifically* $F_{n,k} \leq |\mathcal{C} \cap \mathcal{S}_n| \leq F_{n,k} n^c$*.*
- *All such classes have rational generating functions.*

The Fibonacci Dichotomy

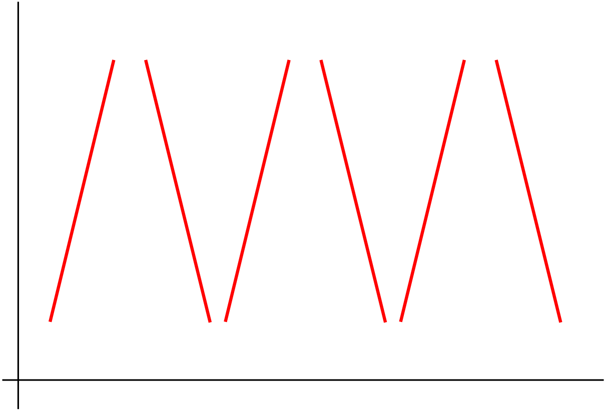
We will illustrate the structural dichotomy that splits the classes of polynomial growth (growth rate 1) from those of properly exponential growth. Throughout, we are considering a class $\mathcal{C}$. The permutation π is called an *up-down* permutation if:

$$\pi(1) < \pi(2) > \pi(3) < \pi(4) > \pi(5) < \cdots$$

Observation

If $\mathcal{C}$ contains only finitely many up-down permutations, then it is contained in a W-class.

What's a *W*-class?



Here there are a fixed number of segments.

Fibonacci Dichotomy (cont'd)

Conversely, if $\mathcal{C}$ contains infinitely many up-down permutations, then the tree of up-down permutations belonging to $\mathcal{C}$ (where the children of a node are its one point extensions), contains an infinite branch.

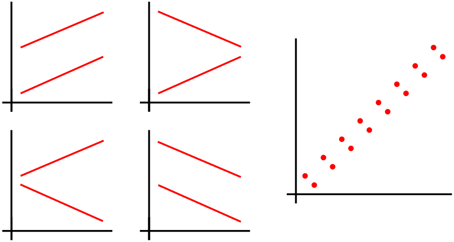
This gives rise to a one to one map $\gamma : \mathbb{N} \longrightarrow \mathbb{R}$ with

$$\gamma(1) < \gamma(2) > \gamma(3) < \gamma(4) > \gamma(5) < \dots$$

such that $\mathcal{C}$ contains every subpermutation of γ.

Enter Ramsey

Consider pairs $(\gamma(2k), \gamma(2k+1))$ and the four element patterns formed by two such pairs. As there are only finitely many possibilities for these patterns, there exists an infinite set of such pairs all creating the same pattern. This implies that $\mathcal{C}$ would contain one of the classes:



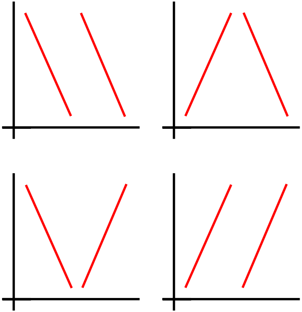
The first four of these have growth rate 2, while the last is enumerated by the Fibonacci numbers.

Back to *W*ork

So now, $\mathcal{C}$ either has at least Fibonacci growth, or is contained in a W-class.

Back to *W*ork

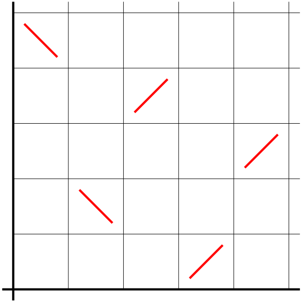
If a segment in a W class is “cut” arbitrarily often, then a pigeonhole argument shows that $\mathcal{C}$ contains one of the subclasses:



all of growth rate 2. What if there is a bound on the number of times each segment is cut?

The end of the story

Since this applies to each segment, $\mathcal{C}$ is contained in a class obtained from a permutation matrix, replacing each non-zero element by an increasing or decreasing segment.



As the segments do not interact, these classes have polynomial growth.

Moving above 2

Theorem

Let $\kappa \sim 2.20557$ be the unique positive root of

$$1 + 2x^2 - x^3 = 0.$$

There are only countably many classes of growth rate less than κ (and these growth rates can be explicitly described as the roots of certain polynomials), and there are uncountably many permutation classes of growth rate κ.

The proof, shows that such small classes have "good structure", while, at κ infinite antichains appear.

V. Vatter, *Small Permutation Classes*, arXiv:0712.4006v2

Here there be dragons

Theorem

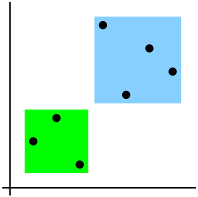
(MA, Steve Linton) There is a perfect set of growth rates in the interval $[2.47, 3]$.

To provide a quick overview of the arguments leading to this result, just a few more definitions are requred.

Sum of permutations

- The *sum* $\alpha \oplus \beta$ of permutations α and β is obtained by stacking the graph of β above and to the right of that of α.

$$231 \oplus 4132 = 231\,7465.$$



- A permutation is *plus indecomposable* if it cannot be written as a sum of shorter permutations.
- A class $\mathcal{C}$ is *plus closed* if

$$\alpha, \beta \in \mathcal{C} \quad \text{implies} \quad \alpha \oplus \beta \in \mathcal{C}.$$

Plus closed classes

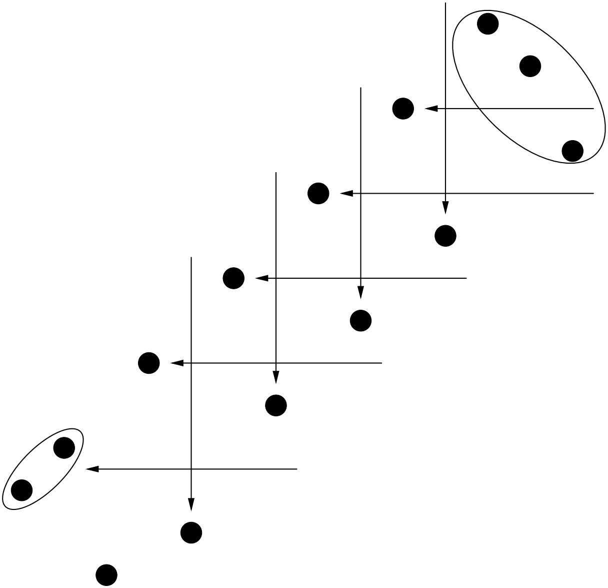
If a class $\mathcal{C}$ is plus closed, let its generating function be $C(t)$ and the generating function of its plus indecomposables be $C^+(t)$. Then of course:

$$C(t) = \frac{1}{1 - C^+(t)}$$

If, for some s inside the radius of convergence of $C^+(t)$, $C^+(s) = 1$, then s will be the radius of convergence of $C(t)$ (and hence the reciprocal of its growth rate.)

The plan is to construct a variety of plus closed classes satisfying this condition including ones whose plus indecomposables are the same up to some (arbitrary) length n.

First catch your antichain



Remaining highlights

- Consider classes whose bases are subsets of this antichain, plus a few extra elements for structural convenience.
- Argue that all such classes are plus closed and satisfy the condition mentioned previously.
- Argue that, for all $\epsilon > 0$ there exists *n* such that if the bases agree through length *n*, then the corresponding radii of convergence are within ϵ.
- Compute actual growth rates of the smallest and largest classes in this group to give the range in which the perfect set lies.

Late breaking news

- Vince Vatter (personal communication, 20/6/2008) has reported an interval of growth rates from 2.51002 to 2.51534.
- It appears to be correct, and uses sharpenings of the previous ideas (allowing one to control the plus indecomposables in the classes directly rather than via a basis.)
- (24/6/2008) This has almost certainly been extended to show that all growth rates above 2.6 exist.

Classes and Subclasses

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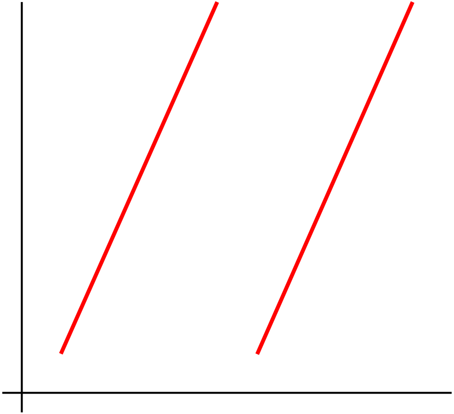
$$\sup\{\mathrm{gr}(\mathcal{D}) \,:\, \mathcal{D} \text{ a proper subclass of } \mathcal{C}\} < \mathrm{gr}(\mathcal{C}).$$

- A class, $\mathcal{C}$ is *smooth* if it is growth rate critical, but not rough.

$\mathrm{Av}(321, 2143, 3142)$ is rough



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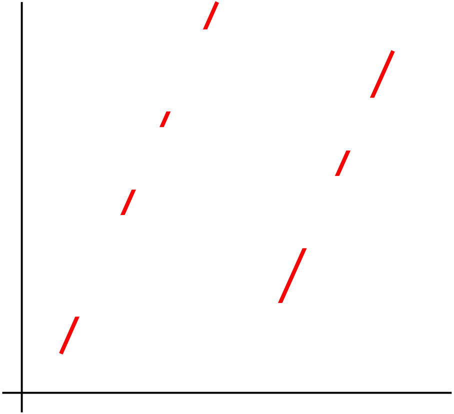
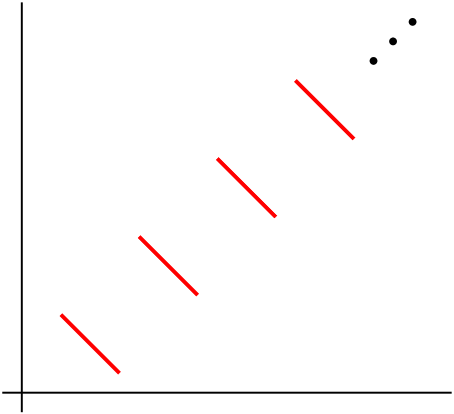


In a subclass, there can be only a bounded number of segments on each side, hence polynomial growth.

Av(312, 231) is smooth



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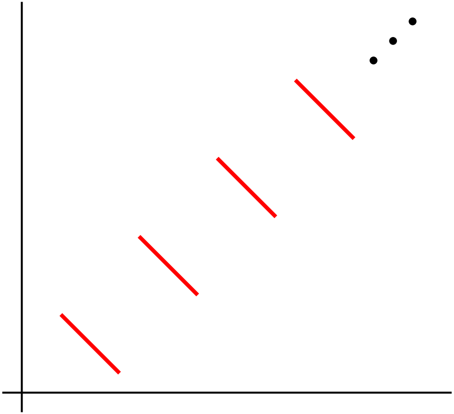

We could take a subclass in which each layer contains at most t elements. As $t \to \infty$ their growth rates converge to 2.

Av(312) is also smooth

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- The generating function of any proper subclass of Av(312) is rational.
- Therefore, it cannot have radius of convergence $1/4$ as the limiting behaviour at $1/4$ is wrong.
- And, it is easy to build subclasses whose growth rates converge to 4.

Commercial announcement

For all your class counting needs, please visit:

http://www.cs.otago.ac.nz/staffpriv/mike/HomePages/classcounter/ClassCounterApplet.html

Shorter URL (and google-ability) coming soon . . .

For instance Av(4231, 35142, 42513, 351624):

1	2	6	23	101	477
2343	11762	59786	306132	1574536	8120782
41957030	217021682	. . .			

$\mathrm{Av}(4231, 35142, 42513, 351624)$

- Enough terms to conjecture (thanks to M. Rubey) that the generating function f satisfies:

  $$(4t^4 - 5t^3 + 6t^2 - t)f^2 + (8t^3 - 2t^2 - 5t + 1)f + 4t^2 + 4t - 1 = 0$$

- This yields $f \in \mathbb{Q}(\sqrt{1-4t})$.
- "Semi-automatic" methods (generating trees/insertion encoding) make the confirmation of this a (moderate) exercise.

Av(4231)

- The last principal class with a four element basis element whose growth rate/generating function is unknown.
- Known to have growth rate at least 9.47 (MA, M. Elder, A. Rechnitzer, P. Westcott, M. Zabrocki, Adv. in Appl. Math., **36**, 2006).
- Obtained by approximating the class with certain others that are recognized by finite state machines.

Terra incognita

- What is the growth rate of Av(4231)?
- Does every permutation class have a true growth rate?
- ~~Is every sufficiently large number the growth rate of a permutation class?~~
- What are the boundaries between structure and chaos in permutation classes?