Combinatorial Gelfand Models

Ron Adin

Bar-Ilan University

Alexander Postnikov

MIT

Yuval Roichman

Bar-Ilan University

FPSAC '08

Valparaiso, Chile

1. Gelfand Models

A *Gelfand model* for a group G is

$$\psi \cong \bigoplus_{\rho \in \mathrm{Irr(G)}} \rho,$$

the multiplicity-free direct sum of **all** irreducible G-representations.

Problem. **[Bernstein-Gelfand-Gelfand ’75]** Given a group G, construct a Gefand model for G.

History:

BGG, Klyachko, Saxl, Verma and others:

Gow, Inglis-Richardson-Saxl, Baddeley, Kawanaka-Matsuyama, Ryan, Aguado-Araujo-Bigeón, Bump-Ginzburg, Kodiyalam-Verma, Sayag ...

2. The Symmetric Group S_n

Let

$$I_n := \{w \in S_n \quad : \quad w^2 = 1\}.$$

Folklore: RSK $\Longrightarrow$

$$\#I_n = \sum_{\rho \in \mathrm{Irr(S_n)}} \dim(\rho).$$

Goal:

Determine a simple combinatorial action on I_n which gives a model for S_n and for related groups and algebras.

Preliminaries

Let S_n be the symmetric group on n letters.

The Coxeter generating set of S_n is

$$S = \{s_i \ : \ 1 \le i < n\},$$

where $s_i := (i, i+1)$.

The descent set of $\pi \in S_n$ is

$$\mathrm{Des}(\pi) := \{0 \le i < n \ : \ \pi(i) > \pi(i+1)\}$$

$$= \{0 \le i < n \ : \ \ell(\pi s_i) < \ell(\pi)\}.$$

3. Signed Conjugation

Let $V_n := \text{span}_{\mathbf{Q}}\{C_w \;:\; w \in I_n\}$.

Define $\rho : S \longrightarrow GL(V_n)$ by

$$\rho(s_i)C_w := \begin{cases} -C_{s_i w s_i}, & s_i w s_i = w \;\wedge\; i \in \text{Des}(w)\ ; \\ C_{s_i w s_i}, & \text{otherwise.} \end{cases}$$

Theorem 1.
ρ determines a Gelfand model for S_n.

Restricting ρ to a Conjugacy Class

Let

$\rho|_{2^k 1^{n-2k}}$ - the restriction of ρ to the conjugacy class of cycle type $2^k 1^{n-2k}$;

S^λ - the irreducible S_n-representation indexed by the partition λ .

Proposition 2:

S^λ appears in $\rho|_{2^k 1^{n-2k}} \iff$

λ has $n - 2k$ odd columns .

Corollary 3:

$$\#\{w \in I_n : \text{fix}(w) = m\}$$

$$= \#\text{SYT}(n) \text{ with } m \text{ odd columns.}$$

Proof of Theorem 1:

Part 1: ρ determines a representation.

For $\pi \in S_n$ and $w \in I_n$ let

$$\mathrm{Pair}(w) := \{2-\text{cycles of } w\}$$

and

$$\mathrm{inv}_w(\pi) := \#\{(i,j) \in \mathrm{Pair}(w) \ : \ i < j,\ \pi(i) > \pi(j)\}.$$

Example. $w = (1,4)(3,5)$, $\pi = [2,5,3,4,1]$. Then $\mathrm{inv}_w(\pi) = 1$.

Lemma

$$\rho(\pi) C_w = (-1)^{\mathrm{inv}_w(\pi)} \cdot C_{\pi w \pi^{-1}}.$$

Proof of Theorem 1:

Part 2: ρ determines a model.

Frobenius-Schur Theorem Let G be a finite group, all of whose representations are real.

Then for every $w \in G$

$$\sum_{\chi \in \mathrm{Irr(G)}} \chi(w) = \#\{u \in G \mid u^2 = w\},$$

Lemma For every $\pi \in S_n$

$$\sum_{w \in I_n} (-1)^{\mathrm{inv}_w(\pi)} = \#\{u \in G \mid u^2 = w\}.$$

4. Wreath Products

Consider $G = \boldsymbol{Z}_r \wr S_n$.

For $r > 2$, $\boldsymbol{Z}_r \wr S_n$ is not real.

Let $\omega := e^{2\pi i/r}$.
For $v \in \boldsymbol{Z}_r \wr S_n$ let $\bar{v}$ be its *complex conjugate.*

Example

$$v = \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & 0 & \omega^0 \\ 0 & \omega^1 & 0 \end{pmatrix} \qquad \bar{v} = \begin{pmatrix} \omega^{-2} & 0 & 0 \\ 0 & 0 & \omega^0 \\ 0 & \omega^{-1} & 0 \end{pmatrix}$$

Theorem 4. For any $\pi \in G = \boldsymbol{Z}_r \wr S_n$,

$$\sum_{\chi \in \mathrm{Irr}(G)} \chi(\pi) = \#\{v \in G \,|\, v \cdot \bar{v} = \pi\},$$

Let

$I_{r,n}$ - the set of symmetric elements in $\boldsymbol{Z}_r \wr S_n$,

i.e., $\{v \in \boldsymbol{Z}_r \wr S_n \ : \ v \cdot \bar{v} = 1\}$

S - the standard generating set of simple complex reflections in $\boldsymbol{Z}_r \wr S_n$.

Theorem 5. There exists a signed two-sided action on $I_{r,n}$

$$\rho(s_i)C_v := \text{sign}(i;\ v) \cdot C_{s_i v s_i}$$

(for all $s_i \in S$, $v \in I_{r,n}$),

which determines a Gelfand model for $\boldsymbol{Z}_r \wr S_n$.

For $v \in \boldsymbol{Z}_r \wr S_n$ let $|v| := (|v_{i,j}|)$. Then

$$\operatorname{sign}(i;\ v) := \begin{cases} -1, & s_i v s_i = v \text{ and } s_i \in \operatorname{Des}(|v|); \\ 1, & \text{otherwise} \end{cases}$$

for all $i > 0$

$$\operatorname{sign}(0;\ v) := \begin{cases} -1, & v_{1,1} = \omega^{-1} \text{ and } r \text{ is even}; \\ 1, & \text{otherwise} \end{cases}$$

Corollary For $r = 2$ (i.e., for the Weyl group of type B)

$$\operatorname{sign}(0;\ v) := \begin{cases} -1, & \text{if } s_0 v s_0 = v \text{ and } s_0 \in \operatorname{Des}(v); \\ 1, & \text{otherwise} \end{cases}$$

5. Hecke Algebra

Let $\mathcal{H}_n(q)$ be the Hecke algebra of S_n,

with set of generators $\{T_i \mid 1 \leq i < n\}$

and defining relations

$$(T_i + q)(T_i - 1) = 0 \qquad (\forall i),$$

$$T_i T_j = T_j T_i \qquad \text{if} \quad |i - j| > 1,$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \qquad (1 \leq i < n - 1).$$

Define a map $\rho_q : S \to GL(V_n)$ by

$$\rho_q(T_s)C_w := \begin{cases} -q\ C_w, & sws = w \wedge s \in \mathrm{Des}(w) \\ C_w, & sws = w \wedge s \notin \mathrm{Des}(w) \\ (1-q)\ C_w + & \\ +q\ C_{sws}, & w <_I sws \\ C_{sws}, & sws <_I w. \end{cases}$$

Theorem 6. ρ_q is a Gelfand model for $\mathcal{H}_n(q)$.

Proof Idea:

By Lusztig's version of Tits' deformation theorem, deforming representation matrices gives "essentially the same" representation.
Now, apply Theorem 1.

By the deformation theorem,
ρ_q may be viewed as an S_n-representation;
the characters are rational functions of $q^{1/2}$.

By discreteness of the S_n character values,
each such function is locally constant,
and is thus constant globally.

By Theorem 1, $\rho_q|_{q=1} = \rho$ is a model for the group algebra of S_n.

$\square$

The Involutive Order on Involutions

Let $g_k := s_1 s_3 \cdots s_{2k-1} \in I_n$.

Define the *involutive length*

of an involution $w \in I_n$ of cycle type $2^k 1^{n-2k}$

$$\hat{\ell}(w) := \min\{\ell(v) |\ w = v g_k v^{-1},\ v \in S_n\},$$

where $\ell(v)$ is the standard length of $v \in S_n$.

Define the *involutive weak order* $\leq_I$ on I_n as the reflexive and transitive closure of the relation:

$$w \prec_I sws \text{ if } \ \exists s \in S \ \ \hat{\ell}(sws) = \hat{\ell}(w) + 1.$$

Characters

Unimodal Permutations

$\pi = 2\ 7\ 15\ 10\ 3 \mid 5\ 13\ 12\ 11 \mid 1\ 9\ 14 \mid 8\ 6\ 4$
is (5,4,3,3)-unimodal.

Let $T_\mu := T_1 \cdots \hat{T}_{\mu_1} T_{\mu_1+1} \cdots \hat{T}_{\mu_1+\mu_2} \cdots\cdots T_{n-1}$.

[R '97, APR '00]

$$\chi_q^\lambda(T_\mu) = \sum_{\{w \mapsto P \mid \ w \text{ is } \mu\text{-unimodal}\}} (-q)^{\mathrm{des}(w)}$$

$$\mathrm{Tr}\ (\psi_k(T_\mu)) = \sum_{\{\ell(w)=k \mid \ w \text{ is } \mu\text{-unimodal}\}} (-q)^{\mathrm{des}(w)}$$

Theorem 7.

$$\mathrm{Tr}\ (\rho_q(T_\mu)) = \sum_{\{w^2=1 \mid \ w \text{ is } \mu\text{-unimodal}\}} (-q)^{\mathrm{des}(w)}$$

6. Open Problems

Question 1: Define a simple action on involutions of a Coxeter group W, which determines a Gelfand model.

Known for S_n, B_n, D_{2n+1}. Open for D_{2n} ....

Question 1': Construct a Gelfand model for complex reflection groups; affine Weyl groups; Iwahori Hecke algebras.

Question 2: Define a simple action on symmetric elements in $GL_n(\boldsymbol{F}_q)$, which determines a Gelfand model.

Restriction in $Z_r \wr S_n$:

Show that the Stanton-White colored RSK is compatible with restriction of ρ.