Formal Power Series and Algebraic Combinatorics
Nankai University, Tianjin, China, 2007

Catalan's intervals and realizers of triangulations

Olivier Bernardi and Nicolas Bonichon

Abstract. The Stanley lattice, Tamari lattice and Kreweras lattice are three remarkable orders defined on the set of Catalan objects of a given size. These lattices are ordered by inclusion: the Stanley lattice is an extension of the Tamari lattice which is an extension of the Kreweras lattice. The Stanley order can be defined on the set of Dyck paths of size n as the relation of *being above*. Hence, intervals in the Stanley lattice are pairs of non-crossing Dyck paths. In a former article, the second author defined a bijection Φ between pairs of non-crossing Dyck paths and the realizers of triangulations (or Schnyder woods). We give a simpler description of the bijection Φ. Then, we study the restriction of Φ to Tamari's and Kreweras' intervals. We prove that Φ induces a bijection between Tamari intervals and minimal realizers. This gives a bijection between Tamari intervals and triangulations. We also prove that Φ induces a bijection between Kreweras intervals and the (unique) realizers of stack triangulations. Thus, Φ induces a bijection between Kreweras intervals and stack triangulations which are known to be in bijection with ternary trees.

Résumé. Les treillis de Stanley, de Tamari et de Kreweras sont trois ordres remarquables définis sur les objets de Catalan d'une taille donnée. Ces treillis sont ordonnés par inclusion : le treillis de Stanley est une extension du treillis de Tamari qui lui-même est une extension du treillis de Kreweras. L'ordre de Stanley peut être défini sur l'ensemble des chemins de Dyck de taille n par la relation d'*être au-dessus de*. Ainsi les intervalles du treillis de Stanley sont les paires de chemins de Dyck qui ne se coupent pas. Dans un article precedent, le second auteur définit une bijection Φ entre les paires de chemins de Dyck qui ne se coupent pas et les réaliseurs des triangulations (ou arbres de Schnyder). Nous donnons une description plus simple de cette bijection Φ. Nous étudions ensuite la restriction de Φ aux intervalles de Tamari et de Kreweras. Nous montrons que Φ induit une bijection entre les intervalles de Tamari et les réaliseurs minimaux. Ceci donne une bijection entre les intervalles de Tamari et les triangulations. Nous prouvons également que Φ induit une bijection entre les intervalles de Kreweras et les réaliseurs (uniques) des triangulations 3-dégénérées. Ainsi, Φ induit une bijection entre les intervalles de Kreweras et les triangulations 3-dégénérées dont on sait qu'elles sont en bijection avec les arbres ternaires.

1. Introduction

A *Dyck path* is a lattice path made of $+1$ and -1 steps that starts from 0, remains non-negative and ends at 0. It is often convenient to represent a Dyck path by a sequence of North-East and South-East steps as is done in Figure 1 (a). The set $\mathbf{D}_n$ of Dyck paths of length $2n$ can be ordered by the relation $P \leq_S Q$ if P stays below Q. This partial order is in fact a distributive lattice on $\mathbf{D}_n$ known as the *Stanley lattice*. The Hasse diagram of the Stanley lattice on $\mathbf{D}_3$ is represented in Figure 2 (a).

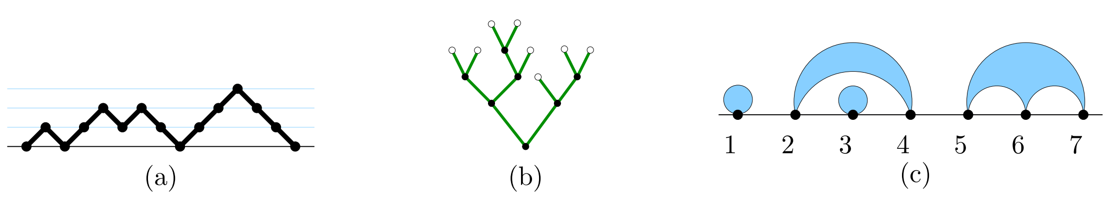

Figure 1. (a) A Dyck path. (b) A binary tree. (c) A non-crossing partition.

2000 *Mathematics Subject Classification.* Primary 05A15; Secondary 05C30.

Key words and phrases. Triangulations, realizers, bijection, Tamari lattice, Kreweras lattice.

The two authors are supported by the project *GeoComp* of the ACI Masses de Données.

It is well known that the Dyck paths of length $2n$ are counted by the n^{th} *Catalan number* $C_n = \frac{1}{n+1}\binom{2n}{n}$. The Catalan sequence is a pervasive guest in enumerative combinatorics. Indeed, beside Dyck paths, this sequence enumerates the binary trees, the plane trees, the non-crossing partitions and over 60 other fundamental combinatorial structures [**Sta99**, Ex. 6.19]. These different incarnations of the Catalan family gave rise to several lattices beside Stanley's. The *Tamari lattice* appears naturally in the study of binary trees where the covering relation corresponds to right rotation. This lattice is actively studied due to its link with the associahedron (Stasheff polytope). Indeed, the Hasse diagram of the Tamari lattice is the 1-skeleton of the associahedron. The *Kreweras lattice* appears naturally in the setting of non-crossing partitions. In the seminal paper [**Kre71**], Kreweras proved that the refinement order on non-crossing partitions defines a lattice. Kreweras lattice appears to support a great deal of mathematics that reach far beyond enumerative combinatorics [**McC06, Sim00**]. Using suitable bijection between Dyck paths, binary trees, non-crossing partitions and plane trees, the three *Catalan lattices* can be defined on the set of plane trees of size n in such way that the Stanley lattice $\mathcal{L}_n^S$ is an extension of the Tamari lattice $\mathcal{L}_n^T$ which in turn is an extension of the Kreweras lattice $\mathcal{L}_n^K$ (see [**Knu06**, Ex. 7.2.1.6 - 26, 27 and 28]). In this paper, we shall find convenient to embed the three Catalan lattices on the set $\mathbf{D}_n$ of Dyck paths. The Hasse diagram of the Catalan lattices on $\mathbf{D}_3$ is represented in Figure 2.

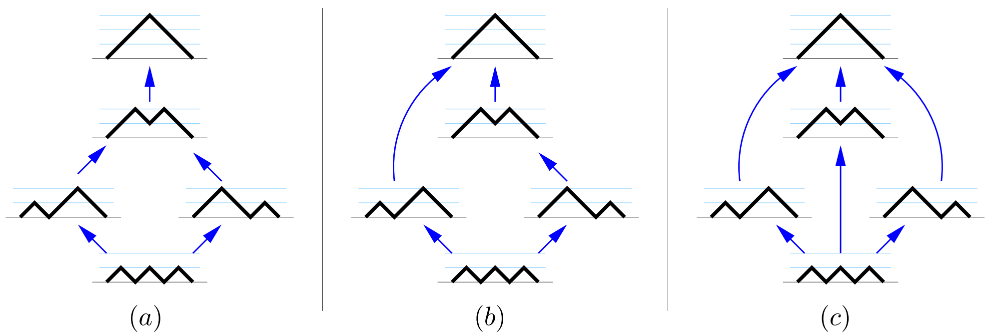

FIGURE 2. Hasse diagrams of the Catalan lattices on the set $\mathbf{D}_3$ of Dyck paths: (a) Stanley lattice, (b) Tamari lattice, (c) Kreweras lattice.

There are closed formulas for the number of *intervals* (i.e. pairs of comparable elements) in each of the Catalan lattices. The intervals of the Stanley lattice are the pairs of non-crossing Dyck paths and the number $|\mathcal{L}_n^S|$ of such pairs can be calculated using the lattice path determinant formula of Lindström-Gessel-Viennot [**GV85**]. It is shown in [**DSCV86**] that

$$|\mathcal{L}_n^S| \;=\; C_{n+2}C_n - C_{n+1}^2 \;=\; \frac{6(2n)!(2n+2)!}{n!(n+1)!(n+2)!(n+3)!}. \tag{1}$$

The intervals of the Tamari lattice were recently enumerated by Chapoton [**Cha06**] using a generating function approach. It was proved that the number of intervals in the Tamari lattice is

$$|\mathcal{L}_n^T| \;=\; \frac{2(4n+1)!}{(n+1)!(3n+2)!}. \tag{2}$$

Chapoton also noticed that (2) is the number of triangulations (i.e. maximal planar graphs) and asked for an explanation. The number $|\mathcal{L}_n^K|$ of intervals of the Kreweras Lattice has an even simpler formula. In [**Kre71**], Kreweras proved by a recursive method that

$$|\mathcal{L}_n^K| \;=\; \frac{1}{2n+1}\binom{3n}{n}. \tag{3}$$

This is also the number of ternary trees and a bijection was exhibited in [**Ede82**].

In [**Bon05**], the second author defined a bijection Φ between the pairs of non-crossing Dyck paths (equivalently, Stanley's intervals) and the *realizers* (or *Schnyder woods*) of triangulations. The main purpose of this article is to study the restriction of the bijection Φ to the Tamari intervals and to the Kreweras intervals. We first give an alternative, simpler, description of the bijection Φ. Then, we prove that the bijection Φ induces a bijection between the intervals of the Tamari lattice and the realizers which are *minimal*. Since every triangulation has a unique *minimal* realizer, we obtain a bijection between Tamari intervals and triangulations. As a corollary, we obtain a bijective proof of Formula (2) thereby answering the question of Chapoton. Turning to the Kreweras lattice, we prove that the mapping Φ induces a bijection between Kreweras intervals and the realizers which are both *minimal* and *maximal*. We then characterize the triangulations having a realizer which is both minimal and maximal and prove that these triangulations are in bijection with ternary trees. This gives a new bijective proof of Formula (3).

The outline of this paper is as follows. In Section 2, we review our notations about Dyck paths and characterize the covering relations for the Stanley, Tamari and Kreweras lattices in terms of Dyck paths. In Section 3, we recall the definitions about triangulations and realizers. We then give an alternative description of the bijection Φ defined in [**Bon05**] between pairs of non-crossing Dyck paths and the realizers. In Section 4, we study the restriction of Φ to the Tamari intervals. Lastly, in Section 5 we study the restriction of Φ to the Kreweras intervals. Due to space limitation, some proofs are either skipped or sketched.

2. Catalan lattices

Dyck paths. A *Dyck path* is a lattice path made of steps $N = +1$ and $S = -1$ that starts from 0, remains non-negative and ends at 0. A Dyck path is said to be *prime* if it remains positive between its start and end. The *size* of a path is half its length and the set of Dyck paths of size n is denoted by $\mathbf{D}_n$.

Let P be a Dyck path of size n. Since P begins by an N step and has n N steps, it can be written as $P = NS^{\alpha_1}NS^{\alpha_2}\dots NS^{\alpha_n}$. We call i^{th} *descent* the subsequence S^{α_i} of P. For $i = 0, 1, \dots, n$ we call i^{th} *exceedence* and denote by $e_i(P)$ the height of the path P after the i^{th} descent, that is, $e_i(P) = i - \sum_{j\leq i}\alpha_j$. For instance, the Dyck path represented in Figure 3 (a) is $P = NS^1NS^0NS^1NS^2NS^0NS^0NS^3$ and $e_0(P) = 0$, $e_1(P) = 0$, $e_2(P) = 1$, $e_3(P) = 1$, $e_4(P) = 0$, $e_5(P) = 1$, $e_6(P) = 2$ and $e_7(P) = 0$. If P, Q are two Dyck paths of size n, we denote $\delta_i(P, Q) = e_i(Q) - e_i(P)$ and $\Delta(P, Q) = \sum_{i=1}^{n}\delta_i(P, Q)$. For instance, if P and Q are respectively the lower and upper paths in Figure 3 (b), the values $\delta_i(P, Q)$ are zero except for $\delta_1(P, Q) = 1$, $\delta_4(P, Q) = 2$ and $\delta_5(P, Q) = 1$.

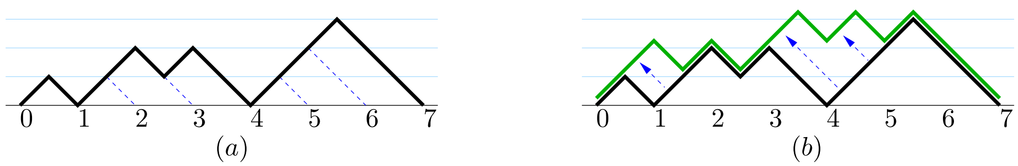

FIGURE 3. (a) Exceedence of a Dyck path. (b) Differences between two Dyck paths.

For $0 \leq i \leq j \leq n$, we write $i \underline{P}\, j$ (resp. $i \underline{\underline{P}}\, j$) if $e_i(P) \geq e_j(P)$ and $e_i(P) \leq e_k(P)$ (resp. $e_i(P) < e_k(P)$) for all $i < k < j$. In other words, $i \underline{P}\, j$ (resp. $i \underline{\underline{P}}\, j$) means that the subpath $NS^{\alpha_{i+1}}NS^{\alpha_{i+2}}\dots NS^{\alpha_j}$ is a Dyck path (resp. prime Dyck path) followed by $e_i(P) - e_j(P)$ S steps. For instance, for the Dyck path P of Figure 3 (a), we have $0 \underline{P}\, 4$, $1 \underline{\underline{P}}\, 4$ and $2 \underline{P}\, 4$ (and many other relations).

We will now define the Stanley, Tamari and Kreweras lattices in terms of Dyck paths. More precisely, we will characterize the covering relation of each lattice in terms of Dyck paths and show that our definitions respects the known hierarchy between the three lattices (the Stanley lattice is a refinement of the Tamari lattice which is refinement of the Kreweras Lattice; see [**Knu06**, Ex. 7.2.1.6 - 26, 27 and 28]).

Stanley lattice. Let $P = NS^{\alpha_1}\dots NS^{\alpha_n}$ and $Q = NS^{\beta_1}\dots NS^{\beta_n}$ be two Dyck paths of size n. We denote by $P \leq_S Q$ if the path P stays below the path Q. Equivalently, $e_i(P) \leq e_i(Q)$ for all $1 \leq i \leq n$. The relation $\leq_S$ defines the *Stanley lattice* $\mathcal{L}_n^S$ on the set $\mathbf{D}_n$. Clearly the path P is covered by the path Q in the Stanley lattice if Q is obtained from P by replacing a subpath SN by NS. Equivalently, there is an index $1 \leq i \leq n$ such that $\beta_i = \alpha_i - 1$, $\beta_{i+1} = \alpha_{i+1} + 1$ and $\beta_k = \alpha_k$ for all $k \neq i, i+1$. The covering relation of the Stanley lattice is represented in Figure 4 (a) and the Hasse Diagram of $\mathcal{L}_3^S$ is represented in Figure 2 (a).

FIGURE 4. Covering relations in (a) Stanley lattice, (b) Tamari lattice.

Tamari lattice. The Tamari lattice has a simple interpretation in terms of binary trees. The set of binary trees can be defined recursively by the following grammar. A binary tree B is either a leaf denoted by $\circ$ or is made of a node and two ordered binary trees in which case we denote $B = (B_1, B_2)$. It is often convenient to draw a binary tree by representing the leaf by a white vertex and the tree $B = (B_1, B_2)$ by a black vertex at the bottom joined to the subtrees B_1 (on the left) and B_2 (on the right). The tree $(((\circ,\circ),((\circ,\circ),\circ)),(\circ,(\circ,\circ)))$ is represented in Figure 5.

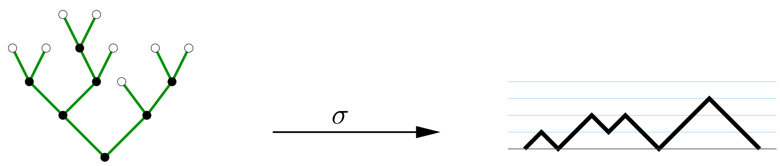

FIGURE 5. The binary tree $(((\circ,\circ),((\circ,\circ),\circ)),(\circ,(\circ,\circ)))$ and its image by the bijection σ.

The set $\mathbf{B}_n$ of binary trees with n nodes has cardinality $C_n = \frac{1}{n+1}\binom{2n}{n}$ and there are well known bijections between the set $\mathbf{B}_n$ and the set $\mathbf{D}_n$. We call σ the bijection defined as follows: the image of the binary tree reduced to a leaf is the empty word and the image of the binary tree $B = (B_1, B_2)$ is the Dyck path $\sigma(B) = \sigma(B_1)N\sigma(B_2)S$. An example is given in Figure 5.

In [**HT72**], Tamari defined a partial order on the set $\mathbf{B}_n$ of binary trees and proved it has a lattice structure. The covering relation for the Tamari lattice is defined has follows: a binary tree B containing a subtree of type $X = ((B_1, B_2), B_3)$ is covered by the binary tree B' obtained from B by replacing X by $(B_1,(B_2, B_3))$. The Hasse diagram of the Tamari lattice on the set of binary trees with 4 nodes is represented in Figure 6 (left).

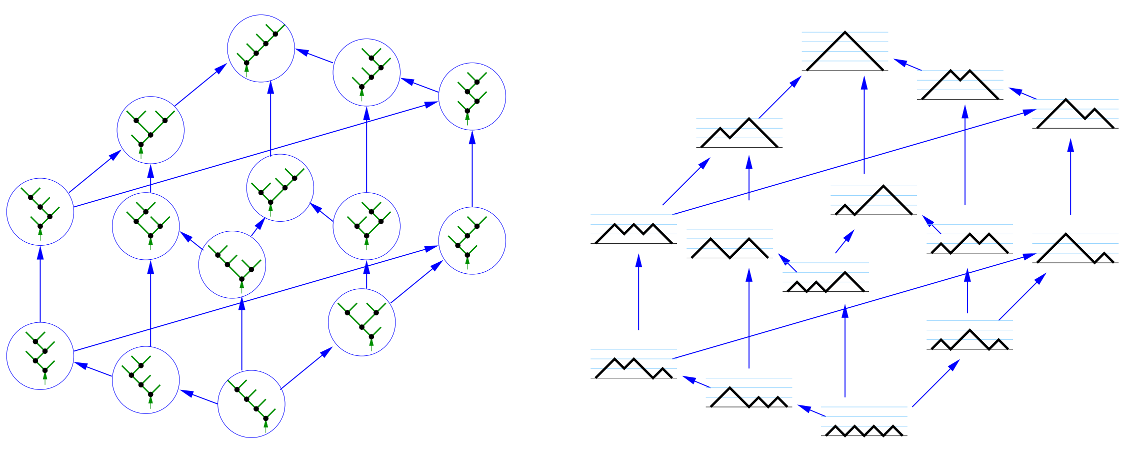

FIGURE 6. Hasse diagram of the Tamari lattice $\mathcal{L}_4^T$.

The bijection σ allows to transfer the Tamari lattice to the set of $\mathbf{D}_n$ Dyck paths. We denote by $\mathcal{L}_n^T$ the image of the Tamari lattice on $\mathbf{D}_n$ and denote by $P \leq_T Q$ if the path P is less than or equal to the path Q

for this order. The Hasse diagram of $\mathcal{L}_4^T$ is represented in Figure 6 (right). The following lemma expresses the covering relation of the Tamari lattice $\mathcal{L}_n^T$ in terms of Dyck paths. This covering relation is illustrated in Figure 4 (b).

LEMMA 2.1. *Let $P = NS^{\alpha_1} \dots NS^{\alpha_n}$ and $Q = NS^{\beta_1} \dots NS^{\beta_n}$ be two Dyck paths. The path P is covered by the path Q in the Tamari lattice $\mathcal{L}_n^T$ if Q is obtained from P by swapping an S step and the prime Dyck subpath following it, that is, there exist two indices $1 \leq i < j \leq n$ with $\alpha_i > 0$ and $i \overset{P}{\equiv} j$ such that $\beta_i = \alpha_i - 1$, $\beta_j = \alpha_j + 1$ and $\beta_k = \alpha_k$ for all $k \neq i, j$.*

Proof (sketch): Let B be a binary tree and let $P = \sigma(B)$. There is a one-to-one correspondence between the subtrees of B and the Dyck subpaths of P which are either a prefix of P or are preceded by an N step.
• If the binary tree B' is obtained from B by replacing a subtree $X = ((B_1, B_2), B_3)$ by $X' = (B_1, (B_2, B_3))$, then the Dyck path $Q = \sigma(B')$ is obtained from P by replacing a subpath $\sigma(X) = \sigma(B_1)N\sigma(B_2)SN\sigma(B_3)S$ by $\sigma(X') = \sigma(B_1)N\sigma(B_2)N\sigma(B_3)SS$; hence by swapping an S step and the prime Dyck subpath following it.
• Suppose conversely that the Dyck path Q is obtained from P by swapping an S step with a prime Dyck subpath NP_3S following it. Then, there are two Dyck paths P_1 and P_2 (possibly empty) such that $W = P_1NP_2SNP_3S$ is a Dyck subpath of P which is either a prefix of P or is preceded by an N step. Hence, the binary tree B contains the subtree $X = \sigma^{-1}(W) = ((B_1, B_2), B_3)$, where $B_i = \sigma^{-1}(P_i)$, $i = 1, 2, 3$. Moreover, the binary tree $B' = \sigma^{-1}(Q)$ is obtained from B by replacing the subtree $X = ((B_1, B_2), B_3)$ by $X' = (B_1, (B_2, B_3)) = \sigma^{-1}(P_1NP_2NP_3SS)$. □

COROLLARY 2.2. *The Stanley lattice $\mathcal{L}_n^S$ is a refinement of the Tamari lattice $\mathcal{L}_n^T$. That is, for any pair of Dyck paths P, Q, $P \leq_T Q$ implies $P \leq_S Q$.*

Kreweras lattice. A partition of $\{1, \dots, n\}$ is *non-crossing* if whenever four elements $1 \leq i < j < k < l \leq n$ are such that i, k are in the same class and j, l are in the same class, then the two classes coincide. The non-crossing partition whose classes are $\{1\}$, $\{2, 4\}$, $\{3\}$, and $\{5, 6, 7\}$ is represented in Figure 7. In this figure, each class is represented by a connected cell incident to the integers it contains.

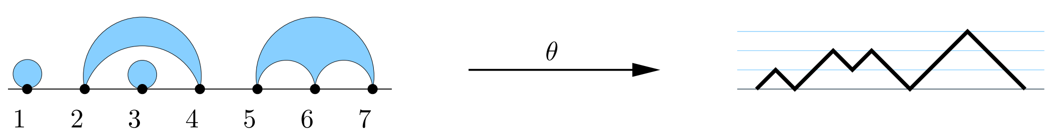

FIGURE 7. A non-crossing partition and its image by the bijection θ.

The set $\mathbf{NC}_n$ of non-crossing partition on $\{1, \dots, n\}$ has cardinality $C_n = \frac{1}{n+1}\binom{2n}{n}$ and there are well known bijections between non-crossing partitions and Dyck paths. We consider the bijection θ defined as follows. The image of a non-crossing partition π of size n by the mapping θ is the Dyck path $\theta(\pi) = NS^{\alpha_1}NS^{\alpha_2} \dots NS^{\alpha_n}$, where α_i is the size of the class containing i if i is maximal in its class and $\alpha_i = 0$ otherwise. An example is given in Figure 7.

In [**Kre71**], Kreweras showed that the partial order of refinement defines a lattice on the set $\mathbf{NC}_n$ of non-crossing partitions. The covering relation of this lattice corresponds to the merging of two parts when this operation does not break the *non-crossing condition*. The Hasse diagram of the Kreweras lattice on the set $\mathbf{NC}_4$ is represented in Figure 8 (left).

The bijection θ allows to transfer the Kreweras lattice on the set $\mathbf{D}_n$ of Dyck paths. We denote by $\mathcal{L}_n^K$ the lattice structure obtained on $\mathbf{D}_n$ and denote by $P \leq_K Q$ if the path P is less than or equal to the path Q for this order. The Hasse diagram of $\mathcal{L}_4^K$ is represented in Figure 8 (right). The following lemma expresses the covering relation of the Kreweras lattice $\mathcal{L}_n^K$ in terms of Dyck paths. This covering relation is represented in Figure 9.

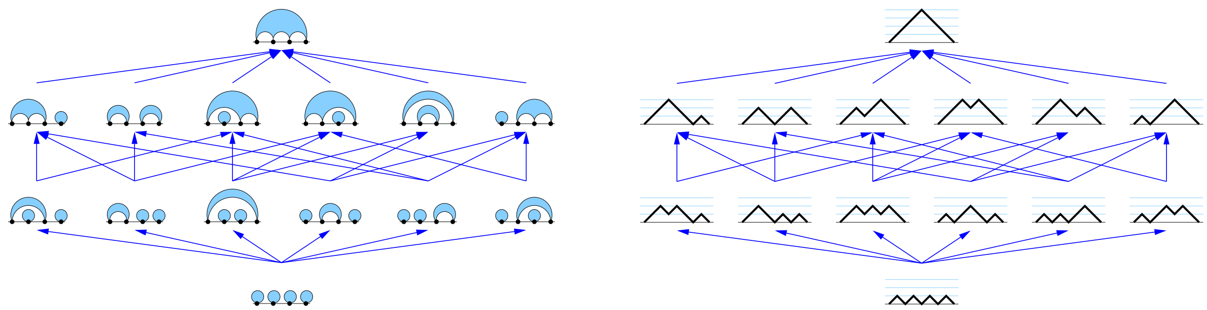

FIGURE 8. Hasse diagram of the Kreweras lattice $\mathcal{L}_4^K$.

FIGURE 9. Two examples of covering relations in Kreweras lattice.

LEMMA 2.3. *Let $P = NS^{\alpha_1} \dots NS^{\alpha_n}$ and $Q = NS^{\beta_1} \dots NS^{\beta_n}$ be two Dyck paths of size n. The path P is covered by the path Q in the Kreweras lattice $\mathcal{L}_n^K$ if Q is obtained from P by swapping a (non-empty) descent with a Dyck subpath following it, that is, there exist two indices $1 \leq i < j \leq n$ with $\alpha_i > 0$ and $i \underline{P} j$ such that $\beta_i = 0$, $\beta_j = \alpha_i + \alpha_j$ and $\beta_k = \alpha_k$ for all $k \neq i, j$.*

Proof (sketch): Let π be a non-crossing partition and let $P = \theta(\pi)$. The mapping which associates with a class c of π its largest element $\max(c)$ is a bijection between the classes of π and the indices of the non-empty descents of P. Moreover, two classes c and c' of π can be merged without breaking the non-crossing condition if and only if $\max(c) \ \underline{P}\ \max(c')$. Thus a non-crossing partition κ covers π in the Kreweras lattice if and only if $Q = \theta(\kappa)$ is obtained from $P = \theta(\pi)$ by swapping a non-empty descent with a Dyck subpaths that follows it. □

COROLLARY 2.4. *The Tamari lattice $\mathcal{L}_n^T$ is a refinement of the Kreweras lattice $\mathcal{L}_n^K$. That is, for any pair P, Q of Dyck paths, $P \leq_K Q$ implies $P \leq_T Q$.*

3. A bijection between Stanley intervals and realizers

In this section, we recall some definitions about triangulations and realizers. Then, we define a bijection between pairs of non-crossing Dyck paths and realizers.

3.1. Triangulations and realizers.

Maps. A *planar map*, or *map* for short, is an embedding of a connected finite planar graph in the sphere considered up to continuous deformation. In this paper, maps have no loop nor multiple edge. The *faces* are the connected components of the complement of the graph. By removing the midpoint of an edge we get two *half-edges*, that is, one dimensional cells incident to one vertex. Two consecutive half-edges around a vertex define a *corner*. If an edge is oriented we call *tail* (resp. *head*) the half-edge incident to the origin (resp. end).

A *rooted* map is a map together with a special half-edge which is not part of a complete edge and is called the *root*. (Equivalently, a rooting is defined by the choice of a corner.) The root is incident to one vertex called *root-vertex* and one face (containing it) called the *root-face*. When drawing maps in the plane the root is represented by an arrow pointing on the root-vertex and the root-face is the infinite one. See Figure 10 for an example. The vertices and edges incident to the root-face are called *external* while the others are called *internal*. From now on, *maps are rooted* without further notice.

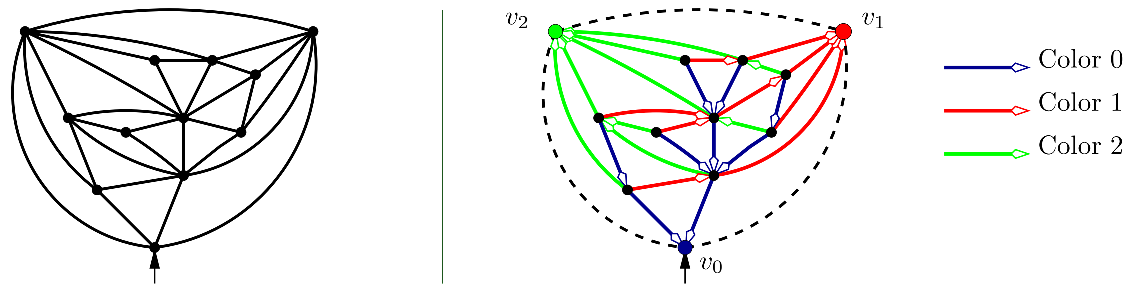

FIGURE 10. A rooted triangulation (left) and one of its realizers (right).

Triangulations. A *triangulation* is a map in which any face *has degree* 3 (has 3 corners). A triangulation has *size* n if it has n internal vertices. The incidence relation between faces and edges together with Euler formula show that a triangulation of size n has $3n$ internal edges and $2n+1$ internal triangles.

In one of its famous *census* paper, Tutte proved by a generating function approach that the number of triangulations of size n is $t_n = \frac{2(4n+1)!}{(n+1)!(3n+2)!}$ [**Tut62**]. A bijective proof of this result was given in [**PS03**].

Realizers. We now recall the notion of *realizer* (or *Schnyder wood*) defined by Schnyder [**Sch89, Sch90**]. Given an edge coloring of a map, we shall call *i-edge* (resp. *i-tail*, *i-head*) an edge (resp. tail, head) of color i.

DEFINITION 3.1 ([**Sch89**]). Let M be a triangulation and let U be the set of its internal vertices. Let v_0 be the root-vertex and let v_1, v_2 be the other external vertices with the convention that v_0, v_1, v_2 appear in counterclockwise order around the root-face.
A *realizer* of M is a coloring of the internal edges in three colors $\{0, 1, 2\}$ such that:

(1) *Tree condition*: for $i = 0, 1, 2$, the i-edges form a tree T_i with vertex set $U \cup \{v_i\}$. The vertex v_i is considered to be the root-vertex of T_i and the i-edges are oriented toward v_i.
(2) *Schnyder condition*: in clockwise order around any internal vertex there is: one 0-tail, some 1-heads, one 2-tail, some 0-heads, one 1-tail, some 2-heads. This situation is represented in Figure 11.

We denote by $R = (T_0, T_1, T_2)$ this realizer.

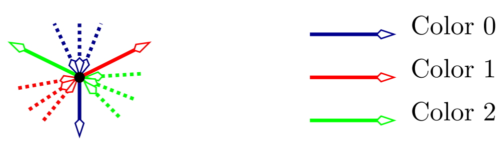

FIGURE 11. Edges coloration and orientation around a vertex in a realizer (Schnyder condition).

A realizer is represented in Figure 10 (right). Let $R = (T_0, T_1, T_2)$ be a realizer. We denote by $\overline{T_0}$ the tree made of T_0 together with the edge (v_0, v_1). For any internal vertex u, we denote by $\mathbf{p}_i(u)$ the parent of u in the tree T_i. A *cw-triangle* (resp. *ccw-triangle*) is a triple of vertices (u, v, w) such that $\mathbf{p}_0(u) = v, \mathbf{p}_2(v) = w$ and $\mathbf{p}_1(w) = u$ (resp. $\mathbf{p}_0(u) = v, \mathbf{p}_1(v) = w$ and $\mathbf{p}_2(w) = u$). A realizer is called *minimal* (resp. *maximal*) if it has no cw-triangle (resp. ccw-triangle). It was proved in [**Men94, Pro93**] that every triangulation has a unique minimal (resp. maximal) realizer. (The appellations *minimal* and *maximal* refer to a natural lattice ordering the realizers of any given triangulation[**Men94, Pro93**].)

3.2. A bijection between pairs of non-crossing Dyck paths and realizers.

In this subsection, we give an alternative (and simpler) description of the bijection defined in [**Bon05**] between realizers and pairs of non-crossing Dyck paths.

We first recall a classical bijection between plane trees and Dyck paths. A *plane tree* is a rooted map whose underlying graph is a tree. Let T be a plane tree. We *make the tour* of the tree T by following its border in clockwise direction starting and ending at the root (see Figure 13 (a)). We denote by $\omega(T)$ the word obtained by making the tour of the tree T and writing N the first time we follow an edge and S the second time we follow this edge. For instance, $w(T) = NNSSNNSNNSNSSNNSSS$ for the tree in Figure 13 (a). It is well known that the mapping ω is a bijection between plane trees with n edges and Dyck paths of size n [**Knu06**].

Let T be a plane tree. Consider the order in which the vertices are encountered while making the tour of T. This defines the *clockwise order around* T (or *preorder*). For the tree in Figure 13 (a) the clockwise order is $v_0 < u_0 < u_1 < \ldots < u_8$. The tour of the tree also defines an order on the set of corners around each vertex v. We shall talk about the *first* (resp. *last*) *corner of* v *around* T.

We are now ready to define a mapping Ψ which associates an ordered pair of Dyck paths to each realizer.

DEFINITION 3.2. Let M be a rooted triangulation of size n and let $R = (T_0, T_1, T_2)$ be a realizer of M. Let $u_0, u_1, \ldots, u_{n-1}$ be the internal vertices of M in clockwise order around T_0. Let $\beta_i, i = 1, \ldots, n-1$ be the number of 1-heads incident to u_i and let β_n be the number of 1-heads incident to v_1. Then $\Psi(R) = (P, Q)$, where $P = \omega^{-1}(T_0)$ and $Q = NS^{\beta_1} \ldots NS^{\beta_n}$.

The image of a realizer by the mapping Ψ is represented in Figure 12.

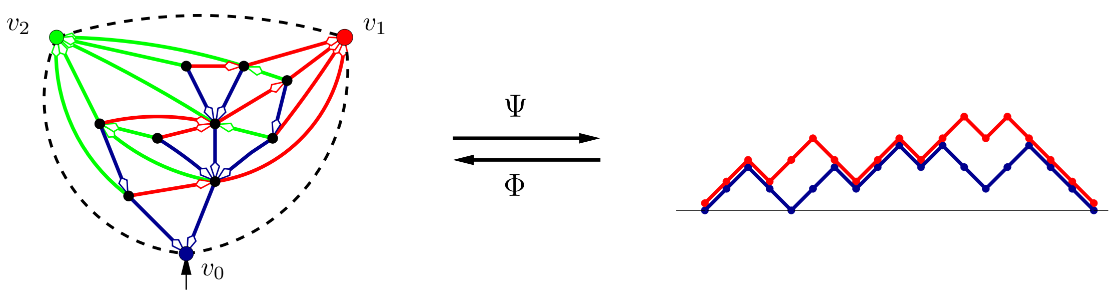

FIGURE 12. The bijections Ψ and Φ.

THEOREM 3.3. *The mapping Ψ is a bijection between realizers of size n and pairs of non-crossing Dyck paths of size n.*

We first prove that the image of a realizer is indeed a pair of non-crossing Dyck paths.

PROPOSITION 3.4. *Let $R = (T_0, T_1, T_2)$ be a realizer of size n and let $(P, Q) = \Psi(R)$. Then, P and Q are both Dyck paths and moreover the path P stays below the path Q.*

We use the following lemmas that we state without proof.

LEMMA 3.5. *Let $R = (T_0, T_1, T_2)$ be a realizer. Then, for any 1-edge e the tail of e is encountered before its head around the tree $\overline{T_0}$.*

LEMMA 3.6. *Let $P = NS^{\alpha_1} \ldots NS^{\alpha_n}$ be a Dyck path and let $T = \omega^{-1}(P)$. Let v_0 be the root-vertex of the tree T and let $u_0, u_1, \ldots, u_{n-1}$ be its other vertices in clockwise order around T. Then, the word obtained by making the tour of T and writing $\boldsymbol{S}^{\beta_i}$ when arriving at the first corner of u_i and $\boldsymbol{N}$ when arriving at the last corner of u_i is $\boldsymbol{W} = \boldsymbol{S}^{\beta_0} \boldsymbol{N}^{\alpha_1} \boldsymbol{S}^{\beta_1} \ldots \boldsymbol{S}^{\beta_{n-1}} \boldsymbol{N}^{\alpha_n}$.*

Proof of Proposition 3.4: We denote $P = NS^{\alpha_1} \ldots NS^{\alpha_n}$ and $Q = NS^{\beta_1} \ldots NS^{\beta_n}$.
• The mapping ω is known to be a bijection between trees and Dyck paths, hence $P = \omega(T)$ is a Dyck path.
• We want to prove that Q is a Dyck path staying above P. Consider the word $\mathbf{W}$ obtained by making the tour of $\overline{T_0}$ and writing $\mathbf{N}$ (resp. $\mathbf{S}$) when we encounter a 1-tail (resp. 1-head). By Lemma 3.6, the word $\mathbf{W}$ is $\mathbf{S}^{\beta_0}\mathbf{N}^{\alpha_1}\mathbf{S}^{\beta_1}\mathbf{N}^{\alpha_2} \ldots \mathbf{S}^{\beta_{n-1}}\mathbf{N}^{\alpha_n}\mathbf{S}^{\beta_n}$. By Lemma 3.5, the word $\mathbf{W}$ is a Dyck path. In particular, $\mathbf{S}^{\beta_0} = 0$ and $\sum_{i=1}^{n} \beta_i = \sum_{i=1}^{n} \alpha_i = n$, hence the path Q returns to the origin. Moreover, for all $i = 1, \ldots, n$, $\delta_i(P, Q) = \sum_{j=1}^{n} \alpha_i - \beta_i \geq 0$. Thus, the path Q stays above P. In particular, Q is a Dyck path.

□

In order to prove Theorem 3.3, we shall define a mapping Φ from pairs of non-crossing Dyck paths to realizers and prove it to be the inverse of Ψ. We first define *prerealizers*.

Definition 3.7. Let M be a map. Let v_0 be the root-vertex, let v_1 be another external vertex and let U be the set of the other vertices. A *prerealizer* of M is a coloring of the edges in two colors $\{0, 1\}$ such that:

(1) *Tree condition*: for $i = 0, 1$, the i-edges form a tree T_i with vertex set $U \cup \{v_i\}$. The vertex v_i is considered to be the root-vertex of T_i and the i-edges are oriented toward v_i.
(2) *Corner condition*: in clockwise order around any vertex $u \in U$ there is: one 0-tail, some 1-heads, some 0-heads, one 1-tail.
(3) *Order condition*: for any 1-edge e the tail of e is encountered before its head around the tree $\overline{T_0}$, where $\overline{T_0}$ is the tree obtained from T_0 by adding the edge (v_0, v_1) at the right of the root.

We denote by $PR = (T_0, T_1)$ this prerealizer.

An example of prerealizer is given in Figure 13 (c).

Lemma 3.8. *Let $PR = (T_0, T_1)$ be a prerealizer. Then, there exists a unique tree T_2 such that $R = (T_0, T_1, T_2)$ is a realizer.*

We now define a mapping Φ from pairs of non-crossing Dyck paths to realizers. This mapping is illustrated by Figure 13. Consider a pair of Dyck paths $P = NS^{\alpha_1} \dots NS^{\alpha_n}$ and $Q = NS^{\beta_1} \dots NS^{\beta_n}$ such that P stays below Q. The image of (P, Q) by the mapping Φ is the realizer $R = (T_0, T_1, T_2)$ obtained as follows.

Step 1. The tree T_0 is $\omega^{-1}(P)$. We denote by v_0 its root-vertex and by $u_0, \dots, u_n$ the other vertices in clockwise order around T_0. We denote by $\overline{T_0}$ the tree obtained from T_0 by adding a new vertex v_1 and an edge (v_0, v_1) at the right of the root.

Step 2. We glue a 1-tail in the last corner of each vertex $u_i, i = 0, \dots, n-1$ and we glue β_i 1-heads in the first corner of each vertex $u_i, i = 1, \dots, n-1$ (if u_i is a leaf we glue the 1-heads before the 1-tail in clockwise order around u_i). We also glue β_n 1-heads in the (unique) corner of v_1. This operation is illustrated by Figure 13 (b).

Step 3. We consider the sequence of 1-tails and 1-heads around $\overline{T_0}$. Let W be the word obtained by making the tour of $\overline{T_0}$ and writing N (resp. S) when we cross a 1-tail (resp. 1-head). By Lemma 3.6, $W = N^{\alpha_1} S^{\beta_1} \dots N^{\alpha_n} S^{\beta_n}$. Since the path P stays below the path Q, we have $\delta_i(P, Q) = \sum_{j \le i} \alpha_j - \beta_j \ge 0$ for all $i = 1, \dots, n$, hence W is a Dyck path. Thus, there exists a unique way of joining each 1-tail to a 1-head that appears after it around the tree $\overline{T_0}$ so that the 1-edges do not intersect (this statement is equivalent to the well-known fact that there is a unique way of matching parenthesis in a well parenthesized word); we denote by T_1 the set of 1-edges obtained in this way. This operation is illustrated in Figure 13 (c).

Step 4. The set T_1 of 1-edges is a tree directed toward v_1 (proof skipped). Hence, by construction, $PR = (T_0, T_1)$ is a prerealizer. By Lemma 3.8, there is a unique tree T_2 such that $R = (T_0, T_1, T_2)$ is a realizer and we define $\Phi(P, Q) = R$.

The mapping Φ is well defined and the image of any pair of non-crossing Dyck paths is a realizer. Conversely, by Proposition 3.4, the image of any realizer by Ψ is a pair of non-crossing Dyck paths. It is clear from the definitions that $\Psi \circ \Phi$ (resp. $\Phi \circ \Psi$) is the identity mapping on pairs of non-crossing Dyck paths (resp. realizers). Thus, Φ and Ψ are inverse bijections between realizers of size n and pairs of non-crossing Dyck paths of size n. This concludes the proof of Theorem 3.3. □

4. Intervals of the Tamari lattice

In the previous section, we defined a bijection Φ between pairs of non-crossing Dyck paths and realizers. Recall that the pairs of non-crossing Dyck paths correspond to the intervals of the Stanley lattice. In this section, we study the restriction of the bijection Φ to the intervals of the Tamari lattice.

Theorem 4.1. *The bijection Φ induces a bijection between the intervals of the Tamari lattice $\mathcal{L}_n^T$ and minimal realizers of size n.*

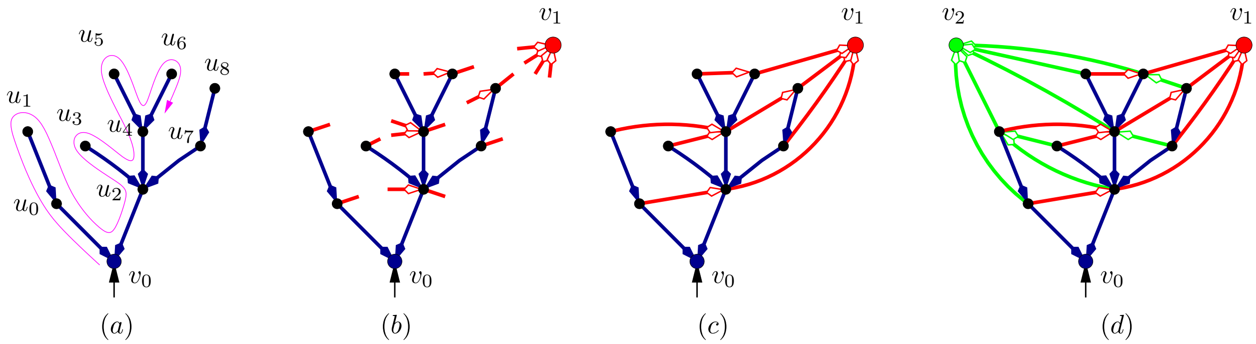

FIGURE 13. Steps of the mapping $\Phi : (P, Q) \mapsto (T_0, T_1, T_2)$. (a) Step 1: build the tree T_0. (b) Step 2: add the 1-tails and 1-heads. (c) Step 3: join the 1-tails and 1-heads together. (d) Step 4: determine the third tree T_2.

Since every triangulation has a unique minimal realizer, Theorem 4.1 implies that the mapping Φ' which associates with a Tamari interval (P, Q) the triangulation underlying $\Phi(P, Q)$ is a bijection. This gives a bijective explanation to the relation between the number of Tamari intervals enumerated in [**Cha06**] and the number of triangulations enumerated in [**Tut62, PS03**].

COROLLARY 4.2. *The number of intervals in the Tamari lattice* $\mathcal{L}_n^T$ *is equal to the number* $\frac{2(4n+1)!}{(n+1)!(3n+2)!}$ *of triangulations of size* n.

The rest of this section is devoted to the proof of Theorem 4.1. We first recall a characterization of minimality given in [**BGH02**] and illustrated in Figure 14.

PROPOSITION 4.3 ([**BGH02**]). *A realizer* $R = (T_0, T_1, T_2)$ *is minimal if and only if for any internal vertex* u*, the vertex* $\boldsymbol{p}_0(\boldsymbol{p}_1(u))$ *is an ancestor of* u *in the tree* T_0.

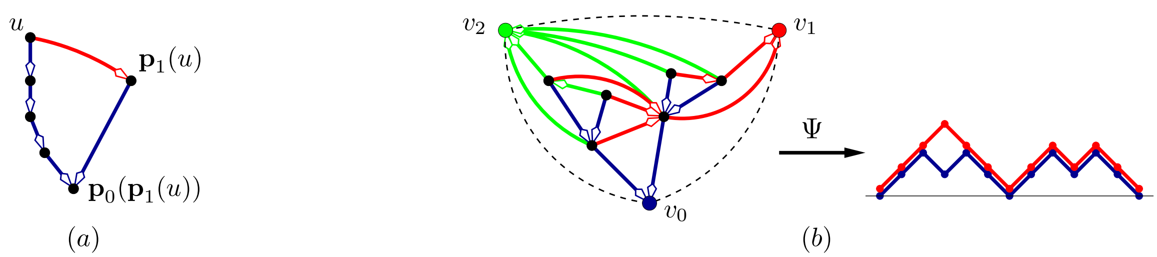

FIGURE 14. (a) Characterization of minimality: $\mathbf{p}_0(\mathbf{p}_1(u))$ is an ancestor of u in T_0. (b) A minimal realizer and its image by Ψ.

Using Proposition 4.3, we obtain the following characterization of the pairs of non-crossing Dyck paths (P, Q) whose image by the bijection Φ is a minimal realizer.

PROPOSITION 4.4. *Let* (P, Q) *be a pair of non-crossing Dyck paths and let* $R = (T_0, T_1, T_2) = \Phi(P, Q)$. *Let* $u_0, \ldots, u_{n-1}$ *be the non-root vertices of* T_0 *in clockwise order. Then, the realizer* R *is minimal if and only if* $\delta_i(P, Q) \leq \delta_j(P, Q)$ *whenever* u_i *is the parent of* u_j *in* $T_0 = \omega^{-1}(P)$.

In order to prove Proposition 4.4, we need to interpret the value of $\delta_i(P, Q)$ is terms of the realizer $R = \Phi(P, Q)$. Let u be an internal vertex of the triangulation underlying the realizer $R = (T_0, T_1, T_2)$. We say that a 1-tail is *available at* u if this tail appears before the first corner of u in clockwise order around T_0 while the corresponding 1-head appears (strictly) after the first corner of u.

LEMMA 4.5. *Let* (P, Q) *be a pair of non-crossing Dyck paths and let* $R = (T_0, T_1, T_2) = \Phi(P, Q)$. *Let* $u_0, \ldots, u_{n-1}$ *be the non-root vertices of* T_0 *in clockwise order. The number of 1-tails available at* u_i *is* $\delta_i(P, Q)$.

Proof of Lemma 4.5: We denote $P = NS^{\alpha_1} \dots NS^{\alpha_n}$ and $Q = NS^{\beta_1} \dots NS^{\beta_n}$. Let $\mathcal{W}$ be the word obtained by making the tour of T_0 and writing $N\mathbf{S}^{\beta_i}$ when arriving at the first corner of u_i and $\mathbf{N}S$ when arriving at the last corner of u_i for $i = 0, \dots, n-1$ (with the convention that $\beta_0 = 0$). By definition of the mapping ω, the restriction of $\mathcal{W}$ to the letters N, S is $\omega(T_0) = P = NS^{\alpha_1} \dots NS^{\alpha_n}$. Therefore, $\mathcal{W} = N\mathbf{S}^{\beta_0}(\mathbf{N}S)^{\alpha_1} N\mathbf{S}^{\beta_1}(\mathbf{N}S)^{\alpha_2} \dots N\mathbf{S}^{\beta_{n-1}}(\mathbf{N}S)^{\alpha_n}$. The prefix of $\mathcal{W}$ written after arriving at the first corner of u_i is $N\mathbf{S}^{\beta_0}(\mathbf{N}S)^{\alpha_1} N\mathbf{S}^{\beta_1} \dots (\mathbf{N}S)^{\alpha_i} N\mathbf{S}^{\beta_i}$. The sub-word $\mathbf{S}^{\beta_0}\mathbf{N}^{\alpha_1}\mathbf{S}^{\beta_1} \dots \mathbf{N}^{\alpha_i}\mathbf{S}^{\beta_i}$ corresponds to the sequence of 1-tails and 1-heads encountered so far ($\mathbf{N}$ for a 1-tail, $\mathbf{S}$ for a 1-head). Thus, the number of 1-tails available at u_i is $\sum_{j \le i} \alpha_j - \beta_j = \delta_i(P, Q)$. □

Proof of Proposition 4.4:

• We suppose that a vertex u_i is the parent of a vertex u_j in T_0 and that $\delta_i(P, Q) > \delta_j(P, Q)$, and we want to prove that the realizer $R = \Phi(P, Q)$ is not minimal. Since u_i is the parent of u_j we have $i < j$ and all the vertices $u_r,\ i < r \le j$ are descendants of u_i. By Lemma 4.5, $\delta_i(P, Q) > \delta_j(P, Q)$ implies that there is a 1-tail t available at u_i which is not available at u_j, hence the corresponding 1-head is incident to a vertex u_l with $i < l \le j$. Let u_k be the vertex incident to the 1-tail t. Since t is available at u_i, the vertex u_k is not a descendant of u_i. But $\mathbf{p}_0(\mathbf{p}_1(u_k)) = \mathbf{p}_0(u_l)$ is either u_i or a descendant of u_i in T_0. Thus, the vertex u_k contradicts the minimality condition given by Proposition 4.3. Hence, the realizer R is not minimal.

• We suppose that the realizer R is not minimal and we want to prove that there exists a vertex u_i parent of a vertex u_j in T_0 such that $\delta_i(P, Q) > \delta_j(P, Q)$. By Proposition 4.3, there exists a vertex u such that $\mathbf{p}_0(\mathbf{p}_1(u))$ is not an ancestor of u in T_0. In this case, the 1-tail t incident to u is available at $u_i = \mathbf{p}_0(\mathbf{p}_1(u))$ but not at $u_j = \mathbf{p}_1(u)$ (since t cannot appear between the first corner of u_i and the first corner of u_j around T_0, otherwise u would be a descendant of u_i). Moreover, any 1-tail t' available at u_j appears before the 1-tail t around T_0 (otherwise, the 1-edge corresponding to t' would cross the 1-edge (u, u_j)). Hence, any 1-tail t' available at u_j is also available at u_i. Thus, there are more 1-tails available at u_i than at u_j. By Lemma 4.5, this implies $\delta_i(P, Q) > \delta_j(P, Q)$. □

Proposition 4.6. . *Let (P, Q) be a pair of non-crossing Dyck paths. Let $T = \omega^{-1}(P)$, let v_0 be the root-vertex of the tree T and let $u_0, \dots, u_{n-1}$ be its other vertices in clockwise order. Then, $P \le_T Q$ if and only if $\delta_i(P, Q) \le \delta_j(P, Q)$ whenever u_i is the parent of u_j.*

We skip the proof of Proposition 4.6. Observe that Propositions 4.4 and Propositions 4.6 clearly imply Theorem 4.1. □

5. Intervals of the Kreweras lattice

In this section, we study the restriction of the bijection Φ to the Kreweras intervals.

Theorem 5.1. *The mapping Φ induces a bijection between the intervals of the Kreweras lattice $\mathcal{L}_n^K$ and realizers of size n which are both minimal and maximal.*

Before commenting on Theorem 5.1, we characterize the realizers which are both minimal and maximal. Recall that a triangulation is *stack* if it is obtained from the map reduced to a triangle by recursively inserting a vertex of degree 3 in one of the (triangular) internal face. An example is given in Figure 15.

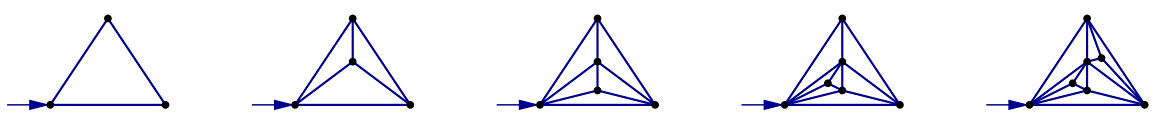

Figure 15. A stack triangulation is obtained by recursively inserting a vertex of degree 3.

Proposition 5.2. *A realizer R is both minimal and maximal if and only if the underlying triangulation M is stack. (In this case, R is the unique realizer of M.)*

The proof of Proposition 5.2 uses the following Lemma.

LEMMA 5.3. *Let M be a triangulation and let $R = (T_0, T_1, T_2)$ be one of its realizers. Suppose that M has an internal vertex v of degree 3 and let M' be obtained from M by removing v (and the incident edges). Then, the restriction of the realizer R to the triangulation M' is a realizer.*

Proof: By Schnyder condition, the vertex v is incident to three tails and no head, hence it is a leaf in each of the trees T_1, T_2, T_3. Thus, the *tree condition* is preserved by the deletion of v. Moreover, deleting v does not deprive any other vertex of an i-tail, hence the *Schnyder condition* is preserved by the deletion of v. □

Proof of Proposition 5.2 (sketch):
• Let M be a stack triangulation and let R be a realizer. We want to prove that R is minimal and maximal, that is, contains neither a cw- nor a ccw-triangle. We proceed by induction on the size of M. If M is reduced to the triangle, the property is obvious. Let M be a stack triangulation not reduced to the triangle. By definition, the triangulation M contains an internal vertex v of degree 3 such that the triangulation M' obtained from M by removing v is stack. By Lemma 5.3, the restriction of the realizer R to M' is a realizer. Hence, by the induction hypothesis, the triangulation M' contains neither a cw- nor a ccw-triangle. Thus, if C is either a cw- or a ccw-triangle of M, then C contains v. But this is impossible since v is incident to no head.
• We want to prove that any realizer of a non-stack triangulation contains either a cw- or a ccw-triangle. It is known that if R contains a directed cycle, then it contains either a cw- or ccw-triangle (proof omitted; see [**Men94**]). Therefore, it is sufficient to prove that any realizer of a non-stack triangulation contains a directed cycle. Given Lemma 5.3, it is sufficient to prove this property for triangulations such that every internal vertex has degree at least 4. It is also sufficient (proof skipped) to prove the property for triangulations without a *separating triangle* (a triangle which is not a face).
Let R be a realizer of a triangulation M without internal vertex of degree 3 nor separating triangle. We want to prove that R contains a directed cycle. Let u be the third vertex of the internal triangle incident to the edge (v_1, v_2). The vertex u is such that $\mathbf{p}_1(u) = v_1$ and $\mathbf{p}_2(u) = v_2$ (see Figure 16). The vertex u has degree at least 4 and is not adjacent to v_0 (otherwise one of the triangles (v_0, v_1, u) or (v_0, v_2, u) contains some vertices, hence is separating). Thus, $u' = \mathbf{p}_0(u) \neq v_0$. Moreover, either $\mathbf{p}_1(u') \neq v_1$ or $\mathbf{p}_2(u') \neq v_2$, otherwise the triangle (v_1, v_2, u') is separating. Let us assume that $u'' = \mathbf{p}_1(u') \neq v_1$ (the other case is symmetrical). By Schnyder condition, the vertex u'' lies inside the cycle C made of the edges (v_0, v_1), (v_1, u) and the 0-path from u to v_0. By Schnyder condition, the 1-path from u'' to v_1 stays strictly inside C. Let C' be the cycle made of the edges (v_1, u), (u, u') and the 1-path from u' to v_1. By Schnyder condition, the 2-path from u'' to v_2 starts inside the cycle C', hence cut this cycle. Let v be the first vertex of C' on the 2-path from u'' to v_2. The vertex v is incident to a 2-head lying inside C', hence by Schnyder condition $v = u$. Thus, the cycle made of the edges (u, u'), (u', u'') and the 2-path from u'' to u is directed. □

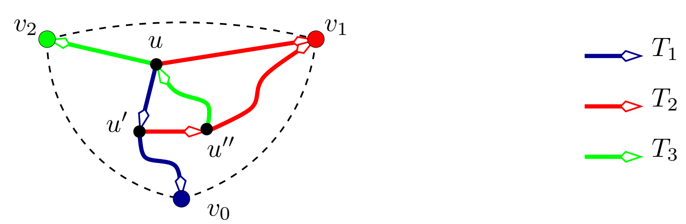

FIGURE 16. The vertices u, $u' = \mathbf{p}_0(u)$ and $u'' = \mathbf{p}_1(u')$.

Given Theorem 5.1 and Proposition 5.2, the mapping Φ induces a bijection between the intervals of the Kreweras lattice and the stack triangulations. Stack triangulations are known to be in bijection with ternary trees (see for instance [**ZZ**]), hence we obtain a new proof that the number of intervals in $\mathcal{L}_n^K$ is $\frac{1}{2n+1}\binom{3n}{n}$. The rest of this section is devoted to the proof of Theorem 5.1. We first recall a characterization of the realizers which are both minimal and maximal. This characterization which is illustrated in Figure 17 follows immediately from the characterizations of minimality and of maximality given in [**BGH02**].

PROPOSITION 5.4 ([**BGH02**]). *A realizer $R = (T_0, T_1, T_2)$ is both minimal and maximal if and only if for any internal vertex u, either $\boldsymbol{p}_0(\boldsymbol{p}_1(u)) = \boldsymbol{p}_0(u)$ or $\boldsymbol{p}_1(\boldsymbol{p}_0(u)) = \boldsymbol{p}_1(u)$.*

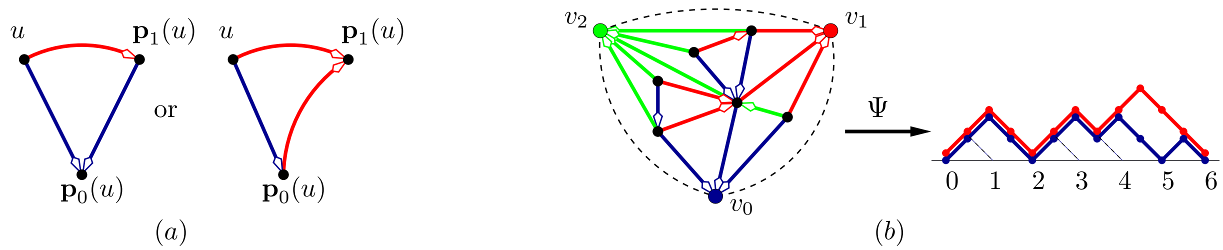

FIGURE 17. (a) Condition for a realizer to be both minimal and maximal: $\mathbf{p}_0(\mathbf{p}_1(u)) = \mathbf{p}_0(u)$ or $\mathbf{p}_1(\mathbf{p}_0(u)) = \mathbf{p}_1(u)$. (b) A minimal and maximal realizer and its image by Ψ.

Let $R = (T_0, T_1, T_2)$ be a realizer of a triangulation M and let u, u' be two vertices distinct from v_0 and v_2. We say that there is a *1-obstruction* between u and u' if there is a 1-edge e such that the tail of e appears before the first corner of u while its head appears strictly between the first corner of u and the first corner of u' around the tree $\overline{T_0}$. This situation is represented in Figure 18. Using Proposition 5.4, we obtain the following property satisfied by realizers which are both minimal and maximal.

LEMMA 5.5. *Let $R = (T_0, T_1, T_2)$ be a minimal and maximal realizer and let $(P, Q) = \Psi(R)$. Let $v_0, u_0, u_1, \ldots, u_n = v_1$ be the vertices of the tree $\overline{T_0}$ in clockwise order. Then, for all indices $0 \leq i < j \leq n$, the relation $i \ \underline{Q}\ j$ holds if and only if the three following properties are satisfied*

(1) *the vertex u_j is an ancestor of u_i in the tree T_1,*
(2) *either $\boldsymbol{p}_1(\boldsymbol{p}_0(u_i)) = u_j$ or $\boldsymbol{p}_0(u_i) = \boldsymbol{p}_0(u_j)$ (with the convention that $\boldsymbol{p}_0(u_n) = v_0$),*
(3) *there is no 1-obstruction between u_i and u_j.*

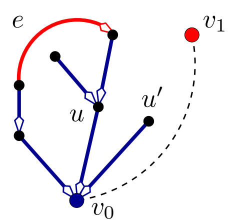

FIGURE 18. A 1-obstruction between the vertices u and u'.

Proof of Theorem 5.1 (sketch): Let $P = NS^{\alpha_1} \ldots NS^{\alpha_n}$ and $Q = NS^{\beta_1} \ldots NS^{\beta_n}$ be two Dyck paths and let $R = (T_0, T_1, T_2) = \Phi(P, Q)$.

• We suppose that $P \leq_K Q$ and we want to prove that the realizer R is minimal and maximal. We proceed by induction on $\Delta(P, Q)$.

- If $\Delta(P, Q) = 0$, then $P = Q$. In this case, the realizer R is minimal and maximal (proof skipped).

- If $\Delta(P, Q) > 0$, there is a Dyck path $Q' = NS^{\beta'_1} \ldots NS^{\beta'_n}$ covered by Q in the Kreweras lattice and such that $P \leq_K Q'$. Since Q' is covered by Q is the Kreweras lattice, there are indices $0 \leq i < j \leq n$ such that $i \ \underline{Q'}\ j$ and $\beta_i = 0$, $\beta_j = \beta'_i + \beta'_j$ and $\beta_k = \beta'_k$ for all $k \neq i, j$ (this situation is represented in Figure 19 (a)). By the induction hypothesis, the realizer $R' = (T'_0, T'_1, T'_2) = \Phi(P, Q')$ is both minimal and maximal. Moreover, by definition of the bijection Φ, the trees T_0 and T'_0 are the same. We use this fact to identify the vertices in the prerealizers $PR = (T_0, T_1)$ and $PR' = (T_0, T'_1)$ that we denote by $v_0, u_0, u_1, \ldots, u_n = v_1$ in clockwise order around $\overline{T_0} = \overline{T'_0}$. We also denote by $\mathbf{p}'_1(u)$ the parent of any vertex u in T'_1.

  • We first prove that *for any vertex v, $\boldsymbol{p}'_1(v) = \boldsymbol{p}_1(v)$ except if $\boldsymbol{p}'_1(v) = u_i$ in which case $\boldsymbol{p}_1(v) = u_j$.* Since $i \ \underline{Q}\ j$, Lemma 5.5 implies that there is no 1-obstruction between u_i and u_j in the realizer R'. Thus, the β'_i 1-heads incident to u_i can be unglued from the first corner of u_i and glued to the first corner of u_j without creating any crossing in the prerealizer $PR' = (T_0, T'_1)$ (the transfer of the β'_i 1-heads is represented in Figure 19 (b)). Let $PR'' = (T_0, T''_1)$ be the colored map obtained. Clearly, $PR'' = (T_0, T''_1)$ satisfies the *tree condition* (T''_1 is a tree), the *corner condition* (the 1-heads are in first corners, the 1-tails are in last corners) and the *order condition* (any 1-tail appears before the corresponding 1-head around $\overline{T_0}$), therefore PR'' is a prerealizer. Moreover, for all $i = 0, \ldots, n$, there are β_i 1-heads incident to the vertex u_i. Thus, by definition of the mapping Φ, the prerealizer PR'' is equal to $PR = (T_0, T_1)$. Since the only difference between the prerealizers PR' and PR is that the 1-heads incident to u_i in PR' are incident to u_j in PR, the property holds.

- We now prove that *the realizer* $R = (T_0, T_1, T_2)$ *is minimal and maximal.* If the realizer R is not both minimal and maximal, there is a vertex u such that $\mathbf{p}_1(u) \neq \mathbf{p}_1(\mathbf{p}_0(u))$ and $\mathbf{p}_0(\mathbf{p}_1(u)) \neq \mathbf{p}_0(u)$. Since the realizer R' is both minimal and maximal, either $\mathbf{p}'_1(u) = \mathbf{p}'_1(\mathbf{p}_0(u))$ or $\mathbf{p}_0(\mathbf{p}'_1(u)) = \mathbf{p}_0(u)$. But $\mathbf{p}'_1(u) \neq \mathbf{p}'_1(\mathbf{p}_0(u))$, otherwise $\mathbf{p}_1(u) = \mathbf{p}_1(\mathbf{p}_0(u))$. Thus, $\mathbf{p}_0(\mathbf{p}'_1(u)) = \mathbf{p}_0(u)$ and $\mathbf{p}'_1(u) = u_i$. Hence, $\mathbf{p}_0(u_i) = \mathbf{p}_0(u)$ and $\mathbf{p}_1(u) = u_j$. Moreover, since $i \,\underline{Q'}\, j$, Lemma 5.5 implies that either $\mathbf{p}_0(u_i) = \mathbf{p}_0(u_j)$ or $\mathbf{p}'_1(\mathbf{p}_0(u_i)) = u_j$. But, if $\mathbf{p}_0(u_i) = \mathbf{p}_0(u_j)$, then $\mathbf{p}_0(u) = \mathbf{p}_0(u_i) = \mathbf{p}_0(u_j) = \mathbf{p}_0(\mathbf{p}_1(u))$ which is forbidden. And, if $\mathbf{p}'_1(\mathbf{p}_0(u_i)) = u_j$, then $\mathbf{p}_1(\mathbf{p}_0(u)) = \mathbf{p}_1(\mathbf{p}_0(u_i)) = \mathbf{p}'_1(\mathbf{p}_0(u_i)) = u_j = \mathbf{p}_1(u)$ which is also forbidden. We reach a contradiction.

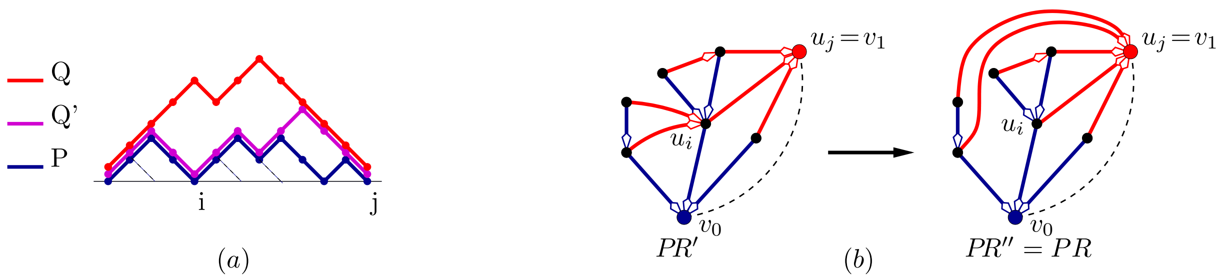

FIGURE 19. (a) The Dyck paths $P \leq_K Q' \leq_K Q$. (b) The prerealizer PR'' is obtained from $PR' = (T_0, T'_1)$ by moving β'_i 1-heads from the first corner of u_i to the first corner of u_j.

• We suppose that the realizer R is minimal and maximal and we want to prove that $P \leq_K Q$. We proceed by induction on $\Delta(P, Q)$. If $\Delta(P, Q) = 0$, then $P = Q$ and the property holds. We suppose now that $\Delta(P, Q) > 0$ and we denote by $v_0, u_0, u_1, \ldots, u_n = v_1$ the vertices of the tree $\overline{T_0}$ in clockwise order.
- We first prove that *there are indices* $0 \leq k < i < j \leq n$ *such that* $\boldsymbol{p}_0(u_k) = \boldsymbol{p}_0(u_i)$ *and* $\boldsymbol{p}_1(u_k) = u_j$. If there are no such indices, then $P = Q$ (proof skipped).
- Let $k < i < j$ be as described in the preceding point with k maximal and i minimal with respect to k (i.e. u_i is the first sibling of u_k appearing after u_k around the tree $\overline{T_0}$). This situation is represented in Figure 20. Observe that no 1-head is incident to u_i in the prerealizer $PR = (T_0, T_1)$ (see Figure 20), hence $\beta_i = 0$,. Let H be the set of 1-heads incident to u_j and such that the corresponding 1-tail is either incident to u_k or to one of its descendants. One can unglue the 1-heads in H from the first corner of u_j and glue them to the first corner of u_i without creating any crossing (see Figure 20). Moreover, the resulting colored map PR' is easily seen to be a prerealizer that we denote by $PR' = (T_0, T'_1)$. Let R' be the realizer corresponding to the prerealizer PR' and let $Q' = NS^{\beta'_1} \ldots NS^{\beta'_n}$ be the Dyck path such that $\Phi(P, Q') = R'$. By definition of Φ, we have $\beta'_i = |H|$, $\beta'_j = \beta_j - |H|$ and $\beta'_l = \beta_l$ for all $l \neq i, j$.
- We now prove that *the realizer* $R' = \Phi(P, Q')$ *is minimal and maximal.* By Proposition 5.4, we only need to prove that for every internal vertex u, either $\mathbf{p}_0(\mathbf{p}'_1(u)) = \mathbf{p}_0(u)$ or $\mathbf{p}'_1(\mathbf{p}_0(u)) = \mathbf{p}'_1(u)$, where $\mathbf{p}'_1(u)$ denotes the parent of u in the tree T'_1. Suppose that there is a vertex u not satisfying this condition. Note first that $u \neq u_k$ since $\mathbf{p}_0(\mathbf{p}'_1(u_k)) = \mathbf{p}_0(u_k)$. Since the realizer R is minimal and maximal, either $\mathbf{p}_0(\mathbf{p}_1(u)) = \mathbf{p}_0(u)$ or $\mathbf{p}_1(\mathbf{p}_0(u)) = \mathbf{p}_1(u)$. Suppose first $\mathbf{p}_0(\mathbf{p}_1(u)) = \mathbf{p}_0(u)$. In this case, the vertex u is a descendant of u_k (otherwise, $\mathbf{p}_0(\mathbf{p}'_1(u)) = \mathbf{p}_0(\mathbf{p}_1(u)) = \mathbf{p}_0(u)$), and $\mathbf{p}'_1(u) = u_j$ (for the same reason). Therefore, $\mathbf{p}_0(u_j) = \mathbf{p}_0(\mathbf{p}_1(u)) = \mathbf{p}_0(u)$ implies that u_j is a descendant of u_k. This is impossible since u_j appears after u_i around $\overline{T_0}$. Suppose now that $\mathbf{p}_1(\mathbf{p}_0(u)) = \mathbf{p}_1(u)$. In this case, the vertex u is a descendant of u_k (otherwise, $\mathbf{p}'_1(\mathbf{p}_0(u)) = \mathbf{p}_1(\mathbf{p}_0(u)) = \mathbf{p}_1(u) = \mathbf{p}'_1(u)$), and $\mathbf{p}_1(\mathbf{p}_0(u)) = \mathbf{p}_1(u) = u_j$ (for the same reason). Thus $\mathbf{p}'_1(\mathbf{p}_0(u)) = \mathbf{p}'_1(u) = u_i$. We reach again a contradiction.
- We now prove that *the Dyck path* Q' *is covered by* Q *in the Kreweras lattice.* By definition of the covering relation in the Kreweras lattice $\mathcal{L}^K$, it suffices to prove that $i \,\underline{Q'}\, j$. Since the realizer R' is minimal and maximal, it suffices to prove that the conditions (1), (2) and (3) of Lemma 5.5 hold. Clearly, there is no

1-obstruction between the vertices u_i and u_j in the realizer R' (see Figure 20), hence condition (3) holds. Moreover, since the realizer R is minimal and maximal, either $\mathbf{p}_0(u_k) = \mathbf{p}_0(u_j)$ or $\mathbf{p}_1(\mathbf{p}_0(u_k)) = u_j$. Thus, either $\mathbf{p}_0(u_i) = \mathbf{p}_0(u_j)$ or $\mathbf{p}_1(\mathbf{p}_0(u_i)) = u_j$, hence condition (2) holds. Let $i = i_1, i_2, \dots, i_s$ be the indices of the siblings of u_k appearing between u_k and u_j in clockwise order around $\overline{T_0}$ (see Figure 20). By the choice of k, we get $\mathbf{p}_1(u_{i_r}) = u_{i_{r+1}}$ for all $r < s$. Moreover, since the realizer R is minimal and maximal, either $\mathbf{p}_0(u_k) = \mathbf{p}_0(u_j)$ or $\mathbf{p}_1(\mathbf{p}_0(u_k)) = u_j$. If either case, we get $\mathbf{p}_1(u_s) = u_j$. Thus, $\mathbf{p}'_1(u_{i_r}) = \mathbf{p}_1(u_{i_r}) = u_{i_{r+1}}$ for all $r < s$, and $\mathbf{p}'_1(u_s) = \mathbf{p}_1(u_s) = u_j$. Hence, u_j is an ancestor of u_i in the tree T'_1, that is, condition (1) holds.

- The realizer $R' = \Phi(P, Q')$ is minimal and maximal, hence by the induction hypothesis $P \leq_K Q'$. Moreover, the path Q' is covered by Q in the Kreweras lattice. Thus, $P \leq_K Q$. □

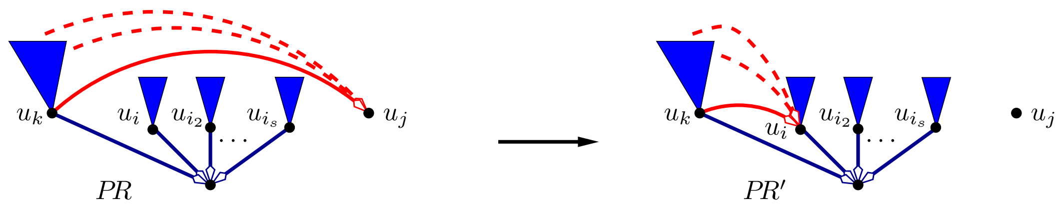

FIGURE 20. The vertices u_k, u_i, u_j in the prerealizer $PR = (T_0, T_1)$ and $PR' = (T_0, T'_1)$.

Acknowledgments: The authors are very grateful to Mireille Bousquet-Mélou for pointing them to the enumerative result of Frédéric Chapoton and to Xavier Viennot for fruitful discussions and suggestions. The authors also thank the anonymous referees for helpful comments.

References

[BGH02] N. Bonichon, C. Gavoille, and N. Hanusse. An information upper bound of planar graphs using triangulation. Technical Report RR-1279-02, LaBRI, University Bordeaux 1, 2002.

[Bon05] N. Bonichon. A bijection between realizers of maximal plane graphs and pairs of non-crossing dyck paths. *Discrete Math.*, 298:104–114, 2005.

[Cha06] F. Chapoton. Sur le nombre d'intervalles dans les treillis de tamari. *Sém. Lothar. Combin.*, 55, 2006.

[DSCV86] M. De Sainte-Catherine and G. Viennot. Enumeration of certain young tableaux with bounded height. *Lecture notes in Math.*, 1234:58–67, 1986.

[Ede82] P.H. Edelman. Multichains, non-crossing partitions and trees. *Discrete Math.*, 40:171–179, 1982.

[GV85] I.M. Gessel and X. Viennot. Binomial determinants, paths and hook formulae. *Adv. Math.*, 58:300–321, 1985.

[HT72] S. Huang and D. Tamari. Problems of associativity: A simple proof of the lattice property of systems ordered by a semi-associative law. *J. Combin. Ser. A*, 13:7–13, 1972.

[Knu06] D. Knuth. *The art of computer programming. Volume 4, Fascicle 4. Generating all trees - History of combinatorial generation.* Addison Wisley, 2006.

[Kre71] G. Kreweras. Sur les partitions non-croisées d'un cycle. *Discrete Math.*, 1(4):333–350, 1971.

[McC06] J. McCammond. Noncrosing partitions in surprising locations. *American Mathematical Monthly*, 113:598–610, 2006.

[Men94] P. Ossona De Mendez. *Orientations bipolaires.* PhD thesis, École des Hautes Études en Sciences Sociales, Paris, 1994.

[Pro93] J. Propp. Lattice structure for orientations of graphs. Manuscript: www.math.wisc.edu/~propp/orient.html, 1993.

[PS03] D. Poulalhon and G. Schaeffer. Optimal coding and sampling of triangulations. In *Automata, Languages and Programming*, 2003. Proceedings of the 30th International Colloquium ICALP'03.

[Sch89] W. Schnyder. Planar graphs and poset dimension. *Order*, pages 323–343, 1989.

[Sch90] W. Schnyder. Embedding planar graphs in the grid. *Symposium on Discrete Algorithms (SODA)*, pages 138–148, 1990.

[Sim00] R. Simion. Noncrossing partitions. *Discrete Math.*, 217:367–409, 2000.

[Sta99] R.P. Stanley. *Enumerative combinatorics, volume 2.* Wadsworth & Brooks/Cole, 1999.

[Tut62] W.T. Tutte. A census of planar triangulations. *Canad. J. Math.*, 14:21–38, 1962.

[ZZ] F. Zickfeld and G. M. Ziegler. Integer realizations of stacked polytopes. http://www.math.tu-berlin.de/ zickfeld/Alacala06-Poster.pdf.

Centre de Recerca Matematica, Bellaterra, 08193 Spain

LaBRI, Université Bordeaux 1, 351 cours de la Libération, 33405 Talence, France