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Refined Enumerations of Totally Symmetric Self-Complementary Plane Partitions and Constant Term Identities

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Abstract. In this paper we give Pfaffian or determinant expressions, and constant term identities for the conjectures in the paper "Self-complementary totally symmetric plane partitions" (*J. Combin. Theory Ser. A* **42**, 277–292) by Mills, Robbins and Rumsey. We also settle a weak version of Conjecture 6 in the paper, i.e., the number of shifted plane partitions invariant under a certain involution is equal to the number of alternating sign matrices invariant under the vertical flip.

Résumé. Dans cet article nous donnons des expressions pfaffiennes ou déterminantales, et des identidtés en termes constants pour les conjectures dans l'article "Self-complementary totally symmetric plane partitions" (*J. Combin. Theory Ser. A* **42**, 277–292) par Mills, Robbins and Rumsey. Nous démontrons aussi une version faible de la Conjecture 6 de cet article, i.e., le nombre de partitions planes décalées invariantes sous certaine involution est égal au nombre de matrices à signes alternants invariantes sous la réflexion verticale.

1. Introduction

In the paper [**31**] Mills, Robbins and Rumsey presented several conjectures on the enumeration of the totally symmetric self-complementary plane partitions. G.E. Andrews ([**2**]) settled the conjecture ([**31**, Conjecture 1]) on the cardinality of the totally symmetric self-complementary plane partitions of size n (see also [**38**]). D. Zeilberger gave a constant term identity of this cardinality in [**41**]. The aim of this paper is to give Pfaffian or determinant expressions for the other conjectures in [**31**]. We also generalize Zeilberger's constant term identity in [**41**].

In [**31**] Mills, Robbins and Rumsey have introduced a class $\mathscr{B}_n$ of triangular shifted plane partitions

$$\begin{array}{cccc} b_{11} & b_{12} & \dots & b_{1,n-1} \\ & b_{22} & \dots & b_{2,n-1} \\ & & \ddots & \vdots \\ & & & b_{n-1,n-1} \end{array}$$

whose parts are $\leq n$, weakly decreasing along rows and columns, and all parts in row i are $\geq n-i$. For example, $\mathscr{B}_3$ consists of the following seven elements.

$$\begin{array}{cc} 3 & 3 \\ & 3 \end{array} \quad \begin{array}{cc} 3 & 3 \\ & 2 \end{array} \quad \begin{array}{cc} 3 & 3 \\ & 1 \end{array} \quad \begin{array}{cc} 3 & 2 \\ & 2 \end{array} \quad \begin{array}{cc} 3 & 2 \\ & 1 \end{array} \quad \begin{array}{cc} 2 & 2 \\ & 2 \end{array} \quad \begin{array}{cc} 2 & 2 \\ & 1 \end{array}$$

Let $\mathscr{T}_n$ denote the set of totally symmetric self-complementary plane partitions of size n (see Section 3). They have established a bijection between $\mathscr{T}_n$ and $\mathscr{B}_n$.

Mills, Robbins and Rumsey also defined the statistics U_r, $r = 1, \dots, n$, (see (3.1)), and conjectured that U_r has the same distribution as the position of the 1 in the top row of an alternating

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sign matrix, and presented several conjectures related to the symmetry and the distribution of U_r. The aim of this paper is to obtain the generating functions for the enumerations concerning these conjectures. Here we briefly recall these conjectures by Mills, Robbins and Rumsey, and present a Pfaffian or determinant expression for each problem. For the definition of the numbers A_n, A_n^k, and etc. and the polynomials $A_n(t)$ and etc., the reader shall refer to the Section 2. It seems that these numbers and polynomials have the standard notation which have appeared concerning the alternating sign matrices (see [**27, 32, 35, 40**]). First of all, Mills, Robbins and Rumsey presented the following conjecture, which we call the refined enumeration of TSSCPPs:

CONJECTURE 1.1. ([**31**, pp.282, Conjecture 2]) Let n be a positive integer. Let $1 \le k \le n$ and $1 \le r \le n$. Then the number of elements b of $\mathscr{B}_n$ such that $U_r(b) = k-1$ would be A_n^k. Namely, $\sum_{b\in\mathscr{B}_n} t^{U_r(b)} = A_n(t)$ would hold.

Let n and N be positive integers, and let $B_n^N(t) = (b_{ij}(t))_{0\le i\le n-1,\ 0\le j\le n+N-1}$ be the $n \times (n+N)$ matrix whose (i,j)th entry is

$$b_{ij}(t) = \begin{cases} \delta_{0,j} & \text{if } i=0,\\ \binom{i-1}{j-i} + \binom{i-1}{j-i-1}t & \text{otherwise.}\end{cases}$$

Especially, when $t=1$, we write B_n^N for $B_n^N(1)$ whose (i,j)th entry is $\binom{i}{j-i}$. Let $\bar{S}_n = (\bar{s}_{ij})_{1\le i,j\le n}$ be the skew-symmetric matrix of size n whose (i,j)th entry $\bar{s}_{ij}$ is equal to $(-1)^{j-i-1}$ for $1 \le i < j \le n$, and let O_n denote the $n\times n$ zero matrix. Let $J_n = (\delta_{i,n+1-j})_{1\le i,j\le n}$ denote the anti-diagonal matrix where $\delta_{i,j}$ stands for the Kronecker delta function. One of the results we obtain for Conjecture 1.1 is the following:

THEOREM 1.2. Let n be a positive integer and let N be an even integer such that $N \ge n-1$. Then

$$\sum_{b\in\mathscr{B}_n} t^{U_r(b)} = \operatorname{Pf}\begin{pmatrix} O_n & J_nB_n^N(t)\\ -{}^tB_n^N(t)J_n & \bar{S}_{n+N}\end{pmatrix}. \tag{1.1}$$

For example, if $n=3$ and $N=2$ then the above Pfaffian looks like as follows.

$$\operatorname{Pf}\left(\begin{array}{ccc|ccccc} 0 & 0 & 0 & 0 & 0 & 1 & 1+t & t\\ 0 & 0 & 0 & 0 & 1 & t & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\ \hline 0 & 0 & -1 & 0 & 1 & -1 & 1 & -1\\ 0 & -1 & 0 & -1 & 0 & 1 & -1 & 1\\ -1 & -t & 0 & 1 & -1 & 0 & 1 & -1\\ -1-t & 0 & 0 & -1 & 1 & -1 & 0 & 1\\ -t & 0 & 0 & 1 & -1 & 1 & -1 & 0 \end{array}\right).$$

In the same paper, they also presented the following conjecture which we call the doubly refined enumeration of TSSCPPs:

CONJECTURE 1.3. ([**31**, pp.284, Conjecture 3]) Let $n \ge 2$ and $1 \le k, l \le n$ be integers. Then the number of elements b of $\mathscr{B}_n$ such that $U_1(b) = k-1$ and $U_2(b) = n-l$ would be $A_n^{k,l}$.

Let n and N be positive integers. Let $B_n^N(t,u) = (b_{ij}(t,u))_{0\le i\le n-1,\ 0\le j\le n+N-1}$ be the $n\times(n+N)$ matrix whose (i,j)th entry is

$$b_{ij}(t,u) = \begin{cases} \delta_{0,j} & \text{if } i=0,\\ \delta_{0,j-i} + \delta_{0,j-i-1}tu & \text{if } i=1,\\ \binom{i-2}{j-i} + \binom{i-2}{j-i-1}(t+u) + \binom{i-2}{j-i-2}tu & \text{otherwise.}\end{cases}$$

Note that, when $u=1$, $B_n^N(t,1)$ is equal to $B_n^N(t)$. Then one form of the Pfaffian expressions for Conjecture 1.3 which we obtain in this paper is following:

THEOREM 1.4. Let n be a positive integer and let N be an even integer such that $N \ge n-1$. If r is an integer such that $2 \le r \le n$, then we have

$$\sum_{b\in\mathscr{B}_n} t^{U_1(b)}u^{U_r(b)} = \operatorname{Pf}\begin{pmatrix} O_n & J_nB_n^N(t,u)\\ -{}^tB_n^N(t,u)J_n & \bar{S}_{n+N}\end{pmatrix}. \tag{1.2}$$

The monotone triangles are known to be in one-to-one correspondence with the alternating sign matrices ([**5, 30**]). Here we arrange our definition following the notation in [**31**]. A *monotone triangle* of size n is, by definition, a triangular array of positive integers

$$\begin{array}{cccc} & & & m_{n,n} \\ & & m_{n-1,n-1} & m_{n-1,n} \\ & \iddots & \vdots & \vdots \\ m_{1,1} & \dots & m_{1,n-1} & m_{1,n} \end{array}$$

subject to the constraints that

(M1) $m_{ij} < m_{i,j+1}$ whenever both sides are defined,
(M2) $m_{ij} \geq m_{i+1,j}$ whenever both sides are defined,
(M3) $m_{ij} \leq m_{i+1,j+1}$ whenever both sides are defined,
(M4) the bottom row $(m_{1,1}, m_{1,2}, \dots, m_{1,n})$ is $(1, 2, \dots, n)$.

Let $\mathscr{M}_n$ denote the set of monotone triangles of size n. Note that, if one removes the bottom row of $m \in \mathscr{M}_n$ and turn it upside-down, then he get an array defined in [**31**].

For $k = 0, 1, \dots, n-1$, let $\mathscr{M}_n^k$ denote the set of monotone triangles with all entries m_{ij} in the first $n-k$ columns equal to their minimum values $j-i+1$. For $k = 0, 1, \dots, n-1$, let $\mathscr{B}_n^k$ be the subset of those b in $\mathscr{B}_n$ such that all b_{ij} in the first $n-1-k$ columns are equal to their maximal values n. Then they also presented the following conjecture:

Conjecture 1.5. ([**31**, pp.287, Conjecture 7]) For $n \geq 2$ and $k = 0, 1, \dots, n-1$, the cardinality of $\mathscr{B}_n^k$ is equal to the cardinality of $\mathscr{M}_n^k$.

Let m, n and k be integers such that $1 \leq m \leq n$ and $0 \leq k \leq n-m$. We define the $n \times n$ skew-symmetric matrix $\bar{L}_n^{(m,k)}(\varepsilon) = (\bar{l}_{ij}^{(m,k)}(\varepsilon))_{1 \leq i,j \leq n}$ as follows: if k is even, then

$$\bar{l}_{ij}^{(m,k)}(\varepsilon) = \begin{cases} (-1)^{j-i-1}\varepsilon & \text{if } 1 \leq i < j \leq n \text{ and } i \leq m+k, \\ (-1)^{j-i-1} & \text{if } m+k < i < j \leq n, \end{cases}$$

else

$$\bar{l}_{ij}^{(m,k)}(\varepsilon) = \begin{cases} (-1)^{j-i-1}\varepsilon & \text{if } 1 \leq i < j \leq m+k, \\ (-1)^{j-i-1} & \text{if } 1 \leq i < j \leq n \text{ and } m+k < j. \end{cases}$$

Then a Pfaffian expression for Conjecture 1.5 which we obtain in this paper is following:

Theorem 1.6. Let n be a positive integer and let $k = 0, 1, \dots, n-1$. Let N be an even integer such that $N \geq k$. The cardinality of $\mathscr{B}_n^k$ is equal to

$$\lim_{\varepsilon \to 0} \varepsilon^{-\lfloor \frac{k}{2} \rfloor} \operatorname{Pf} \begin{pmatrix} O_n & B_n^N J_{n+N} \\ -J_{n+N} {}^t B_n^N & \bar{L}_{n+N}^{(n,k)}(\varepsilon) \end{pmatrix} \tag{1.3}$$

Here $\lfloor x \rfloor$ stands for the floor function, i.e. the greatest integer less than or equal to x.

Mills, Robbins and Rumsey also have introduced two involutions ρ and γ of $\mathscr{B}_n$ onto itself, and conjectured that they correspond to the half turn and the vertical flip of the alternating sign matrices. Here we are mainly concerned with γ, which is defined as follows. Let $b = (b_{ij})_{1 \leq i \leq j \leq n-1}$ be an element of $\mathscr{B}_n$ and let b_{ij} be a part of b off the main diagonal. Then the *flip* of the part b_{ij} is the operation of replacing b_{ij} by b'_{ij} where

$$b'_{ij} + b_{ij} = \min(b_{i-1,j}, b_{i,j-1}) + \max(b_{i,j+1}, b_{i+1,j}). \tag{1.4}$$

When the part is in the main diagonal, the flip of a part b_{ii} is the operation replacing b_{ii} by b'_{ii} where

$$b'_{ii} + b_{ii} = b_{i-1,i} + b_{i,i+1}. \tag{1.5}$$

An operation π_r is defined to be a map $\mathscr{B}_n \to \mathscr{B}_n$ where $\pi_r(b)$ is the result of flipping all the $b_{i,i+r-1}$, $1 \leq i \leq n-r$. Let

$$\gamma = \pi_1 \pi_3 \pi_5 \cdots, \tag{1.6}$$

which is an involution since π_1, π_3, ... commute each other (see ([**31**, pp.286])). Let $\mathscr{B}_n^\gamma$ denotes the set of elements in $\mathscr{B}_n$ invariant under γ.

CONJECTURE 1.7. ([**31**, pp.286, Conjecture 6]) Let $n \geq 0$ be an integer and r, $1 \leq r \leq 2n-1$, be an integer. Then the number of elements of $\mathscr{B}_{2n+1}$ with $\gamma(b) = b$ and $U_2(b) = r-1$ would be the same as the number of $(2n+1)\times(2n+1)$ alternating sign matrices with $a_{r+1,1} = 1$ and invariant under the vertical flip. Namely, $\sum_{b\in\mathscr{B}^{\gamma}_{2n+1}} t^{U_2(b)} = A^{\mathrm{VS}}_{2n+1}(t)$ would hold.

Here we show that Conjecture 1.7 reduce to the evaluation of the determinant in the following theorem:

THEOREM 1.8. Let $n \geq 2$ be a positive integer. Let $\det R^{\mathrm{o}}_n(t) = (R^{\mathrm{o}}_{i,j})_{0\leq i,j\leq n}$ be the $n\times n$ matrix where

$$R^{\mathrm{o}}_{i,j} = \binom{i+j-1}{2i-j} + \left\{\binom{i+j-1}{2i-j-1} + \binom{i+j-1}{2i-j+1}\right\} t + \binom{i+j-1}{2i-j} t^2$$

with the convention that $R^{\mathrm{o}}_{0,0} = R^{\mathrm{o}}_{0,1} = 1$. Then we obtain

$$\sum_{b\in\mathscr{B}^{\gamma}_{2n+1}} t^{U_2(b)} = \det R^{\mathrm{o}}_n(t). \tag{1.7}$$

We are not able to evaluate this determinant in general at this point, but, if $t = 1$, then we can reduce the evaluation of the determinant to the Andrews-Burge determinant [**3**] and prove the weak version of Conjecture 1.7.

THEOREM 1.9. Let $n \geq 2$ be a positive integer. Then the number of elements of $\mathscr{B}_{2n+1}$ with $\gamma(b) = b$ is the same as the number of $(2n-1)\times(2n-1)$ alternating sign matrices invariant under the vertical flip.

2. The numbers and polynomials

First we recall the numbers and polynomials related to the alternating sign matrices (cf. [**5, 27, 30, 32, 33, 35, 40, 42**]). Let A_n denote the number defined by

$$A_n = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}. \tag{2.1}$$

This number is famous for the alternating sign matrix conjecture (cf. [**5**]). The number of totally symmetric self-complementary plane partitions was conjectured to be A_n in [**31**, pp.282, Conjecture 1], and settled in [**38**, p.p.127, Theorem 8.3] and [**2**] (see also [**1, 3**]), Another proof has appeared in [**22**] to describe several determinant techniques have been developed in it. Let n be a positive number and let $1 \leq r \leq n$. Set A^r_n to be the number

$$A^r_n = \frac{\binom{n+r-2}{n-1}\binom{2n-r-1}{n-1}}{\binom{2n-2}{n-1}} A_{n-1} = \frac{\binom{n+r-2}{n-1}\binom{2n-1-r}{n-1}}{\binom{3n-2}{n-1}} A_n. \tag{2.2}$$

The number has appeared to describe the distribution of the position of the 1 in the top row of an alternating sign matrix (see [**25, 27, 32, 42**]). We also define the polynomial $A_n(t) = \sum_{r=1}^{n} A^r_n t^{r-1}$. Let n be a positive integer and let $\omega = e^{2i\pi/3}$. Let $A_n(t,u)$ denote the polynomial defined by

$$A_n(t,u) = \frac{\{\omega^2(\omega+t)(\omega+u)\}^{n-1}}{3^{n(n-1)/2}} s^{(2n)}_{\delta(n-1,n-1)}\left(\frac{1+\omega t}{\omega+t}, \frac{1+\omega u}{\omega+u}, 1, \dots, 1\right) \tag{2.3}$$

where $s^{(n)}_{\lambda}(x_1,\dots,x_n)$ stands for the Schur function in the n variables $x_1, \dots, x_n$, corresponding to the partition λ, and $\delta(n-1,n-1) = (n-1,n-1,n-2,n-2,\dots,1,1)$. Let $A^{k,l}_n$ denote the coefficient of $t^{k-1}u^{n-l}$ in $A_n(t,u)$, i.e. $A_n(t,u) = \sum_{k,l=1}^{n} t^{k-1}u^{n-l}$ (See [**9, 32, 40**]).

Let A^{VS}_{2n+1} be the number defined by

$$A^{\mathrm{VS}}_{2n+1} = (-3)^{n^2} \prod_{\substack{1\leq i,j\leq 2n+1\\ 2|j}} \frac{3(j-i)+1}{j-i+2n+1} = \frac{1}{2^n}\prod_{k=1}^{n} \frac{(6k-2)!(2k-1)!}{(4k-1)!(4k-2)!} \tag{2.4}$$

and let $A^{\mathrm{VS},\,r}_{2n+1}$ be the number given by

$$A^{\mathrm{VS},\,r}_{2n+1} = \frac{A^{\mathrm{VS}}_{2n-1}}{(4n-2)!} \sum_{k=1}^{r} (-1)^{r+k} \frac{(2n+k-2)!(4n-k-1)!}{(k-1)!(2n-k)!}. \tag{2.5}$$

This number A^{VS}_{2n+1} is equal to the number of vertically symmetric alternating sign matrices of size $2n+1$ (see [**27, 32, 35**]). For example, the first few terms of A^{VS}_{2n+1} are 1, 3, 26, 646 and 45885. We also define the polynomial $A^{\mathrm{VS}}_{2n+1}(t)$ by $A^{\mathrm{VS}}_{2n+1}(t) = \sum_{r=1}^{2n} A^{\mathrm{VS},\, r}_{2n+1} t^{r-1}$.

3. Definitions and bijections

In this section we study three classes of (shifted) plane partitions which are denoted by $\mathscr{T}_{n,m}$, $\mathscr{B}_{n,m}$ and $\mathscr{P}_{n,m}$, and we establish bijections between them. The set $\mathscr{B}_{n,m}$ is a generalization of the set $\mathscr{B}_n$ defined in [**31**], and the set $\mathscr{P}_{n,m}$ is newly defined in this paper. First of all we have to recall the basic definitions and notation concerning plane partitions. For the general theory of plane partitions the reader may consult [**5, 29, 36, 37, 38**].

A *plane partition* is an array $\pi = (\pi_{ij})_{i,j\geq 1}$ of nonnegative integers such that π has finite support (i.e. finitely many nonzero entries) and is weakly decreasing in rows and columns. If $\sum_{i,j\geq 1} \pi_{ij} = n$, then we write $|\pi| = n$ and say that π is a plane partition of n, or π has *weight* n. A *part* of a plane partition $\pi = (\pi_{ij})_{i,j\geq 1}$ is a positive entry $\pi_{ij} > 0$. The *shape* of π is the ordinary partition λ for which π has λ_i nonzero parts in the ith row. We denote the shape of π by $\mathrm{sh}(\pi)$.

Consider the elements of $\mathbb{P}^3$, regarded as the lattice points of $\mathbb{R}^3$ in the positive orthant. The *Ferrers graph* $F(\pi)$ of π is the set of all lattice points $(i,j,k) \in \mathbb{P}^3$ such that $k \leq \pi_{ij}$. A subset F of $\mathbb{P}^3$ is a Ferrers graph if and only if it satisfies

$$x_1 \leq x_2,\ y_1 \leq y_2,\ z_1 \leq z_2 \text{ and } (x_2, y_2, z_2) \in F \Rightarrow (x_1, y_1, z_1) \in F.$$

Hereafter we identify a plane partition and its Ferrers graph, and write π for $F(\pi)$. The symmetric group S_3 is acting on $\mathbb{P}^3$ as permutations of the coordinate axes. A plane partition is said to be *totally symmetric* if its Ferrers graph is mapped to itself under all 6 permutations in S_3.

Definition 3.1. Let m and $n \geq 1$ be nonnegative integers. Let $\mathscr{P}_{n,m}$ denote the set of plane partitions $c = (c_{ij})_{1\leq i,j}$ subject to the constraints that

(C1) c has at most n columns;
(C2) c is column-strict and each part in the jth column does not exceed $n+m-j$.

We call an element of $\mathscr{P}_{n,m}$ a *restricted column-strict plane partition* (abbreviated to RCSPP). When $m = 0$, we write $\mathscr{P}_n$ for $\mathscr{P}_{n,0}$. If a part in the jth column of c is equal to $n+m-j$ (that can happen only in the first row, i.e. $c_{1j} = n+m-j$), we call the part a *saturated part*.

Let μ be a strict partition. A *shifted plane partition* τ of *shifted shape* μ is an arbitrary filling of the cells of μ with nonnegative integers such that each entry is weakly decreasing in rows and columns. In this paper we allow parts to be zero for shifted plane partitions of a fixed shifted shape μ.

Definition 3.2. (See [**22**, Theorem 1]). Let m and $n \geq 1$ be nonnegative integers. Let $\mathscr{B}_{n,m}$ denote the set of shifted plane partitions $b = (b_{ij})_{1\leq i\leq j}$ subject to the constraints that

(B1) the shifted shape of b is $(n+m-1, n+m-2, \ldots, 2, 1)$;
(B2) $\max\{n-i, 0\} \leq b_{ij} \leq n$ for $1 \leq i \leq j \leq n+m-1$.

When $m = 0$, we write $\mathscr{B}_n$ for $\mathscr{B}_{n,0}$. In this paper we call an element of $\mathscr{B}_{n,m}$ a *triangular shifted plane partition* (abbreviated to TSPP).

For a $b = (b_{ij})_{1\leq i\leq j\leq n+m-1}$ in $\mathscr{B}_{n,m}$ and integers $r = 1, \ldots, n+m$, let

$$U_r(b) = \sum_{t=1}^{n+m-r} (b_{t,t+r-1} - b_{t,t+r}) + \sum_{t=n+m-r+1}^{n+m-1} \chi\{b_{t,n+m-1} > n-t\}. \tag{3.1}$$

Here $\{\ldots\}$ has value 1 when the statement "$\ldots$" is true and 0 otherwise, and we use the convention that $b_{i,n} = n-i$ for all i and $b_{0,j} = n$ for all j. It is easy to check that each of these functions U_r can vary between 0 and $n+m-1$ as b varies over $\mathscr{B}_{n,m}$. We put $\overline{U}_r(b) = n+m-1-U_r(b)$.

A plane partition $\pi \subseteq X_{r,s,t} := [r] \times [s] \times [t]$ is (r,s,t)-self-complementary if we have, for all $p \in X_{r,s,t}$, $p \in \pi$ if and only if $\sigma_{r,s,t}(p) \notin \pi$. Let $\mathscr{T}_n$ denote the set of all plane partitions which is contained in the cube X_{2n}, $(2n,2n,2n)$-self-complementary and totally symmetric. We can define a subset $\mathscr{T}_{n,m}$ of $\mathscr{T}_n$, and can construct a bijection between $\mathscr{T}_{n,m}$ and $\mathscr{P}_n$ and a bijection between

$\mathscr{T}_{n,m}$ and $\mathscr{B}_n$. But here we don't mention the detail of $\mathscr{T}_{n,m}$ and the bijections, and only state the bijection between $\mathscr{P}_n$ and $\mathscr{B}_n$ obtained as Corollary.

Let $c = (c_{ij})_{1\le i\le n+m, 1\le j\le n}$ be a RCSPP in $\mathscr{P}_{n,m}$ and let k be a positive integer. Let $c_{\ge k}$ denote the plane partition formed by the parts $\ge k$. Let

$$\theta_i(c_{\ge k}) = \sharp\{l : c_{i,l} \ge k\} \tag{3.2}$$

denote the length of the ith row of $c_{\ge k}$, i.e. the rightmost column containing a letter $\ge k$ in the ith row of c.

Theorem 3.3. Let m and $n \ge 1$ be nonnegative integers and $c = (c_{ij})_{1\le i\le n+m, 1\le j\le n}$ be a RCSPP in $\mathscr{P}_{n,m}$. Associate to the array $c = (c_{ij})_{1\le i\le n+m, 1\le j\le n}$ the array $b = (b_{ij})_{1\le i\le j\le n+m-1}$ defined by

$$n - b_{ij} = \theta_{n+m-j}(c_{\ge 1-i+j}) \tag{3.3}$$

with $1 \le i \le j \le n+m-1$. Then b is in $\mathscr{B}_{n,m}$, and this mapping $\varphi_{n,m}$, which associate to a RCSPP c the TSPP $b = \varphi_{n.m}(c)$, is a bijection of $\mathscr{P}_{n,m}$ onto $\mathscr{B}_{n,m}$.

Furthermore we identify each element in $\mathscr{P}_{n,m}$ and each element in $\mathscr{B}_{n,m}$ by the bijection $\varphi_{n,m}$ defined in Theorem 3.3, and we define $U_r(c) = U_r(\varphi_{n,m}(c))$ and $\overline{U}_r(c) = \overline{U}_r(\varphi_{n,m}(c))$ for $c \in \mathscr{P}_{n,m}$. The following theorem enable us to compute $\overline{U}_r(c)$ directly.

Theorem 3.4. Let m and $n \ge 1$ be nonnegative integers and let $c \in \mathscr{P}_{n,m}$. Then $\overline{U}_r(c)$ is the number of parts equal to r plus the number of saturated parts less than r, i.e.

$$\overline{U}_r(c) = \sharp\{(i,j) : c_{ij} = r\} + \sharp\{1 \le k < r : c_{1,n+m-k} = k\}. \tag{3.4}$$

Especially $\overline{U}_1(c)$ is the number of 1's in c and $\overline{U}_{n+m}(c)$ is the number of saturated parts in c. It is also easy to see that $\overline{U}_{n+m-1}(c) = \overline{U}_{n+m}(c)$ since, if a part of $c \in \mathscr{P}_{n,m}$ is equal to $n+m-1$, then it is saturated.

A classical method to prove that a Schur function is symmetric is to define involutions s_i on tableaux which swaps the number of i's and $(i-1)$'s, for each i. This is well-known as the Bender-Knuth involution ([**4**]). We can define a twisted Bender-Knuth involution $\widetilde{\pi}_r : \mathscr{P}_{n,m} \to \mathscr{P}_{n,m}$, and show that it correspond to the involution π_r of $\mathscr{B}_{n,m}$. This involution $\widetilde{\pi}_r$ is a "Bender-Knuth involution", which swaps r's and unsaturated $r-1$'s. In fact, if it did convert a saturated $r-1$ of c into r, the resulting plane partition would violate the axiom of $\mathscr{P}_{n,m}$. We see the exact definition and an example below.

In the following we use the convention that each row of $c \in \mathscr{P}_{n,m}$ is followed by the appropriate number of 0's as the ith row of c has $n+m-i$ entries. Let $1 \le r \le n+m$ and $c \in \mathscr{P}_{n,m}$. Consider the parts of c equal to r or $r-1$. Since c is column-strict, some columns of c will contain neither r nor $r-1$, while some others will contain one r and one $r-1$. These columns we ignore. We also ignore an $r-1$ in column $n+m-r+1$, i.e. we ignore a saturated part which is equal to $r-1$ because a saturated $r-1$ can't be changed to r. The remaining parts equal to r or $r-1$ occur once in each column. Assume row i has a certain number k of r's followed by a certain number l of $r-1$'s. Note that we don't count an $r-1$ if it is saturated so that a saturated $r-1$ always remains untouched. For example, the three consecutive rows $i-1$, i and $i+1$ of c could look as above. In row i, convert the k r's and l $r-1$'s to l r's and k $r-1$'s. It is easy to see that the

$$\begin{array}{r||ccc|ccccccc|ccc|}
i-1 & \vdots & & \vdots & & & & & & & & r & \dots & r \\
i & r & \dots & r & r & \dots & r & r-1 & \dots & r-1 & & r-1 & \dots & r-1 \\
i+1 & r-1 & \dots & r-1 & & & & & & & & & &
\end{array}$$

resulting array satisfies the axioms (C1) and (C2). Define an operation $\widetilde{\pi}_r : \mathscr{P}_{n,m} \to \mathscr{P}_{n,m}$ by $c \mapsto \widetilde{\pi}_r(c)$ where $\widetilde{\pi}_r(c)$ is the result of swapping r's and $r-1$'s in row i of c by this twisted rule for $1 \le i \le n+m-r$. For example, if $n = 6$, $m = 0$ and $r = 2$, then the left below RCSPP c

corresponds to the right below RCSPP $\widetilde{\pi}_2(c)$ by $\widetilde{\pi}_2$.

$$c = \begin{array}{|c|c|c|c|c|}\hline \boldsymbol{\mathit{5}} & 3 & 1 & 1 & \boldsymbol{\mathit{1}} \\ \hline 3 & 2 \\ \cline{1-2} 2 & 1 \\ \cline{1-2} \end{array} \qquad \widetilde{\pi}_2(c) = \begin{array}{|c|c|c|c|c|}\hline \boldsymbol{\mathit{5}} & 3 & 2 & \boldsymbol{\mathit{2}} & \boldsymbol{\mathit{1}} \\ \hline 3 & 2 \\ \cline{1-2} 1 & 1 \\ \cline{1-2} \end{array}$$

The involution $\widetilde{\pi}_1$ is well-defined when $m = 0, 1$, and the following theorem holds.

THEOREM 3.5. Let m and $n \geq 1$ be non-negative integers and let $1 \leq r \leq n + m$. Assume $m = 0$ or 1 if $r = 1$. Then we have

$$\pi_r\left(\varphi_{n,m}(c)\right) = \varphi_{n,m}\left(\widetilde{\pi}_r(c)\right).$$

Thus we define the corresponding involution $\widetilde{\gamma} : \mathscr{P}_{n,m} \to \mathscr{P}_{n,m}$, $m = 0, 1$, by

$$\widetilde{\gamma} = \widetilde{\pi}_1\widetilde{\pi}_3\widetilde{\pi}_5 \cdots \tag{3.5}$$

where the product is over all $\widetilde{\pi}_i$ with i odd and $\leq n$. Let $\mathscr{P}^{\widetilde{\gamma}}_{n,m}$ denote the set of c in $\mathscr{P}_{n,m}$ invariant under $\widetilde{\gamma}$. Hereafter we assume $m = 0$ and we write $\mathscr{P}^{\widetilde{\gamma}}_{n}$ for $\mathscr{P}^{\widetilde{\gamma}}_{n,0}$. For example, if $n = 7$, the following RCSPP in $\mathscr{P}_7$ is invariant under $\widetilde{\gamma}$.

$$\begin{array}{|c|c|c|c|c|}\hline 5 & 5 & 3 & 2 & 1 \\ \hline 4 & 4 & 1 \\ \cline{1-3} 3 & 3 \\ \cline{1-2} 2 & 2 \\ \cline{1-2} 1 \\ \cline{1-1} \end{array}$$

4. The generating functions

From here we define several skew-symmetric matrices which play an important role in the applications. Let n be a positive integer. Let $S_n = (s_{ij})_{1 \leq i,j \leq n}$ be the skew-symmetric matrix of size n whose (i,j)th entry s_{ij} is 1 for $1 \leq i < j \leq n$, and let $\bar{S}_n$ be as defined in Section 1.

Let n and N be positive integers, and let m be a nonnegative integer. Let $B^N_{n,m}(t,u) = (b^{(m)}_{ij}(t,u))_{0 \leq i \leq n-1,\ 0 \leq j \leq n+N-1}$ be the $n \times (n+N)$ matrix whose (i,j)th entry is

$$b^{(m)}_{ij}(t,u) = \begin{cases} \delta_{0,j} & \text{if } i+m=0, \\ \binom{i+m-1}{j-i} + \binom{i+m-1}{j-i-1}tu & \text{if } i+m=1, \\ \binom{i+m-2}{j-i} + \binom{i+m-2}{j-i-1}(t+u) + \binom{i+m-2}{j-i-2}tu & \text{otherwise.} \end{cases} \tag{4.1}$$

For example,

$$B^2_{3,0}(t,u) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & tu & 0 & 0 \\ 0 & 0 & 1 & t+u & tu \end{pmatrix}.$$

We define the $n \times (n+N)$ matrices $B^N_{n,m}(t) = B^N_{n,m}(t,1)$ and $B^N_{n,m} = B^N_{n,m}(1)$. Then the (i,j)th entry of $B^N_{n,m}(t)$ is

$$b^{(m)}_{ij}(t) = \begin{cases} \delta_{0,j} & \text{if } i+m=0, \\ \binom{i+m-1}{j-i} + \binom{i+m-1}{j-i-1}t & \text{otherwise.} \end{cases} \tag{4.2}$$

and the (i,j)th entry of $B^N_{n,m}$ is $\binom{i+m}{j-i}$ where the row index runs $0 \leq i \leq n-1$ and the column index runs $0 \leq j \leq n+N-1$. When $m = 0$, these $B^N_{n,m}(t,u)$, $B^N_{n,m}(t)$ and $B^N_{n,m}$ agree with $B^N_n(t,u)$, $B^N_n(t)$ and B^N_n introduced in Section 1. The following corollary (i) gives a Pfaffian expression for the doubly refined TSSCPP conjecture (Conjecture 1.3).

Theorem 4.1. Let m and $n \geq 1$ be non-negative integers, and let N be an even integer such that $N \geq n+m-1$.
(i) If r is a positive integer such that $2 \leq r \leq n+m$, then the generating function for all plane partitions $c \in \mathscr{P}_{n,m}$ with the weight $t^{\overline{U}_1(c)}u^{\overline{U}_r(c)}$ is

$$\sum_{c\in\mathscr{P}_{n,m}} t^{\overline{U}_1(c)}u^{\overline{U}_r(c)} = \operatorname{Pf}\begin{pmatrix} O_n & J_n B^N_{n,m}(t,u) \\ -{}^tB^N_{n,m}(t,u)J_n & \bar{S}_{n+N}\end{pmatrix}. \tag{4.3}$$

(ii) If r is a positive integer such that $1 \leq r \leq n+m$, then the generating function for all plane partitions $c \in \mathscr{P}_{n,m}$ with the weight $t^{\overline{U}_r(c)}$ is given by

$$\sum_{c\in\mathscr{P}_{n,m}} t^{\overline{U}_r(c)} = \operatorname{Pf}\begin{pmatrix} O_n & J_n B^N_{n,m}(t) \\ -{}^tB^N_{n,m}(t)J_n & \bar{S}_{n+N}\end{pmatrix}. \tag{4.4}$$

(iii) If r is a positive integer such that $1 \leq r \leq n+m$, then the generating function for all plane partitions $c \in \mathscr{P}_{n,m}$ with the weight $t^{\overline{U}_r(c)}$ is given by

$$\sum_{c\in\mathscr{P}^k_{n,m}} t^{\overline{U}_r(c)} = \lim_{\varepsilon\to 0}\varepsilon^{-\lfloor\frac{k}{2}\rfloor}\operatorname{Pf}\begin{pmatrix} O_n & J_n B^N_{n,m}(t) \\ -{}^tB^N_{n,m}(t)J_n & \bar{L}^{(n,k)}_{n+N}(\varepsilon)\end{pmatrix}. \tag{4.5}$$

Especially, when $t=1$, the number of elements of $\mathscr{P}^k_{n,m}$ is equal to

$$\lim_{\varepsilon\to 0}\varepsilon^{-\lfloor\frac{k}{2}\rfloor}\operatorname{Pf}\begin{pmatrix} O_n & J_n B^N_{n,m} \\ -{}^tB^N_{n,m}J_n & \bar{L}^{(n,k)}_{n+N}(\varepsilon)\end{pmatrix}. \tag{4.6}$$

(iv) Let $n \geq 2$ be a positive integer. Let $\det R^{\circ}_n(t) = (R^{\circ}_{i,j})_{0\leq i,j\leq n}$ be the $n\times n$ matrix where

$$R^{\circ}_{i,j} = \binom{i+j-1}{2i-j} + \left\{\binom{i+j-1}{2i-j-1} + \binom{i+j-1}{2i-j+1}\right\}t + \binom{i+j-1}{2i-j}t^2 \tag{4.7}$$

with the convention that $R^{\circ}_{0,0} = R^{\circ}_{0,1} = 1$. Then we have

$$\sum_{b\in\mathscr{B}^{\gamma}_{2n+1}} t^{U_2(b)} = \sum_{c\in\mathscr{P}^{\tilde{\gamma}}_{2n+1}} t^{\overline{U}_2(c)} = \det R^{\circ}_n(t). \tag{4.8}$$

Especially, when $t=1$, this determinant reduces to

$$\det R^{\circ}_n(1) = \det_{0\leq i,j\leq n-1}\left(\binom{i+j+1}{2i-j+1}\right). \tag{4.9}$$

From the Andrews-Burge determinant [**3**], we obtain

$$\det_{0\leq i,j\leq n-1}\left(\binom{i+j+1}{2i-j+1}\right) = \frac{1}{2^{n-1}}\prod_{k=1}^{n-1}\frac{(2k+4)_k\left(2k+\frac{5}{2}\right)_{k-1}}{(k)_k\left(k+\frac{5}{2}\right)_{k-1}} = \frac{1}{2^n}\prod_{k=1}^{n}\frac{(6k-2)!(2k-1)!}{(4k-1)!(4k-2)!},$$

where $(A)_j = A(A+1)\cdots(A+j-1)$. This proves that the number of shifted plane partitions $b \in \mathscr{B}_{2n+1}$ invariant under the involution γ equals A^{VS}_{2r+1} (i.e. Conjecture 6 in [**31**] is true when $t=1$).

5. Constant term identities

In [**41**], D. Zeilberger proved the following constant term identity. Let D be the sum of all the $n\times n$ minors of the $n\times(2n+m-1)$ matrix X given by

$$X_{ij} = \binom{m+i}{j-i}, \qquad 0\leq i\leq n-1, \quad 0\leq j\leq 2n+m-2,$$

and let C be the constant term of

$$\prod_{1\leq i\leq j\leq n}\left(1-\frac{x_i}{x_j}\right)\prod_{i=1}^{n}\left(1+\frac{1}{x_i}\right)^{m+n-i}\prod_{i=1}^{n}\frac{1}{1-x_i}\prod_{1\leq i\leq j\leq n}\frac{1}{1-x_ix_j},$$

then $D=C$ holds. The aim of this section is to give a generalization of this constant term identity, which gives the constant term identities for the conjectures we treat.

There are well-known identities called Littlewood's identity and Cauchy's identity which read

$$\sum_{\lambda} s_{\lambda}^{(n)}(\boldsymbol{x}) = \prod_{i=1}^{n} \frac{1}{1-x_i} \prod_{1\le i<j\le n} \frac{1}{1-x_ix_j}, \tag{5.1}$$

$$\sum_{\lambda \text{ even}} s_{\lambda}^{(n)}(\boldsymbol{x}) s_{\lambda}^{(n)}(\boldsymbol{y}) = \prod_{i=1}^{n} \frac{1}{1-x_i^2} \prod_{1\le i,j\le n} \frac{1}{1-x_iy_j}, \tag{5.2}$$

where $s_{\lambda}^{(n)}(\boldsymbol{x})$ denotes the Schur function in the n variables $\boldsymbol{x}=(x_1,\dots,x_n)$ corresponding to the partition λ (see [**29**, I, 5]). I.G. Macdonald obtained the bounded version of (5.1):

$$\sum_{\substack{\lambda \\ \lambda_1\le k}} s_{\lambda}^{(n)}(\boldsymbol{x}) = \frac{\det(x_i^{j-1}-x_i^{k+2n-j})_{1\le i,j\le n}}{\prod_{i=1}^{n}(1-x_i)\prod_{1\le i<j\le n}(x_j-x_i)(1-x_ix_j)}, \tag{5.3}$$

(see [**29**, I, 5, Ex.16]). In this section we give a surprising relation between these identities and constant term identities enumerating the totally symmetric plane partitions.

Let us fix the notation. If we write $h_i^{(m)}(t,u;x) = \sum_{j\ge 0} b_{ij}^{(m)}(t,u)x^{j-i}$ where $b_{ij}^{(m)}(t)$ is as in (4.1), then we have

$$h_i^{(m)}(t,u;x) = \begin{cases} 1 & \text{if } m+i=0, \\ 1+tux & \text{if } m+i=1, \\ (1+x)^{m+i-2}(1+tx)(1+ux) & \text{if } m+i\ge 2. \end{cases} \tag{5.4}$$

We also write $h_i^{(m)}(t;x)$ for $h_i^{(m)}(t,1;x)$, and $h_i^{(m)}(z) = h_i^{(m)}(1;x) = (1+z)^{m+i}$. Let $\mathrm{CT}_{\boldsymbol{x}} f(\boldsymbol{x})$ denote the constant term of a polynomial $f(\boldsymbol{x})$ in the variable $\boldsymbol{x}=(x_1,\dots,x_n)$. The following theorem gives the main results of this section.

Theorem 5.1. Let m and $n\ge 1$ be non-negative integers.
(i) If r is an integer such that $2\le r\le n+m$, then $\sum_{c\in\mathscr{P}_{n,m}} t^{\overline{U}_1(c)}u^{\overline{U}_r(c)}$ is equal to

$$\mathrm{CT}_{\boldsymbol{x}} \prod_{1\le i<j\le n}\left(1-\frac{x_i}{x_j}\right)\prod_{i=1}^{n} h_{i-1}^{(m)}(t,u;x_i^{-1})\prod_{i=1}^{n}\frac{1}{1-x_i}\prod_{1\le i<j\le n}\frac{1}{1-x_ix_j}. \tag{5.5}$$

(ii) If r is an integer such that $1\le r\le n+m$, then $\sum_{c\in\mathscr{P}_{n,m}} t^{\overline{U}_r(c)}$ is equal to

$$\mathrm{CT}_{\boldsymbol{x}} \prod_{1\le i<j\le n}\left(1-\frac{x_i}{x_j}\right)\prod_{i=1}^{n} h_{i-1}^{(m)}(t;x_i^{-1})\prod_{i=1}^{n}\frac{1}{1-x_i}\prod_{1\le i<j\le n}\frac{1}{1-x_ix_j}. \tag{5.6}$$

(iii) If r is an integer such that $1\le r\le n+m$, then $\sum_{c\in\mathscr{P}_{n,m}^{k}} t^{\overline{U}_r(c)}$ is equal to

$$\mathrm{CT}_{\boldsymbol{x}} \prod_{1\le i<j\le n}\left(1-\frac{x_i}{x_j}\right)\prod_{i=1}^{n} h_{i-1}^{(m)}(t;x_i^{-1})\frac{\det(x_i^{j-1}-x_i^{k+2n-j})_{1\le i,j\le n}}{\prod_{i=1}^{n}(1-x_i)\prod_{1\le i<j\le n}(x_j-x_i)(1-x_ix_j)}. \tag{5.7}$$

Especially, when $t=1$, the number of elements of $\mathscr{P}_{n,m}^{k}$ is equal to

$$\mathrm{CT}_{\boldsymbol{x}} \prod_{1\le i<j\le n}\left(1-\frac{x_i}{x_j}\right)\prod_{i=1}^{n}\left(1+\frac{1}{x_i}\right)^{m+i-1}\frac{\det(x_i^{j-1}-x_i^{k+2n-j})_{1\le i,j\le n}}{\prod_{i=1}^{n}(1-x_i)\prod_{1\le i<j\le n}(x_j-x_i)(1-x_ix_j)}. \tag{5.8}$$

(iv) Then $\sum_{c\in\mathscr{P}_{2n+1}^{\tilde{\gamma}}} t^{\overline{U}_2(c)}$ is equal to

$$\begin{aligned}\mathrm{CT}_{\boldsymbol{x}}\mathrm{CT}_{\boldsymbol{y}} &\prod_{1\le i<j\le n}\left(1-\frac{x_i}{x_j}\right)\prod_{1\le i<j\le n}\left(1-\frac{y_i}{y_j}\right)\prod_{i=2}^{n}\left(1+\frac{1}{x_i}\right)^{i-2}\left(1+\frac{t}{x_i}\right) \\ &\times\prod_{j=2}^{n}\left(1+\frac{1}{y_j}\right)^{j-2}\left(1+\frac{t}{y_j}\right)\prod_{j=1}^{n}(1+y_j)\prod_{i,j=1}^{n}\frac{1}{1-x_iy_j}.\end{aligned} \tag{5.9}$$

Especially, when $t = 1$, the number of elements of $\mathscr{P}_{2n+1}^{\widetilde{\gamma}}$ is equal to

$$\begin{aligned}\mathrm{CT}_{\boldsymbol{x}}\mathrm{CT}_{\boldsymbol{y}} &\prod_{1\le i<j\le n}\left(1-\frac{x_i}{x_j}\right)\prod_{1\le i<j\le n}\left(1-\frac{y_i}{y_j}\right)\\ &\times\prod_{i=1}^{n}\left(1+\frac{1}{x_i}\right)^{i-1}\prod_{j=1}^{n}\left(1+\frac{1}{y_j}\right)^{j-1}\prod_{j=1}^{n}(1+y_j)\prod_{i,j=1}^{n}\frac{1}{1-x_iy_j}.\end{aligned} \tag{5.10}$$

Christian Krattenthaler has obtained an equivalent result to (5.8) in [**24**] concerning Conjecture 1.5 (i.e. Conjecture 7 of [**31**]).

Hereafter, we restrict our attention to the case where $m = 0$. Thus (5.5) reads

$$\begin{aligned}\mathrm{CT}_{\boldsymbol{x}} &\prod_{1\le i<j\le n}\left(1-\frac{x_i}{x_j}\right)\times\left(1+\frac{tu}{x_2}\right)\prod_{i=3}^{n}\left\{\left(1+\frac{t}{x_i}\right)\left(1+\frac{u}{x_i}\right)\left(1+\frac{1}{x_i}\right)^{i-3}\right\}\\ &\times\prod_{i=1}^{n}\frac{1}{1-x_i}\prod_{1\le i<j\le n}\frac{1}{1-x_ix_j}\end{aligned} \tag{5.11}$$

when $n \ge 3$, and (5.6) reads

$$\mathrm{CT}_{\boldsymbol{x}}\prod_{1\le i<j\le n}\left(1-\frac{x_i}{x_j}\right)\cdot\prod_{i=2}^{n}\left\{\left(1+\frac{t}{x_i}\right)\left(1+\frac{1}{x_i}\right)^{i-2}\right\}\cdot\prod_{i=1}^{n}\frac{1}{1-x_i}\prod_{1\le i<j\le n}\frac{1}{1-x_ix_j} \tag{5.12}$$

when $n \ge 2$. Here we regard $\prod_{i=1}^{n}\frac{1}{1-x_i}\prod_{1\le i<j\le n}\frac{1}{1-x_ix_j}$ as a formal power series in the variable $x_1, \dots, x_n$. Meanwhile, (5.7) reads

$$\begin{aligned}\mathrm{CT}_{\boldsymbol{x}} &\prod_{1\le i<j\le n}\left(1-\frac{x_i}{x_j}\right)\cdot\prod_{i=2}^{n}\left\{\left(1+\frac{t}{x_i}\right)\left(1+\frac{1}{x_i}\right)^{i-2}\right\}\\ &\times\frac{\det(x_i^{j-1}-x_i^{k+2n-j})_{1\le i,j\le n}}{\prod_{i=1}^{n}(1-x_i)\prod_{1\le i<j\le n}(x_j-x_i)(1-x_ix_j)}\end{aligned} \tag{5.13}$$

when $n \ge 2$. Here $\frac{\det(x_i^{j-1}-x_i^{k+2n-j})_{1\le i,j\le n}}{\prod_{i=1}^{n}(1-x_i)\prod_{1\le i<j\le n}(x_j-x_i)(1-x_ix_j)}$ is a polynomial in $x_1, \dots, x_n$ since it is a finite sum of the Schur polynomials. We conclude that Conjecture 1.1 reduce to the computation of the constant term (5.12), Conjecture 1.3 reduce to the computation of the constant term (5.11), Conjecture 1.5 reduce to the computation of the constant term (5.13), and Conjecture 1.7 reduce to the computation of the constant term (5.9), respectively.

Now we illustrate these constant terms when $n = 3$. Put

$$F(t,u,\boldsymbol{x})=\left(1-\frac{x_1}{x_2}\right)\left(1-\frac{x_1}{x_2}\right)\left(1-\frac{x_2}{x_3}\right)\left(1+\frac{tu}{x_2}\right)\left(1+\frac{t}{x_3}\right)\left(1+\frac{u}{x_3}\right)$$

and

$$G(t,\boldsymbol{x})=F(t,1,\boldsymbol{x})=\left(1-\frac{x_1}{x_2}\right)\left(1-\frac{x_1}{x_2}\right)\left(1-\frac{x_2}{x_3}\right)\left(1+\frac{t}{x_2}\right)\left(1+\frac{t}{x_3}\right)\left(1+\frac{1}{x_3}\right).$$

Then (5.11) becomes

$$\begin{aligned}&\mathrm{CT}_{\boldsymbol{x}}F(t,u,\boldsymbol{x})(1+x_1+x_1^2+\cdots)(1+x_2+x_2^2+\cdots)(1+x_3+x_3^2+\cdots)\\ &\qquad\times(1+x_1x_2+x_1^2x_2^2+\cdots)(1+x_1x_3+x_1^2x_3^2+\cdots)(1+x_2x_3+x_2^2x_3^2+\cdots),\end{aligned}$$

which equals $1+t+u+tu+t^2u+tu^2+t^2u^2$. Similarly, (5.12) becomes

$$\begin{aligned}&\mathrm{CT}_{\boldsymbol{x}}G(t,\boldsymbol{x})(1+x_1+x_1^2+\cdots)(1+x_2+x_2^2+\cdots)(1+x_3+x_3^2+\cdots)\\ &\qquad\times(1+x_1x_2+x_1^2x_2^2+\cdots)(1+x_1x_3+x_1^2x_3^2+\cdots)(1+x_2x_3+x_2^2x_3^2+\cdots),\end{aligned}$$

which equals $2+3t+2t^2$. If $k = 1$, then (5.13) reads

$$\mathrm{CT}_{\boldsymbol{x}}G(t,\boldsymbol{x})\frac{\det(x_i^{j-1}-x_i^{7-j})_{1\le i,j\le 3}}{\prod_{i=1}^{3}(1-x_i)\prod_{1\le i<j\le 3}(x_j-x_i)(1-x_ix_j)}=\mathrm{CT}_{\boldsymbol{x}}G(t,\boldsymbol{x})(1+x_1)(1+x_2)(1+x_3),$$

which equals $2 + 2t + t^2$. Lastly, (5.9) reads

$$\mathrm{CT}_{\boldsymbol{x}}\mathrm{CT}_{\boldsymbol{y}} G(t,\boldsymbol{x})G(t,\boldsymbol{y}) \prod_{j=1}^{3}(1+y_j) \prod_{i,j=1}^{3} \frac{1}{1-x_i y_j}$$
$$= \mathrm{CT}_{\boldsymbol{x}}\mathrm{CT}_{\boldsymbol{y}} G(t,\boldsymbol{x})G(t,\boldsymbol{y})(1+y_1)(1+y_2)(1+y_3)(1+x_1y_1+x_1^2y_1^2+\cdots)(1+x_1y_2+x_1^2y_2^2+\cdots)\ldots$$

which equals $3 + 6t + 8t^2 + 6t^3 + 3t^4$. Note that, to derive (5.9), we need a long argument. We don't have enough space to state the details (see [**16**]).

6. Concluding Remarks

Here we mainly present the definitions and results, and we don't have enough space to present the proofs. To prove the results, we need various techniques of plane partitions, lattice path methods, symmetric functions, and combinatorics of tableaux. The interested reader can find the proofs in [**15, 16**]. We also should note that the computations of the Pfaffians and determinants are still open. As the reader may notice, the conjectures have so simple forms.

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