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A combinatorial classification of skew Schur functions

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Abstract. We present a single operation for constructing skew diagrams whose corresponding skew Schur functions are equal. This combinatorial operation naturally generalises and unifies all results of this type to date. Moreover, our operation suggests a closely related condition that we conjecture is necessary and sufficient for skew diagrams to yield equal skew Schur functions.

Résumé. Nous présentons une opération simple pour construire des diagrammes gauches dont les fonctions gauches de Schur correspondantes sont égales. Cette opération combinatoire généralise et unifie, d'une manière naturelle, tous les résultats connus de ce type. D'ailleurs, notre opération suggère une condition que nous conjecturons est nécessaire et suffisante pour que les diagrammes gauches produisent des fonctions gauches de Schur égales.

1. Introduction

Littlewood-Richardson coefficients arise in a variety of areas of mathematics and therefore not only knowing how to calculate them, but also knowing relations between them, is of importance. More precisely, given partitions λ, μ, ν, the Littlewood-Richardson coefficient $c^{\lambda}_{\mu\nu}$ arises most prominently in the following three places. Firstly, in the representation theory of the symmetric group, given Specht modules S^{μ} and S^{ν} we have

$$(S^{\mu} \otimes S^{\nu}) \uparrow^{S_n} = \bigoplus_{\lambda} c^{\lambda}_{\mu\nu} S^{\lambda}. \tag{1.1}$$

Secondly, considering the cohomology $H^*(Gr(k,n))$ of the Grassmannian, the cup product of Schubert classes σ_μ and σ_ν is given by

$$\sigma_\mu \cup \sigma_\nu = \sum_{\lambda} c^{\lambda}_{\mu\nu} \sigma_\lambda.$$

Lastly, in the algebra of symmetric functions the skew Schur function $s_{\lambda/\mu}$ can be expressed in terms of the basis of Schur functions, s_ν, via

$$s_{\lambda/\mu} = \sum_{\nu} c^{\lambda}_{\mu\nu} s_\nu. \tag{1.2}$$

Consequently, knowledge about $c^{\lambda}_{\mu\nu}$ impacts a number of fields. Examples of knowledge gleaned so far about $c^{\lambda}_{\mu\nu}$ include a variety of ways to compute them, such as the Littlewood-Richardson rule [**6, 13, 16, 17**], inequalities among them that arise from studying eigenvalues of Hermitian matrices [**5**], instances when they evaluate to zero [**9**], and polynomiality properties that they satisfy [**3, 4, 10**]. However, one natural aspect that has yet to be fully exploited is that of equivalence classes of equal coefficients. One way to approach this would be to use (1.2) and ask when two skew Schur functions are equal. This avenue is worth pursuing since it was recently shown that computing the coefficients $c^{\lambda}_{\mu\nu}$ is #P-complete [**8**].

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Returning to representation theory, there exist two polynomial representations of $GL_N(\mathbb{C})$ known as Schur modules and Weyl modules. These modules do not form a set of irreducible modules, and so a natural line of enquiry would be to ascertain when two of them are isomorphic. Since these modules are determined up to isomorphism by their characters, we simply need to discover when two characters are equal. It so happens that when the modules are indexed by skew diagrams D then the characters are precisely the skew Schur function s_D on N variables. In this case we therefore need only determine when two skew Schur functions are equal.

The question of skew Schur function equality arises naturally in one other place – the algebra of symmetric functions. As noted earlier, the skew Schur functions are not a basis for the symmetric functions and a question currently considered to be intractable is to find all relations among them. In [**11**] it was shown that the more specific goal of deriving all binomial syzygies between skew Schur functions could be attained by answering the question of equality, and therefore for this reason and those cited above this is what we will attempt. In order to do this, we define the following equivalence relation.

DEFINITION. For two skew diagrams D and D' we say they are *skew-equivalent* if $s_D = s_{D'}$, and denote this by $D \sim D'$.

Our question of when two skew Schur functions are equal then reduces to classifying the equivalence classes of $\sim$.

It should be noted that we are not the first to investigate this. In [**1**] skew-equivalence was completely characterized for the subset of skew diagrams known as ribbons (or border strips or rim hooks). Their classification involved a composition of ribbons α and β to form $\alpha \circ \beta$, and they were able to show that the size of every equivalence class of $\circ$ is a specific power of 2. The idea behind composition operations is that they allow us to construct new equivalences from equivalences involving smaller skew diagrams. For example, the results in [**1**] tell us that if $\alpha \sim \alpha'$ and $\beta \sim \beta'$, then $\alpha \circ \beta \sim \alpha' \circ \beta'$. The composition $\circ$ was generalised in [**11**] to include more general skew diagrams D and yielded compositions $\alpha \circ D$ and $D \circ \beta$. A new composition of skew diagrams denoted by $\alpha \circ_\omega D$ for ribbons α, ω and skew diagram D was also introduced, as was the concept of ribbon staircases. These constructions successfully explained almost all skew-equivalences for skew diagrams with up to 18 cells, but unfortunately 6 skew-equivalences evaded the authors. In this paper we unify all the above constructions into one construction $D \circ_W E$ for skew diagrams D, E and W. This composition not only provides us with an explanation for all skew-equivalences discovered to date, and thus suggests necessary and sufficient conditions for skew-equivalence, but also affords us the possibility to conjecture that all equivalence classes are a specific power of 2 in size. More precisely, this paper is structured as follows.

In the next section we review the necessary preliminaries such as skew diagrams and symmetric functions. In Section 3 we describe how to compose two skew diagrams D and E with respect to a third, W, to obtain $D \circ_W E$. For ribbons α, β, ω and a skew diagram D we discuss how our composition generalises the composition $\alpha \circ \beta$ of [**1**] and generalises the compositions $\alpha \circ D$, $D \circ \beta$ and $\alpha \circ_\omega D$ plus the notion of ribbon staircases found in [**11**]. It is also in this section that we state our central theorem, Theorem 3.21, which is the key to proving our sufficient condition for skew-equivalence. Finally in Section 4, as a consequence of Theorem 3.21, Theorem 4.2 gives our sufficient condition for skew-equivalence. We propose in Conjecture 4.5 that a closely related condition is necessary and sufficient for skew-equivalence, and that the size of every equivalence class is a specific power of 2. We also derive some conditions under which D is skew-equivalent to its transpose in Proposition 4.1 and conjecture the that converse is also true.

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2. Preliminaries

2.1. Diagrams. Before we embark on studying skew Schur functions, we need to recall the following combinatorial constructions. We say a *partition*, λ, of n is a list of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$ whose sum is n. We denote this by $\lambda \vdash n$, and we call k the *length* of λ, denoting it by $\ell(\lambda)$. For convenience we denote the unique partition of 0 by $\emptyset$. To every partition λ we can associate a subset of $\mathbb{Z}^2$ called a *diagram* that consists of λ_i left-justified cells in row i. By abuse of notation we also denote this diagram by

λ. In the example below, the symbol $\times$ denotes a cell, although in what follows we may choose to denote cells by numbers, letters or boxes for further clarity.

EXAMPLE 2.1.

$$(3,2,2,1) = \begin{array}{ccc} \times & \times & \times \\ \times & \times & \\ \times & \times & \\ \times & & \end{array} .$$

Using this convention for constructing diagrams, we locate cells in the diagram by their row and column indices (i,j), where $i \leq \ell(\lambda)$ and $j \leq \lambda_1$. Moreover, if a cell is contained in row i and column j of a diagram then we say $c(i,j) = j - i$ is the *content* or *diagonal* of the cell. We will often use navigational terminology to refer to cells of a diagram. For example, the *southeast border* consists of those cells (i,j) such that $(i+1, j+1)$ is not an element of the diagram. A cell (i,j) is said to be *strictly north* of a cell (i', j') if $i < i'$, while (i,j) is said to be *one position northwest* of (i', j') if $(i,j) = (i'-1, j'-1)$.

Now consider two diagrams λ and μ such that $\ell(\lambda) \geq \ell(\mu)$ and $\lambda_i \geq \mu_i$ for all $i \leq \ell(\mu)$, which we denote by $\mu \subseteq \lambda$. If we locate the cells of μ in the northwest corner of the set of cells of λ then the *skew diagram* λ/μ is the array of cells contained in λ but not in μ, where $\lambda/\emptyset = \lambda$. As an example of a skew diagram we have

$$(3,2,2,1)/(2,1) = \begin{array}{ccc} & & \times \\ & \times & \\ \times & \times & \\ \times & & \end{array} .$$

For convenience we will often refer to generic skew diagrams by capital letters such as D. We will call the number of cells in D the *size* of D and denote it by $|D|$. We also consider two skew diagrams to be equal as subsets of the plane if one can be obtained from the other by the addition or deletion of empty rows or columns, or by vertical or horizontal translation.

Any subset of the cells of D that itself forms a skew diagram is said to be a *subdiagram* of D. If two cells (i,j) and (i',j') satisfy $|i - i'| + |j - j'| = 1$ then we say that they are *adjacent* and we similarly say that two subdiagrams D_1 and D_2 are adjacent if there exists a cell in D_1 adjacent to a cell in D_2. This concept will play a fundamental role in the pages to follow, but now we will use it to define what it means to be a connected skew diagram. A skew diagram is said to be *connected* if for every cell d with another cell strictly north or east of it, there exists a cell adjacent to d either to the north or to the east. A connected skew diagram is called a *ribbon* (or *border strip* or *rim hook*) if it does not contain the subdiagram $\lambda = (2,2)$.

Given any connected skew diagram D there exist two natural subdiagrams of D, both of which are ribbons. The first is denoted by nw_D and is the ribbon that starts at the southwesternmost cell of D, traverses the northwest border of D, and ends at the northeasternmost cell of D. The second is denoted by se_D and is the ribbon that starts at the southwesternmost cell of D, traverses the *southeast* border of D, and ends at the northeasternmost cell of D. Since our goal is to extend results previously proved for ribbons, it will often be helpful to decompose a skew diagram into ribbons. Given a connected skew diagram D, the *southeast decomposition* of D is unique up to reordering and is defined as follows: we choose the first ribbon to be se_D. Now consider D with se_D removed and iterate the procedure on the remaining skew diagram. If this skew diagram is no longer connected then iterate on each of the connected components. This procedure clearly results is a disjoint collection of ribbons whose union is D. We can similarly define the *northwest decomposition* by utilising nw_D.

To close this subsection, we recall two symmetries on a skew diagram D. The first of these is the *transpose* or *conjugate* of D, denoted D^t, which is obtained by reflecting D along the diagonal that runs from northwest to southeast through all cells with content 0. The second is the *antipodal rotation* of D, denoted D^*, which is obtained by rotating D by 180 degrees in the plane.

2.2. The algebra of symmetric functions. The algebra of symmetric functions has many facets to it, and in this section we review the pertinent details required for our results. More information on this fascinating algebra can be found in [**7, 12, 14**].

Let Λ^n be the set of all formal power series $\mathbb{Z}[x_1, x_2, \ldots]$ in countably many variables that are homogeneous of bounded degree n in the x_i, and invariant under all permutations of the variables. Then the *algebra* *of symmetric functions* is

$$\Lambda := \bigoplus_{n\geq 0} \Lambda^n$$

where $\Lambda_0 = \text{span}\{1\} = \mathbb{Z}$. It transpires that Λ is a polynomial algebra in the *complete symmetric functions*, which are defined for all integers $r > 0$ by

$$h_r := \prod_{1\leq i_1\leq i_2\leq\cdots\leq i_r} x_{i_1}x_{i_2}\cdots x_{i_r},$$

and $h_0 = 1$. To obtain a $\mathbb{Z}$-basis for Λ, let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition of n and let

$$h_\lambda := h_{\lambda_1}\cdots h_{\lambda_k},$$

for which we find $\{h_\lambda\}_{\lambda\vdash n}$ is a $\mathbb{Z}$-basis for Λ^n. However, for the reasons cited in the introduction, it is arguable that the most important $\mathbb{Z}$-basis of Λ is that consisting of the Schur functions, which we now define as a subset of the skew Schur functions.

Given a skew diagram D, we say that T is a *semistandard Young tableau* if T is a filling of the cells of D with positive integers such that:

- the entries in the rows weakly increase when read from west to east; and
- the entries in the columns strictly increase when read from north to south.

The *skew Schur function* s_D is then

$$s_D := \sum_T x^T \tag{2.1}$$

where the sum ranges over all semistandard Young tableaux of shape D, and

$$x^T := \prod_{(i,j)\in D} x_{T_{ij}}.$$

Moreover, the skew Schur function is a *Schur function* if for $D = \lambda/\mu$ we have that $\mu = \emptyset$. In this case we usually write $s_D = s_\lambda$, which yields another description of Λ as $\Lambda = \oplus_{n\geq 0}\Lambda^n$ where $\Lambda^n = \text{span}\{s_\lambda | \lambda \vdash n\}$.

3. Compositions of skew diagrams

It is now time to recall our equivalence relation that was defined for ribbons in [**1**] and generalised in [**11**].

DEFINITION 3.1. For two skew diagrams D and D' we say they are *skew-equivalent* if $s_D = s_{D'}$, and denote this by $D \sim D'$.

The goal of this paper is to classify skew-equivalence by a condition that is both necessary and sufficient. Fortunately the number of skew-equivalences we need to classify is greatly reduced due to

PROPOSITION 3.1. [**11**, Section 6] *Understanding the equivalence relation $\sim$ on all skew diagrams is equivalent to understanding $\sim$ among* connected *skew diagrams.*

Consequently, we will assume in future that all skew diagrams are connected unless otherwise stated.

Our approach throughout will be to use known skew-equivalences to construct skew-equivalences for larger skew diagrams. Our basic building blocks will be the skew-equivalences of the following proposition, which is not hard to prove using the symmetry of s_D and its definition in terms of tableaux (2.1).

PROPOSITION 3.2. [**14**, Exercise 7.56(a)] *For any skew diagram D, $D^* \sim D$.*

The other main ingredient, and the focus of this paper, is a way to put these building blocks together to construct more complex skew-equivalences. More specifically, we wish to define a notion of composition $D \circ E$ for skew diagrams D and E. Then if $D \sim D'$ and $E \sim E'$, our hope will be that $D \circ E \sim D' \circ E'$. Since we wish to generalise and unify the three main operations of [**11**], some care needs to be taken when defining our composition operation, and some preliminary work is in order.

Definition 3.2. Given skew diagrams W and E, we say that W lies in the top (resp. bottom) of E if W appears as a connected subdiagram of E that includes the northeasternmost (resp. southwesternmost) cell of E.

Given two skew diagrams E_1 and E_2 and a skew diagram W lying in the top of E_1 and the bottom of E_2, the *amalgamation of* E_1 *and* E_2 *along* W, denoted by $E_1 \amalg_W E_2$, is the new skew diagram obtained from the disjoint union of E_1 and E_2 by identifying the copy of W in the top of E_1 with the copy of W in the bottom of E_2.

If W lies in both the top and bottom of E, then we will let W_{ne} (resp. W_{sw}) denote the copy of W in the top (resp. bottom) of E. We can also define

$$E^{\amalg_W m} = \underbrace{E \amalg_W E \amalg_W \cdots \amalg_W E}_{m \text{ factors}} := ((\cdots(E \amalg_W E) \amalg_W E) \amalg_W \cdots \amalg_W E.$$

Example 3.3. The skew diagram E given by

$$E = \begin{array}{cccccccc} & & & & & & \times & \times \\ & & & \times & \times & \times & \times & \times \\ & \times & \times & \times & \times & & & \\ \times & \times & \times & \times & & & & \end{array}$$

has

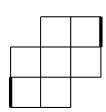
$$W = \begin{array}{ccc} & \times & \times \\ \times & \times & \times \end{array}$$

lying in its top and bottom. We see that

$$E \amalg_W E = \begin{array}{ccccccccccccc} & & & & & & & & & & & w & w \\ & & & & & & & & \times & \times & w & w & w \\ & & & & & & w & w & \times & \times & & & \\ & & & \times & \times & w & w & w & \times & & & & \\ & w & w & \times & \times & & & & & & & & \\ w & w & w & \times & & & & & & & & & \end{array},$$

where we use the symbol w to denote the cells of copies of W. Notice that $V = \begin{array}{ccc} & \times & \times \\ \times & \times & \end{array}$ also lies in the top and bottom of E, and that $E \amalg_V E$ is the same skew diagram as $E \amalg_W E$.

Example 3.4. For complete generality, we will also say that when $W = \emptyset$, W lies in the top and bottom of any skew diagram E. In this case, will will identify W_{sw} with the west edge of the southwesternmost cell of E. Similarly, we will identify W_{ne} with the east edge of the northeasternmost cell of E. For example, if $E = (3,3,2)/(1)$, then W_{sw} and W_{ne} would be identified with the thicker edges as shown.



Then $E \amalg_\emptyset E$ is the skew diagram $(6,6,5,3,2)/(4,3,1)$.

Now is a good time to introduce some assumptions on E and W that we will need for our results to hold.

Hypotheses 3.5. *Suppose that* E *is a skew diagram having* W *lying in its top and bottom. We assume that* E *and* W *satisfy the following conditions:*

(I) W *is maximal in the following sense: there does not exist a skew diagram* $W' \supsetneq W$ *that occupies the same set of diagonals as* W *and that also lies in the top and bottom of* E.
(II) W_{ne} *and* W_{sw} *are separated by at least one diagonal. In other words, there is at least one diagonal between* W_{ne} *and* W_{sw} *that intersects neither* W_{ne} *nor* W_{sw}.
(III) *The complement in* E *of either copy of* W *is a connected skew diagram.*

Remark 3.6. Analogues of Hypotheses II and III are also necessary for the results in [**11**, Section 7.2]. Notice that the V of Example 3.3 fails to satisfy Hypothesis I. As we saw, however, $E \amalg_V E = E \amalg_W E$, and it is true in general that we lose no generality in the skew-equivalences that we obtain when we impose Hypothesis I – it will just make the statements of some of our results simpler.

Hypotheses II and III tell us much about the structure of E. Let O denote the subdiagram of E that results when we delete both copies of W. We will write $E = WOW$ to mean that W lies in the top and bottom of E and that O is the subdiagram of E that results when we delete both copies of W. Since E is assumed to be connected, Hypotheses II and III tell us that O is a non-empty connected skew diagram.

Let us say that the lower (resp. upper) copy of W is *horizontally attached to* O if the southwesternmost (resp. northeasternmost) cell of O has a cell of W one position to its west (resp. east). Similarly, we say that the lower (resp. upper) copy of W is *vertically attached to* O if the southwesternmost (resp. northeasternmost) cell of O has a cell of W one position to its south (resp. north). Since W is a skew diagram and E is connected, each copy of W in E is either horizontally or vertically attached to O, but not both. Therefore, we are in one of the following four cases:

(a) Both copies of W are horizontally attached to O, written $E = W \rightarrow O \rightarrow W$.
(b) Both copies of W are vertically attached to O, written $E = W \uparrow O \uparrow W$.
(c) The lower copy of W is horizontally attached to O, while the upper copy of W is vertically attached to O, written $E = W \rightarrow O \uparrow W$.
(d) The lower copy of W is vertically attached to O, while the upper copy of W is horizontally attached to O, written $E = W \uparrow O \rightarrow W$.

We are almost ready to define composition of general skew diagrams. One issue that lengthens the definition of the composition of D and E with respect to W is that the definition varies according to the cases (a), (b), (c) and (d) above. As justification for this variation, consider the following diagrams that can be created, starting with two copies E_1 and E_2 of E:

(A) Position E_2 so that the lower copy of W in E_2 is one position northwest of the upper copy of W in E_1.
(B) Position E_2 so that the lower copy of W in E_2 is one position southeast of the upper copy of W in E_1.
(C) Form $E_1 \amalg_W E_2$ and translate an extra copy of W one position southeast from $E_1 \cap E_2$.
(D) Form $E_1 \amalg_W E_2$ and translate an extra copy of W one position northwest from $E_1 \cap E_2$.

The key observation is that in each of the four cases (a), (b), (c) and (d), exactly one of these four diagrams is a skew diagram, namely the diagram with the corresponding letter label. This observation effectively consists of sixteen assertions, and we leave their checking as an exercise for the reader that will reinforce the ideas introduced so far. In each of the four cases (a), (b), (c) and (d), we let $E_1 \cdot_W E_2$ denote the skew diagram constructed in (A), (B), (C) and (D) respectively. See Figure 1 for an illustration.

REMARK 3.7. We see that there is a fundamental difference between the set-up for the cases $W \rightarrow O \rightarrow W$, $W \uparrow O \uparrow W$ and the cases $W \rightarrow O \uparrow W$, $W \uparrow O \rightarrow W$. In a certain sense, this is to be expected, since it turns out that $W \rightarrow O \rightarrow W$ and $W \uparrow O \uparrow W$ are involved in generalising the composition and amalgamated composition operations of [**11**], while $W \rightarrow O \uparrow W$ and $W \uparrow O \rightarrow W$ are involved in generalising the ribbon staircase operation. The real strength of our framework will be highlighted by the statements of the results that follow, where all four cases can be treated as one.

We are finally ready to define the composition of general skew diagrams.

DEFINITION 3.8. For skew diagrams D, E with $E = WOW$ subject to Hypotheses 3.5, we define the composition $D \circ_W E$ with respect to W as follows. Every cell d of D will contribute a copy of E, denoted E_d, in the plane. The set of copies $\{E_d \mid d \in D\}$ are combined according to the following rules:

(a), (b) Suppose $E = W \rightarrow O \rightarrow W$ or $E = W \uparrow O \uparrow W$.
  (i) If d is one position west of d' in D, then E_d and $E_{d'}$ appear in the form $E_d \amalg_W E_{d'}$.
  (ii) If d is one position south of d' in D, then E_d and $E_{d'}$ appear in the form $E_d \cdot_W E_{d'}$.

(c), (d) If $E = W \rightarrow O \uparrow W$ then we consider the northwest ribbon decomposition of D, while if $E = W \uparrow O \rightarrow W$ then we consider the southeast ribbon decomposition of D.
  (i) If d is one position west of d' on the same ribbon in D, then E_d and $E_{d'}$ appear in the form $E_d \amalg_W E_{d'}$.
  (ii) If d is one position south of d' on the same ribbon in D, then E_d and $E_{d'}$ appear in the form $E_d \cdot_W E_{d'}$.
  (iii) If d is one position southeast of d' in D, then E_d appears one position southeast of $E_{d'}$.

Additionally, we will use the convention that $\emptyset \circ_W E = W$.

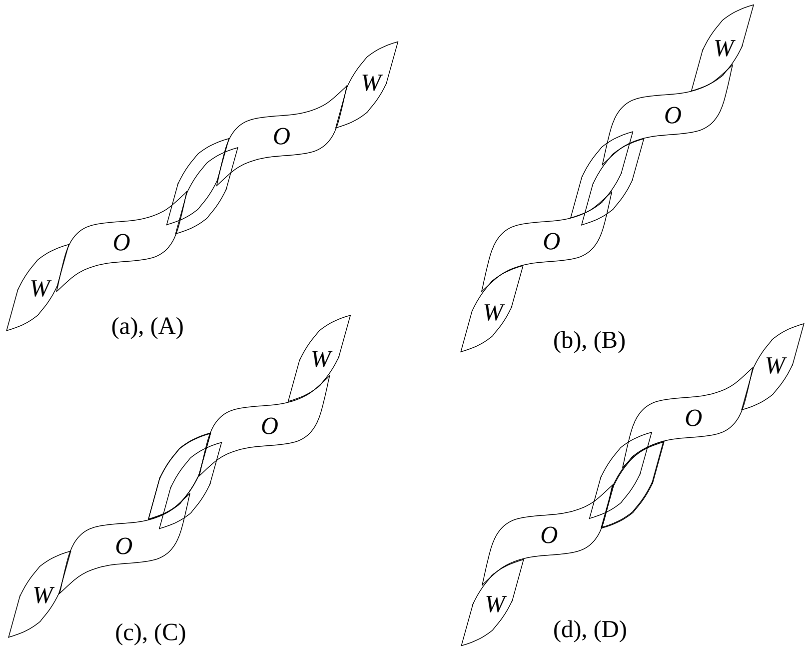


FIGURE 1. $E_1 \cdot_W E_2$ in the four cases

EXAMPLE 3.9. Identifying the cells of D with integers, and labelling the cells of the copies of W in E with the letter w, suppose

$$D = \begin{array}{cc} 1 & 2 \\ 3 & 4 \end{array} \quad \text{and} \quad E = \begin{array}{cccc} & & & w \\ w & \times & \times & w \\ w & \times & & \end{array}.$$

Then $E = W \!\rightarrow\! O \!\rightarrow\! W$ and so $D \circ_W E$ is the skew diagram

$$\begin{array}{ccccccccc} & & & & & & & & 2 \\ & & & & & \times & 2 & 2 & 2 \\ & & 1 & 1 & 1 & \times & \times & & \\ & & 1 & \times & 4 & 4 & 4 & & \\ 3 & 3 & 3 & \times & 4 & & & & \\ 3 & 3 & & & & & & & \end{array}$$

where a cell is labelled by $\times$ if it is an element of E_d for more than one $d \in D$, and otherwise is labelled by d when it is an element of E_d.

Alternatively, if

$$E = \begin{array}{ccc} & & w \\ & & w \\ w & \times & \times \\ w & \times & \end{array},$$

then $E = W \!\rightarrow\! O \uparrow W$ and so $D \circ_W E$ is the skew diagram

$$\begin{array}{ccccccc} & & & & & & 2 \\ & & & & & & 2 \\ & & & & \times & 2 & 2 \\ & & & & \times & \times & \\ & & \times & 1 & 1 & 4 & \\ & & \times & \otimes & 4 & 4 & \\ 3 & 3 & 3 & \otimes & 4 & & \\ 3 & 3 & & & & & \end{array},$$

where $\otimes$ denotes an element of both E_4 and $E_3 \cdot_W E_1$.

EXAMPLE 3.10. If W is empty, then referring to Example 3.4, it is natural to consider E to be of the form $W \to O \to W$. If at least one of D and E is a ribbon, $\circ_\emptyset$ becomes the composition denoted simply by $\circ$ in [**11**]. When both D and E are ribbons, $D \circ_\emptyset E$ also corresponds to $D \circ E$ of [**1**]. When neither D nor E is a ribbon, $\circ_\emptyset$ behaves like $\circ$, except that we allow overlaps to occur among copies of E. To see this in action, take D as in the previous example, and let $E = \begin{smallmatrix}\times & \times \\ \times & \times\end{smallmatrix}$. Then

$$D \circ_\emptyset E = \begin{array}{ccccc} & & & 2 & 2 \\ & 1 & 1 & 2 & 2 \\ & 1 & \times & 4 & \\ 3 & 3 & 4 & 4 & \\ 3 & 3 & & & \end{array}.$$

We note in passing that this is the same skew diagram that appears in [**11**, Remark 7.9], and was the first motivating example for the work of the current article.

EXAMPLE 3.11. The previous example demonstrates how one of the three main operations of [**11**] is obtained as a special case of our composition operation. The other two operations are also obtained as special cases: the amalgamated composition operation corresponds to certain cases of $D \circ_W E$ with D and W non-empty ribbons and E of the form $W \to O \to W$ or $W \uparrow O \uparrow W$. On the other hand, if E is a ribbon of the form $W \to O \uparrow W$ or $W \uparrow O \to W$, then $D \circ_W E$ is a ribbon staircase construction.

The following observation is of obvious importance.

LEMMA 3.12. *For skew diagrams D and $E = WOW$, $D \circ_W E$ is a skew diagram.*

We can now work directly towards the statement of Theorem 3.21, which expresses $s_{D \circ_W E}$ in terms of s_D and s_E, and thus serves as the foundation for all our skew-equivalence proofs. As mentioned in Example 3.10, a new feature of our definition of $D \circ_W E$ is that now we allow overlaps among the copies of E. At some point, we must obtain an understanding of, and account for, these overlaps. This motivates the following definition of the skew diagrams $\overline{W}$ and $\overline{O}$.

DEFINITION 3.13. Consider the infinite skew diagram

$$\boldsymbol{E} := E^{\amalg_W \infty} = \cdots \amalg_W E \amalg_W E \amalg_W \cdots. \tag{3.1}$$

For every copy O_1 of O in $\boldsymbol{E}$ we define

$$\overline{O_1} = \{(i,j) \in O_1 \mid (i+1, j+1) \in O_1\}.$$

For every copy W_1 of W we define

$$\overline{W_1} = \{(i,j) \in \boldsymbol{E} \mid (i+1, j+1) \in W_1\} \cup \{(i,j) \in W_1 \mid (i+1, j+1) \in \boldsymbol{E}\}.$$

Clearly, every copy O_1 of O defines the same diagram $\overline{O_1}$, which we denote simply by $\overline{O}$. Similarly, we define $\overline{W}$.

EXAMPLE 3.14. If

$$E = \begin{array}{cccccc} & & & \bar{\times} & \bar{w} & \bar{w} \\ & & \bar{\times} & \times & w & w \\ \bar{w} & \bar{w} & \times & \times & & \\ w & w & \times & & & \end{array},$$

then the top row comprises one copy of $\overline{W}$, while the single $\bar{\times}$ on the second row denotes $\overline{O}$. Part of a second copy of $\overline{W}$ is also shown.

Let us make two observations about Definition 3.13. Firstly, suppose we construct a second copy of $\boldsymbol{E}$ which is the translation of $\boldsymbol{E}$ one position northwest. Then $\overline{O}$ and $\overline{W}$ are exactly the shapes that form the overlap of the two copies of $\boldsymbol{E}$. Secondly, $\overline{O}$ and $\overline{W}$ are skew diagrams but neither one need be connected.

It turns out that we will need one further assumption about the structure of E. We conjecture below that this final assumption encompasses exactly what we need for our expression for $s_{D \circ_W E}$ to hold. In $\boldsymbol{E} = E^{\amalg_W \infty}$, Hypothesis II tells us that no two copies of $\overline{W}$ will be adjacent.

HYPOTHESIS 3.15. *Suppose that $E = WOW$. Assume that E satisfies the following condition:*

(IV) *In* $\boldsymbol{E}$*, no copy of* $\overline{O}$ *is adjacent to a copy of* $\overline{W}$*.*

Remark 3.16. In [**11**], $\overline{W}$ is always empty, so this hypothesis is not necessary.

The final construction required for our main results is a map on symmetric functions that will give an algebraic interpretation of the diagrammatic operation $\circ_W$. We note that the definition below is the natural generalisation of [**11**, Definition 7.17].

Definition 3.17. Let E and W be skew diagrams such that $E = WOW$. Consider the following map of sets

$$\begin{array}{ccc} \Lambda & \stackrel{(-)\circ_W s_E}{\longrightarrow} & \Lambda \\ 0 & \longmapsto & 0 \\ f & \longmapsto & f \circ_W s_E \end{array}$$

that consists of the composition of the following two maps $\Lambda \to \Lambda[t] \to \Lambda$ if $f \neq 0$. If we think of Λ as the polynomial algebra $\mathbb{Z}[h_1, h_2, \ldots]$, then we can temporarily grade Λ and $\Lambda[t]$ by setting $\deg(t) = \deg(h_r) = 1$ for all r. The first map $\Lambda \to \Lambda[t]$ then homogenises a polynomial in the h_r with respect to the above grading, using the variable t as the homogenisation variable.

Meanwhile the second map is given by

$$\begin{array}{ccc} \Lambda[t] & \longrightarrow & \Lambda \\ h_r & \longmapsto & s_{E\amalg_W r} \\ t & \longmapsto & s_W. \end{array}$$

For example, if $f = h_1h_2h_3 - (h_3)^2 - h_2h_4 + h_6$, then its image under the first map is $h_1h_2h_3 - (h_3)^2t - h_2h_4t + h_6t^2$. Therefore,

$$f \circ_W s_E = s_E s_{E\amalg_W 2} s_{E\amalg_W 3} - (s_{E\amalg_W 3})^2 s_W - s_{E\amalg_W 2} s_{E\amalg_W 4} s_W + s_{E\amalg_W 6}(s_W)^2.$$

Remark 3.18. If $f = s_D$ for some skew diagram D, then we see that there is a nice way to think of $f \circ_W s_E$ in terms of the Jacobi-Trudi decomposition matrix for s_D. Specifically, we homogenise by writing each entry of the form 1 in the Jacobi-Trudi matrix as h_0. Then we replace h_r by $E^{\amalg_W r}$ for $r \geq 0$. With the convention that $E^{\amalg_W 0} = W$, we now have that $s_D \circ_W s_E$ is simply the determinant of the resulting matrix. The reader is invited to check that the example above corresponds to this rule applied to the case of $f = s_D$ with $D = (4, 2, 2)/(1, 1)$.

Let $\widehat{D}$ (resp. $\widetilde{D}$) denote the subset of elements of D that have another element of D one position to their south (resp. southeast). Notice that $|\widehat{D}| = |\widetilde{D}| + r - 1$, where r is the number of rows in D. For symmetric functions f and g, we will write $f = \pm g$ to mean that either $f = g$ or $f = -g$.

We are finally ready to put everything together and start reaping the rewards of our hard work.

Conjecture 3.19. *For any skew diagram* D*, and a skew diagram* E *satisfying Hypotheses I – IV, we have*

$$s_{D\circ_W E}\,(s_{\overline{W}})^{|\widehat{D}|}\,(s_{\overline{O}})^{|\widetilde{D}|} = \pm\,(s_D \circ_W s_E)\,. \tag{3.2}$$

The sign on the right-hand side is a plus sign if $E = W\!\to O\!\to W$ *or* $E = W \uparrow O \uparrow W$*, and otherwise depends only on* D*. Furthermore, if* E *does not satisfy Hypothesis IV, then there exists a skew diagram* D *for which* (3.2) *fails to hold.*

We can prove (3.2) when E is of the form $W\!\to O \uparrow W$ or $W \uparrow O\!\to W$. However, our proof techniques require one further assumption for the two other forms of E.

Hypothesis 3.20. *If* $E = W\!\to\! O\!\to\! W$ *or* $E = W \uparrow O \uparrow W$ *then we assume that:*

(V) *In* E*, at least one copy of* W *has just one cell adjacent to* O*.*

Theorem 3.21. *For any skew diagram* D*, and a skew diagram* E *satisfying Hypotheses I – V, we have*

$$s_{D\circ_W E}\,(s_{\overline{W}})^{|\widehat{D}|}\,(s_{\overline{O}})^{|\widetilde{D}|} = \pm\,(s_D \circ_W s_E)\,. \tag{3.3}$$

The sign on the right-hand side is a plus sign if $E = W\!\to O\!\to W$ *or* $E = W \uparrow O \uparrow W$*, and otherwise depends only on* D*.*

REMARK 3.22. We can think of $(s_{\overline{W}})^{|\widehat{D}|}\,(s_{\overline{O}})^{|\widetilde{D}|}$ as the term introduced by the overlaps in $D \circ_W E$. For example, if W is empty and either D or E is a ribbon, then there are no overlaps among copies of E. We obtain

$$s_{D\circ_\emptyset E} = s_D \circ_\emptyset s_E, \tag{3.4}$$

which is (7.2) and Proposition 7.4 in [**11**].

While the major proof in this work is that of Theorem 3.21, our main target has been the following result, which serves as a mechanism for building skew-equivalences.

THEOREM 3.23. *Suppose we have skew diagrams D, D' with $D \sim D'$, and $E = WOW$ satisfying Hypotheses I – V. Then*

$$D' \circ_W E \;\sim\; D \circ_W E \;\sim\; D \circ_{W^*} E^*. \tag{3.5}$$

EXAMPLE 3.24. [**11**, Section 9] contains a list of the 6 skew-equivalences involving skew diagrams with at most 18 cells that are not explained by the results there. Using the equivalences of Theorem 3.23, we can now explain these equivalences. As an example, consider $D = \begin{smallmatrix}\times & \times\\ \times & \end{smallmatrix}$ and $D' = D^*$. Letting

$$E = \begin{array}{ccc} & w & w\\ \times & \times & \\ w & w & \end{array},$$

the first equivalence of (3.5) gives

$$\begin{array}{cccccc} &&&&\times&\times\\ &&\times&\times&\times&\\ &\times&\times&\times&\times&\\ &\times&\times&&&\\ \times&\times&&&&\\ \times&\times&&&& \end{array} \;\sim\; \begin{array}{cccccc} &&&&\times&\times\\ &&&\times&\times&\\ &&&\times&\times&\\ &\times&\times&\times&&\\ \times&\times&\times&\times&&\\ \times&\times&&&& \end{array},$$

which is the first of the 6 equivalences.

4. Concluding remarks

We wish to conclude by making a remark about Conjecture 3.19 and by introducing two further conjectures.

4.1. Removing Hypothesis V. As noted in Conjecture 3.19, we do not believe that Hypothesis V is necessary for Theorem 3.21 to hold. To prove the first assertion of the conjecture, we need to consider skew diagrams E such as

$$E = \begin{array}{cccc} &&\times&w\\ &\times&\times&w\\ w&\times&&\\ w&\times&& \end{array}.$$

Using [**2**], we can check that Conjecture 3.19 still holds if $D = \begin{smallmatrix}\times & \times\\ \times & \end{smallmatrix}$ or if we put D^* in place of D. Since $s_D = s_{D^*}$, we conclude that $s_{D\circ_W E} = s_{D^*\circ_W E}$, i.e.

$$\begin{array}{ccccccccc} &&&&&&&\times&\times\\ &&&&&&\times&\times&\times\\ &&&&\times&\times&\times&&\\ &&&\times&\times&\times&\times&&\\ &&\times&\times&&&&&\\ &&\times&\times&&&&&\\ &\times&\times&\times&&&&&\\ \times&\times&&&&&&&\\ \times&\times&&&&&&& \end{array} \;\sim\; \begin{array}{ccccccccc} &&&&&&&\times&\times\\ &&&&&&\times&\times&\times\\ &&&&&\times&\times&&\\ &&&&&\times&\times&&\\ &&&&\times&\times&\times&&\\ &&\times&\times&\times&&&&\\ &\times&\times&\times&\times&&&&\\ \times&\times&&&&&&&\\ \times&\times&&&&&&& \end{array}. \tag{4.1}$$

Since E does not satisfy Hypothesis V, this equivalence does not follow from Theorem 3.23. On the other hand, skew-equivalences such as these are explained by Conjecture 3.19. However, since this particular equivalence is not amenable to our current techniques, it seems that some new ideas will be necessary in order to prove Conjecture 3.19.

4.2. Skew diagrams equivalent to their transpose. It turns out in practice that there are many skew-equivalences of the form $F \sim F^t$. The following result gives an explanation for this.

PROPOSITION 4.1. *Suppose $E = WOW$ satisfies Hypotheses I – IV with $E^t = E$, $W^t = W$ and $W \neq \emptyset$. Then for any skew diagram D,*

$$(D \circ_W E)^t \sim D \circ_W E.$$

Certainly, if $F = F^t$ then $F \sim F^t$. We conjecture that the appropriate converse to Proposition 4.1 is also true.

CONJECTURE 4.1. *Suppose a skew diagram F has the property that $F \sim F^t$, with $F \neq F^t$. Then there exists a skew diagram $E = WOW$ satisfying Hypotheses I – IV and a skew diagram D such that $F = D \circ_W E$ with $E^t = E$, $W^t = W$ and $W \neq \emptyset$.*

4.3. Necessary and sufficient conditions for skew-equivalence.
The strongest result of [**1**] gives necessary and sufficient conditions for two ribbons to be skew-equivalent. The overarching goal of [**11**] and the current paper has been to make progress towards extending this result to general skew diagrams. We are now in a position to conjecture such necessary and sufficient conditions.

First, let us state a result that follows by induction from Theorem 3.23.

THEOREM 4.2. *Suppose we have skew diagrams $E_1, E_2, \ldots, E_r$ such that for $i = 2, \ldots, r$, $E_i = W_i O_i W_i$ satisfies Hypotheses I – V. Let E_1' denote either E_1 or E_1^*, and for each $i = 2, \ldots, r$, let E_i' and W_i' denote either E_i and W_i, or E_i^* and W_i^*. Then*

$$((\cdots(E_1 \circ_{W_2} E_2) \circ_{W_3} E_3) \cdots) \circ_{W_r} E_r \;\sim\; \left(\left(\cdots\left(E_1' \circ_{W_2'} E_2'\right) \circ_{W_3'} E_3'\right) \cdots\right) \circ_{W_r'} E_r'.$$

Next, let us recall Theorem 4.1 from [**1**] in our notation, where it was also shown that the $\circ_\emptyset$ operation is associative when applied to ribbons.

THEOREM 4.3. ([**1**, Theorem 4.1]) *Two ribbons α and β satisfy $\alpha \sim \beta$ if and only if, for some r,*

$$\alpha = \alpha_1 \circ_\emptyset \alpha_2 \circ_\emptyset \cdots \circ_\emptyset \alpha_r \quad \text{and} \quad \beta = \beta_1 \circ_\emptyset \beta_2 \circ_\emptyset \cdots \circ_\emptyset \beta_r,$$

where, for each i, α_i and β_i are ribbons with either $\beta_i = \alpha_i$ or $\beta_i = \alpha_i^*$. The skew-equivalence class of α will contain 2^r elements, where r is the number of factors α_i in the irreducible factorisation of α such that $\alpha_i \neq \alpha_i^*$.

It transpires that the concept of irreducible factorisation of [**1**] can be extended to arbitrary skew diagrams.

DEFINITION 4.4. Given a *factorisation* of a skew diagram $F = D \circ_W E$, where $E = WOW$ satisfies Hypotheses I – IV, we say that the factorisation is *trivial* if the factorisation is any one of the following:

(i) $(1) \circ_W F$;
(ii) $F \circ_\emptyset (1)$;
(iii) $\emptyset \circ_F E$.

We say the factorisation is *minimal* if it is non-trivial and, among all factorisations of F, W and then E occupies the minimum number of diagonals.

A factorisation $((\cdots(E_1 \circ_{W_2} E_2) \circ_{W_3} E_3) \cdots) \circ_{W_r} E_r$ is called *irreducible* if:

- E_1 only admits trivial factorisations;
- for $i = 2, \ldots, r$ we have $E_i = W_i O_i W_i$ satisfies Hypotheses I – IV;
- each factorisation $D_i \circ_{W_i} E_i$ is minimal, where

$$D_i = ((\cdots(E_1 \circ_{W_2} E_2) \circ_{W_3} E_3) \cdots) \circ_{W_{i-1}} E_{i-1}.$$

We now state our main conjecture, of which Theorem 4.3 implies a very special case.

CONJECTURE 4.5. *Two skew diagrams E and E' satisfy $E \sim E'$ if and only if, for some r,*

$$\begin{aligned} E &= ((\cdots(E_1 \circ_{W_2} E_2) \circ_{W_3} E_3) \cdots) \circ_{W_r} E_r \quad \text{and} \\ E' &= \left(\left(\cdots\left(E_1' \circ_{W_2'} E_2'\right) \circ_{W_3'} E_3'\right) \cdots\right) \circ_{W_r'} E_r', \end{aligned}$$

where

- $E_1, E_2, \ldots, E_r$ *are skew diagrams;*
- *for* $i = 2, \ldots, r$, $E_i = W_i O_i W_i$ *satisfies Hypotheses I – IV;*
- $E'_1 = E_1$ *or* $E'_1 = E^*_1$*;*
- *for* $i = 2, \ldots, r$, E'_i *and* W'_i *denote either* E_i *and* W_i*, or* E^*_i *and* W^*_i*.*

The skew-equivalence class of E *will contain* 2^r *elements, where* r *is the number of factors* E_i *in any irreducible factorisation of* E *such that* $E_i \neq E^*_i$*.*

REMARK 4.6. It was conjectured [**11**, Conjecture 9.2] that skew-equivalence classes have size a power of 2, and with our construction we can now predict precisely which power. For an example of a skew-equivalence class of size greater than 4, consider $(E \circ_W E) \circ_W E$, where $E = (2, 1)$ and $W = (1)$. One can check that the 8 skew diagrams of the form $(E' \circ_W E'') \circ_W E'''$, with E', E'' and E''' each equal to E or E^*, are all different.

To conclude we present evidence in favour of Conjecture 4.5. Observe that the "if" direction of Conjecture 4.5 would follow from Conjecture 3.19 in the same way that Theorem 4.2 follows from Theorem 3.21. The only difference is that Hypothesis V is absent in the conjectures. To support both the converse direction and the skew-equivalence class sizes, we have verified that the conjecture holds for all skew diagrams with at most 20 cells.

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