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The Combinatorics of the Garsia-Haiman Modules for Hook Shapes (Extended Abstract)

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Abstract. Several bases of the Garsia-Haiman modules for hook shapes are given, as well as combinatorial decomposition rules for these modules. These bases and rules extend the classical ones for the coinvariant algebra of type A. We also exhibit algebraic decompositions of the Garsia-Haiman modules for hook shapes that correspond to the combinatorial interpretation of the modified Macdonald polynomial that has recently been proved by Haglund, Haiman, and Loehr [**20, 21**].

1. Outline

The Garsia-Haiman module Δ_μ was introduced in [**12**], in an attempt to prove a conjecture of Macdonald, and indeed played a major role in the solution [**23**]. When the shape μ has a single row, this module is isomorphic to the coinvariant algebra of type A. Our goal here is to understand the structure of the *dual Garsia-Haiman module* Δ_μ^* when μ is a hook shape $(k, 1^{n-k})$.

A family of bases for the dual Garsia-Haiman module of hook shape is presented. This family includes the *k-th Artin basis*, the *k-th descent basis*, the *k-th Haglund basis* and the *k-th Schubert basis*. While the first basis appears in [**14**], the others are new and have interesting applications.

The k-th Haglund basis realizes Haglund's statistics in the hook case. The k-th descent basis extends the well known Garsia-Stanton descent basis for the coinvariant algebra. The advantage of the k-th descent basis is that the S_n-action on it may be described explicitly. This description implies combinatorial rules for decomposing the bi-graded components of the module into Solomon descent representations and into irreducibles. In particular, a constructive proof of a formula due to Stembridge is deduced.

The rest of the paper is organized as follows. Preliminaries and background are given in Section 2. Bases for the dual Garsia-Haiman module of hook shape are presented in Section 3. An explicit formula for the action of the Coxeter generators on the k-th descent basis and the resulting combinatorial rules for decomposing the bi-graded homogeneous components are described in Section 4. Proofs of two of the main theorems are sketched in Sections 5 and 6. Relations with the combinatorial interpretation of the modified Macdonald polynomial that was recently proved by Haglund, Haiman, and Loehr [**20, 21**] are discussed in Section 7.

This is an extended abstract; complete proofs and more details will be given in [**4**].

2. Background

In 1988, I. G. Macdonald [**27**] introduced a remarkable new basis for the space of symmetric functions. The elements of this basis are denoted $P_\lambda(\overline{x}; q, t)$, where λ is a partition, $\overline{x}$ is a vector of indeterminates, and q, t are parameters. The $P_\lambda(\overline{x}; q, t)$'s, which are now called "*Macdonald polynomials*", specialize to many of

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the well-known bases for the symmetric functions, by suitable choices of the parameters q and t. In fact, we can obtain in this manner the Schur functions, the Hall-Littlewood symmetric functions, the Jack symmetric functions, the zonal symmetric functions, the zonal spherical functions, and the elementary and monomial symmetric functions.

Given a cell s in the Young diagram (drawn according to the French convention) of a partition λ, let $leg_\lambda(s)$, $leg'_\lambda(s)$, $arm_\lambda(s)$, and $arm'_\lambda(s)$ denote the number of squares that lie above, below, to the right, and to left of s in λ, respectively. For each partition λ, define

$$h_\lambda(q,t) := \prod_{s\in\lambda}(1-q^{arm_\lambda(s)}t^{leg_\lambda(s)+1})$$

For a partition $\lambda=(\lambda_1,\ldots,\lambda_k)$ (where $\lambda_1\geq\ldots\geq\lambda_k>0$) let $n(\lambda):=\sum_{i=1}^k(i-1)\lambda_i$. Macdonald introduced the (q,t)-Kostka polynomials $K_{\lambda,\mu}(q,t)$ via the equation

$$J_\mu(\overline{x};q,t)=h_\mu(q,t)P_\mu(\overline{x};q,t)=\sum_\lambda K_{\lambda,\mu}(q,t)s_\lambda[X(1-t)],$$

and conjectured that they are polynomials in q and t with non-negative integer coefficients.

In an attempt to prove Macdonald's conjecture, Garsia and Haiman [**12**] introduced the so-called *modified Macdonanld polynomials* $\tilde{H}_\mu(\overline{x};q,t)$ as

$$\tilde{H}_\mu(\overline{x};q,t)=\sum_\lambda \tilde{K}_{\lambda,\mu}(q,t)s_\lambda(\overline{x}),$$

where $\tilde{K}_{\lambda,\mu}(q,t):=t^{n(\mu)}K_{\lambda,\mu}(q,1/t)$. Their idea was that $\tilde{H}_\mu(\overline{x};,q,t)$ is the Frobenius image of the character generating function of a certain bi-graded module Δ_μ under the diagonal action of the symmetric group S_n. To define Δ_μ, assign $(row, column)$-coordinates to squares in the first quadrant, so that the lower left-hand square has coordinates (1,1), the square above it has coordinates (2,1), the square to its right has coordinates (1,2), etc. The first (row) coordinate of a square w is denoted $row(w)$, and the second (column) coordinate of w is the denoted $col(w)$. Given a partition $\mu\vdash n$, let μ also denote the corresponding Young diagram, drawn according to the French convention; it consists of all the squares with coordinates (i,j) such that $1\leq i\leq\ell(\mu)$ and $1\leq j\leq\mu_i$. For example, for $\mu=(4,2,2)$, the labeling of squares is depicted in Figure 1.

(3,1)	**(3,2)**		
(2,1)	**(2,2)**		
(1,1)	**(1,2)**	**(1,3)**	**(1,4)**

FIGURE 1. Labeling of the cells of a partition.

Fix an ordering $w_1,\ldots,w_n$ of the squares of μ, and let

$$\Delta_\mu(x_1,\ldots,x_n;y_1,\ldots,y_n):=\det\left(x_i^{row(w_j)-1}y_i^{col(w_j)-1}\right)_{i,j}.$$

For example,

$$\Delta_{4,2,2}(x_1,\ldots,x_8;y_1,\ldots,y_8)=\det\begin{pmatrix}1 & y_1 & y_1^2 & y_1^3 & x_1 & x_1y_1 & x_1^2 & x_1^2y_1\\ 1 & y_2 & y_2^2 & y_2^3 & x_2 & x_2y_2 & x_2^2 & x_2^2y_2\\ \vdots & & & & & & & \vdots\\ 1 & y_8 & y_8^2 & y_8^3 & x_8 & x_8y_8 & x_8^2 & x_8^2y_8\end{pmatrix}$$

Now let Δ_μ be the vector space of polynomials spanned by all the partial derivatives of $\Delta_\mu(x_1,\ldots,x_n;y_1,\ldots,y_n)$. The symmetric group S_n acts on Δ_μ diagonally, where for any polynomial $P(x_1,\ldots,x_n;y_1,\ldots,y_n)$ and any permutation $\sigma\in S_n$,

$$P(x_1,\ldots,x_n;y_1,\ldots,y_n)^\sigma:=P(x_{\sigma_1},\ldots,x_{\sigma_n};y_{\sigma_1},\ldots,y_{\sigma_n}).$$

The bi-degree (h,k) of a monomial $x_1^{p_1}\cdots x_n^{p_n}y_1^{q_1}\cdots y_n^{q_n}$ is defined by $h:=\sum_{i=1}^n p_i$ and $k:=\sum_{i=1}^n q_i$. Let $\Delta_\mu^{(h,k)}$ denote space of homogeneous polynomials of degree (h,k) in Δ_μ. Then

$$\Delta_\mu = \bigoplus_{(h,k)} \Delta_\mu^{(h,k)}.$$

The S_n-action clearly preserves the bi-degree, so that S_n acts on each homogeneous component $\Delta_\mu^{(h,k)}$. The character of the S_n-action on $\Delta_\mu^{(h,k)}$ can be decomposed as

$$\chi^{\Delta_\mu^{(h,k)}} = \sum_{\lambda\vdash n} c_{\lambda,\mu}^{(h,k)}\chi^\lambda,$$

where χ^λ is the irreducible character of S_n indexed by the partition λ and the $c_{\lambda,\mu}^{(h,k)}$'s are non-negative integers. Garsia and Haiman conjectured [**12**] that, as an S_n-module, Δ_μ carries the regular representation. This conjecture was eventually proved by Haiman [**23**], by exploiting the algebraic geometry of the Hilbert Scheme.

3. Bases

Consider the inner product $\langle\ ,\ \rangle$ on the polynomial ring $Q_n=\mathbf{Q}[x_1,\dots,x_n,y_1,\dots,y_n]$ defined as follows: for any two polynomials $f,g\in Q_n$, $\langle f,g\rangle$ is the constant term of

$$f(\partial_{x_1},\dots,\partial_{x_n};\partial_{y_1},\dots,\partial_{y_n})g.$$

Let Δ_μ^* be the module dual to Δ_μ with respect to $\langle\ ,\ \rangle$.

3.1. The k-th Descent Basis. The *descent set* of a permutation $\pi\in S_n$ is

$$\mathrm{Des}(\pi):=\{i\,|\,\pi(i)>\pi(i+1)\}.$$

Garsia and Stanton [**17**] associated with each $\pi\in S_n$ the *descent monomial*

$$a_\pi := \prod_{i\in \mathrm{Des}(\pi)} (x_{\pi(1)}\cdots x_{\pi(i)}) = \prod_{j=1}^{n-1} x_{\pi(j)}^{|\mathrm{Des}(\pi)\cap\{j,\dots,n-1\}|}.$$

Using Stanley-Reisner rings, Garsia and Stanton [**17**] showed that the set $\{a_\pi\,|\,\pi\in S_n\}$ forms a basis for the coinvariant algebra of type A. See also [**36**] and [**5**].

Definition 3.1. *For every integer $1\le k\le n$ and permutation $\pi\in S_n$ define*

$$d_i^{(k)}(\pi) := \begin{cases} |\mathrm{Des}(\pi)\cap\{i,\dots,k-1\}|, & \text{if } 1\le i<k;\\ 0, & \text{if } i=k;\\ |\mathrm{Des}(\pi)\cap\{k,\dots,i-1\}|, & \text{if } k<i\le n.\end{cases}$$

Definition 3.2. *For every integer $1\le k\le n$ and permutation $\pi\in S_n$ define the k-th* descent monomial

$$\begin{aligned} a_\pi^{(k)} &:= \prod_{\substack{i\in\mathrm{Des}(\pi)\\ i\le k-1}} (x_{\pi(1)}\cdots x_{\pi(i)}) \cdot \prod_{\substack{i\in\mathrm{Des}(\pi)\\ i\ge k}} (y_{\pi(i+1)}\cdots y_{\pi(n)}) \\ &= \prod_{i=1}^{k-1} x_{\pi(i)}^{d_i^{(k)}(\pi)} \cdot \prod_{i=k+1}^{n} y_{\pi(i)}^{d_i^{(k)}(\pi)}. \end{aligned}$$

Note that $a_\pi^{(n)}=a_\pi$, the Garsia-Stanton descent monomial.

Theorem 3.3. *For every $1\le k\le n$, the set of k-th descent monomials $\{a_\pi^{(k)}\mid \pi\in S_n\}$ forms a basis for the dual Garsia-Haiman module $\Delta^*_{(k,1^{n-k})}$.*

Two proofs of Theorem 3.3 are given in [**4**]. In Section 5.2 of this extended abstract we sketch a proof via a straightening algorithm. This proof implies

Corollary 3.4. $\Delta^*_{(k,1^{n-k})}\cong\mathbf{Q}[\bar{x},\bar{y}]/I^+_{(k,1^{n-k})}$*, where the ideal $I^+_{(k,1^{n-k})}$ is generated by*

(1) *the elementary symmetric functions $e_i(x_1,\dots,x_n)$ $(1\le i\le n)$ and $e_i(y_1,\dots,y_n)$ $(1\le i\le n)$;*

(2) *the monomials* $x_{i_1}\cdots x_{i_k}$ $(1 \le i_1 < \cdots < i_k \le n)$ *and* $y_{i_1}\cdots y_{i_{n-k+1}}$ $(1 \le i_1 < \cdots < i_{n-k+1} \le n)$*; and*
(3) *the monomials* $x_i y_i$ $(1 \le i \le n)$.

This result has been obtained, in a different form, by J.-C. Aval [**7**, Theorem 2].

3.2. The k-th Artin and Haglund Bases. The second proof of Theorem 3.3 is sketched in Section 6.1. This proof applies a generalized version of the Garsia-Haiman kicking process. This construction is extended to a rich family of bases.

Let n be a positive integer and $1 \le k \le n$. For every positive integer n, denote $[n] := \{1,\ldots,n\}$. For every subset $A = \{i_1,\ldots,i_k\} \subseteq A$ denote $\bar{x}_A := x_{i_1},\ldots,x_{i_k}$ and $\bar{y}_A := y_{i_1},\ldots,y_{i_k}$. Denote $\bar{x} := \bar{x}_{[n]} = x_1,\ldots,x_n$ and $\bar{y} := \bar{y}_{[n]} = y_1,\ldots,y_n$.

Let $c \in [n]$ and let a A be a subset $\{a_1,\ldots,a_{k-1}\}$ of size $k-1$ of $[n]\setminus c$. For any such a pair (A,c) let B_A be an arbitrary basis of the coinvariant algebra of S_{k-1} acting on $\mathbf{Q}[\bar{x}_A]$; let $\bar{A} := [n]\setminus(A\cup\{c\})$ and let $C_{\bar{A}} = C_{[n]\setminus(A\cup j)}$ be a basis of the coinvariant algebra of S_{n-k} acting on $\mathbf{Q}[\bar{y}_{\bar{A}}]$.

For every pair (A,c) define a monomial in $\mathbf{Q}[\bar{x},\bar{y}]$,

$$m_{(A,c)} := \prod_{\{i\in A|\ i>c\}} x_i \prod_{\{j\in\bar{A}\ |\ j<c\}} y_j.$$

Then

THEOREM 3.5. *The set*

$$\bigcup_{A,c} m_{(A,c)}\ B_A\ C_{\bar{A}}$$

forms a basis for the dual Garsia-Haiman module $\Delta^*_{(k,1^{n-k})}$.

DEFINITION 3.6. *For every integer* $1 \le k \le n$ *and permutation* $\pi \in S_n$ *define*

$$\mathrm{inv}_i^{(k)}(\pi) := \begin{cases} |\{j\ :\ i < j \le k \textit{and}\ \pi(i) > \pi(j)\}|, & \textit{if } 1 \le i < k;\\ 0, & \textit{if } i = k;\\ |\{j\ :\ k \le j < i \textit{ and } \pi(j) < \pi(i)\}|, & \textit{if } k < i \le n.\end{cases}$$

For every integer $1 \le k \le n$ *and permutation* $\pi \in S_n$ *define the* k-th Artin monomial

$$b_\pi^{(k)} \quad := \quad \prod_{i=1}^{k-1} x_{\pi(i)}^{\mathrm{inv}_i^{(k)}(\pi)} \cdot \prod_{i=k+1}^{n} y_{\pi(i)}^{\mathrm{inv}_i^{(k)}(\pi)}.$$

and the k-th Haglund monomial

$$c_\pi^{(k)} \quad := \quad \prod_{i=1}^{k-1} x_{\pi(i)}^{d_i^{(k)}(\pi)} \cdot \prod_{i=k+1}^{n} y_{\pi(i)}^{\mathrm{inv}_i^{(k)}(\pi)}.$$

Interesting special cases of Theorem 3.5 are the following.

COROLLARY 3.7. *Each of the following sets :* $\{a_\pi^{(k)} \mid \pi \in S_n\}$, $\{b_\pi^{(k)} \mid \pi \in S_n\}$ *and* $\{c_\pi^{(k)} \mid \pi \in S_n\}$ *form a basis for the dual Garsia-Haiman module* $\Delta^*_{(k,1^{n-k})}$.

REMARK 1.

1. Garsia and Haiman [12] showed that $\{b_\pi^{(k)}\ :\ \pi \in S_n\}$ is a basis for $\Delta^*_{(k,1^{n-k})}$. Other bases of $\Delta^*_{(k,1^{n-k})}$ were also constructed by J.-C. Aval [**7**] and E. Allen [**5, 6**]. They used completely different methods. Aval constructed a basis of the form of an explicitly described set of partial differential operators applied to $\Delta_{(k,1^{n-k})}$ and Allen constructed a basis for $\Delta^*_{(k,1^{n-k})}$ out his theory of bitableaux.
2. It should be noted that the last basis corresponds to Haglund's maj-inv statistics for the Hilbert series of $\Delta^*_{(k,1^{n-k})}$ that is implied by his combinatorial interpretation for the modified Macdonald polynomial $\tilde{H}_{(k,1^{n-k})}(\bar{x};q,t)$; see Section 7 below.
3. Choosing B_A and $C_{\bar{A}}$ in Theorem 3.5 to be the Schubert bases of the coinvariant algebras of S_{k-1} (acting on $\mathbf{Q}[\bar{x}_A]$) and of S_{n-k} (acting on $\mathbf{Q}[\bar{y}_{\bar{A}}]$), respectively, gives the *k-th Schubert basis.* One may study the Hecke algebra actions on this basis along the lines drawn in [**2**].

4. Representations

4.1. Decomposition into Descent Representations. The set of elements in a Coxeter group having a fixed descent set carries a natural representation of the group, called a descent representation. Descent representations of Weyl groups were first introduced by Solomon [**32**] as alternating sums of permutation representations. This concept was extended to arbitrary Coxeter groups, using a different construction, by Kazhdan and Lusztig [**25**] [**24**, §7.15]. For Weyl groups of type A, these representations also appear in the top homology of certain (Cohen-Macaulay) rank-selected posets [**34**]. Another description (for type A) is by means of zig-zag diagrams [**18, 16**]. A new construction of descent representations for Weyl groups of type A, using the coinvariant algebra as a representation space, is given in [**1**].

For every subset $A \subseteq \{1, \ldots, n-1\}$ let

$$S_n^A := \{\ \pi \in S_n \mid \mathrm{Des}(\pi) = A\ \}$$

be the corresponding *descent class*; denote by ρ^A the corresponding *descent representation* of S_n.

Define $1 \le i < n$ to be a *descent* in a standard Young tableau T if $i+1$ lies strictly above and weakly to the left of i (in French notation). Denote the set of all descents in T by $\mathrm{Des}(T)$.

The following theorem is well known.

Theorem 4.1. *For any subset $A \subseteq [n-1]$ and partition $\mu \vdash n$, the multiplicity in the descent representation ρ^A of the irreducible S_n-representation corresponding to μ is*

$$m_\mu^A := \#\{T \in SYT(\mu) \,|\, \mathrm{Des}(T) = A\},$$

the number of standard Young tableaux of shape μ with descent set A.

Definition 4.2. *A bipartition (i.e., a pair of partitions) $\lambda = (\mu, \nu)$ is called an (n,k)-bipartition if μ has at most $k-1$ parts and ν has at most $n-k$ parts.*
For a permutation $\pi \in S_n$ and a corresponding k-descent basis element $a_\pi^{(k)} = \prod_{i=1}^{k-1} x_{\pi(i)}^{d_i} \cdot \prod_{i=k+1}^{n} y_{\pi(i)}^{d_i}$, let

$$\lambda(m) := (\lambda_x(m), \lambda_y(m)) := ((d_1, d_2, \ldots, d_{k-1}), (d_n, d_{n-1}, \ldots, d_{k+1}))$$

be its exponent bipartition.

For an (n,k) bipartition $\lambda = (\mu, \nu)$ let

$$J_\lambda^{(k)\trianglelefteq} := \mathrm{span}_{\mathbf{Q}}\{a_\pi^{(k)} + I^+_{(k,1^{n-k})} \,|\, \pi \in S_n,\ \lambda(a_\pi^{(k)}) \trianglelefteq \lambda\},$$

where $\trianglelefteq$ is the dominance order on bipartitions (see Definition 5.6.1). Let

$$J_\lambda^{(k)\triangleleft} := \mathrm{span}_{\mathbf{Q}}\{a_\pi^{(k)} + I^+_{(k,1^{n-k})} \,|\, \pi \in S_n,\ \lambda(a_\pi^{(k)}) \triangleleft \lambda\}$$

be subspaces of the module $\Delta^*_{(k,1^{n-k})}$, and let

$$R_\lambda^{(k)} := J_\lambda^{(k)\trianglelefteq} / J_\lambda^{(k)\triangleleft}.$$

Proposition 4.3. *$J_\lambda^{(k)\trianglelefteq}$, $J_\lambda^{(k)\triangleleft}$ and thus $R_\lambda^{(k)}$ are S_n-invariant.*

Lemma 4.4. *Let $\lambda = (\mu, \nu)$ be an (n,k) bipartition. Then*

(1) $$R_\lambda^{(k)} \ne \{0\} \iff (1 \le i < k-1) \quad \mu_i - \mu_{i+1} \in \{0,1\} \text{ and } (1 \le \mathrm{i} < \mathrm{n-k}) \quad \nu_{\mathrm{i+1}} - \nu_{\mathrm{i}} \in \{0,1\}.$$

If these conditions hold then a basis for $R_\lambda^{(k)}$ is

$$\{a_\pi^{(k)} + I^+_{(k,1^{n-k})} \,|\, \mathrm{Des}(\pi) = A_\lambda\}.$$

where

(2) $$A_\lambda := \{1 \le i < n \mid \mu_i = \mu_{i+1} + 1 \text{ or } \nu_{\mathrm{n-i+1}} = \nu_{\mathrm{n-i}} + 1\ \}.$$

THEOREM 4.5. *The S_n-action on $R_\lambda^{(k)}$ is given by*

$$s_j(a_\pi^{(k)}) = \begin{cases} a_{s_j\pi}^{(k)}, & \text{if } |\pi^{-1}(j+1) - \pi^{-1}(j)| > 1; \\ a_\pi^{(k)}, & \text{if } \pi^{-1}(j+1) = \pi^{-1}(j) + 1; \\ -a_\pi^{(k)} - \sum_{\sigma \in A_j(\pi)} a_\sigma^{(k)}, & \text{if } \pi^{-1}(j+1) = \pi^{-1}(j) - 1. \end{cases}$$

Here $s_j = (j, j+1)$ $(1 \le j < n)$ are the Coxeter generators of S_n, $\{a_\pi^{(k)} + I^+_{(k,1^{n-k})} \mid \pi \in S_\lambda\}$ is the descent basis of $R_\lambda^{(k)}$, and for $\pi \in S_\lambda$ with $\pi^{-1}(j+1) = \pi^{-1}(j) - 1$ we define

$$\begin{array}{rcl} t & := & \pi^{-1}(j+1), \\ m_1 & := & \max\{i \in \mathrm{Des}(\pi) \cup \{0\} \mid i \le t-1\}, \\ m_2 & := & \min\{i \in \mathrm{Des}(\pi) \cup \{n\} \mid i \ge t+1\} \end{array}$$

(so that $\pi(t) = j+1$, $\pi(t+1) = j$, and $\{m_1+1, \ldots, m_2\}$ is the maximal interval containing t and $t+1$ on which $s_j\pi$ is increasing); and let $A_j(\pi)$ be the set of all $\sigma \in S_n$ satisfying

1. $(i \le m_1 \text{ or } i \ge m_2 + 1) \Longrightarrow \sigma(i) = \pi(i)$;
2. *the sequences $(\sigma(m_1+1), \ldots, \sigma(t))$ and $(\sigma(t+1), \ldots, \sigma(m_2))$ are increasing;*
3. $\sigma \notin \{\pi, s_j\pi\}$ *(i.e., $\{\sigma(t), \sigma(t+1)\} \ne \{j, j+1\}$).*

EXAMPLE 4.6. *Let $\pi = 2416573 \in S_7$ and $j = 5$. Then:*

$$j = 5,\ j+1 = 6;\ t = 4,\ t+1 = 5;$$

$$\mathrm{Des}(\pi) = \{2,4,6\};\ m_1 = 2,\ m_2 = 6;\ s_j\pi = 24\underline{1567}3;$$

$$A_j(\pi) = \{24\underline{1756}3,\ 24\underline{5617}3,\ 24\underline{5716}3,\ 24\underline{6715}3\}.$$

Note that $|A_j(\pi)| = \binom{m_2 - m_1}{t - m_1} - 2 = \binom{4}{2} - 2 = 4.$

COROLLARY 4.7. *The S_n representation on $R_\lambda^{(k)}$ is independent of k.*

THEOREM 4.8. *Let $\lambda = (\mu, \nu)$ be an (n,k) bipartition. $R_\lambda^{(k)}$ is isomorphic as an S_n-module to the corresponding Solomon descent representation determined by the descent class $\{\pi \in S_n \mid \mathrm{Des}(\pi) = A_\lambda\}$, defined in Lemma 4.4 above.*

Proof. By Theorem 4.5 together with Lemma 4.4, for every Coxeter generator s_i, the representation matrices of s_i on $R_\lambda^{(k)}$ and on $R_\lambda^{(n)}$ with respect to the corresponding k-th and n-th descent monomials respectively are identical. By [**1**, Theorem 4.1], the multiplicity of the irreducible S_n-representation corresponding to μ in $R_\lambda^{(n)}$ is $m_{S,\mu} := \#\{T \in SYT(\mu) \mid \mathrm{Des}(T) = A_\lambda\}$, the number of standard Young tableaux of shape μ and descent set A_λ. Theorem 4.1 completes the proof. □

Let $R_{t_1,t_2}^{(k)}$ be the (t_1, t_2)-th homogeneous component of $\Delta^*_{(k,1^{n-k})}$.

COROLLARY 4.9. *For every $0 \le t_1$, $0 \le t_2$ and $0 \le k \le n$ the (t_1,t_2)-th homogeneous component of $\Delta^*_{(k,1^{n-k})}$ is decomposed into a direct sum of Solomon descent representations as follows:*

$$R_{t_1,t_2}^{(k)} \cong \bigoplus_S R_\lambda^{(k)},$$

where the sum is over all (n,k) bipartitions $\lambda = (\mu, \nu)$ with $\mu_{i+1} - \mu_i \in \{0,1\}$ $(\forall i)$, $\nu_{i+1} - \nu_i \in \{0,1\}$ $(\forall i)$ and

$$\sum_{\mu_i > \mu_{i+1} \text{ and } i<k} i = t_1 \qquad \sum_{\nu_i < \nu_{i+1} \text{ and } i \ge k} (n-i) = t_2.$$

4.2. Decomposition into Irreducibles. A classical theorem of Lusztig and Stanley gives the multiplicity of the irreducibles in the homogeneous component of the coinvariant algebra of type A. For a standard Young tableau T define

$$\mathrm{maj}(T) := \sum_{i\in \mathrm{Des}(T)} i$$

where $\mathrm{Des}(T)$ is the descent of T, defined in previous Subsection.

THEOREM 4.10. [**33**, Prop. 4.11] *The multiplicity of the irreducible S_n-representation S^λ in the k-th homogeneous component of the coinvariant algebra of type A is*

$$\#\{T \in SYT(\lambda) \,|\, \mathrm{maj}(T) = k\},$$

where $SYT(\lambda)$ is the set of all standard Young tableaux of shape λ.

In 1994, Stembridge [**35**] gave an explict combinatorial interpretation of the (q,t)-Kostka polynomials for hook shape. Stembridge's result implies the following extension of Lusztig-Stanley theorem.

For a standard Young tableau T define

$$\mathrm{maj}_{i,j}(T) := \sum_{\substack{r\in \mathrm{Des}(T)\\ i\le r<j}} r$$

and

$$\mathrm{comaj}_{i,j}(T) := \sum_{\substack{r\in \mathrm{Des}(T)\\ i\le r<j}} (n-r).$$

THEOREM 4.11. *The multiplicity of the irreducible S_n-representation S^λ in the (h,h') level of $\Delta_{(k,1^{n-k})}$ (bi-graded by total degrees in the x-s and y-s) is*

$$\chi_\lambda^{(h,h')} = \#\{T \in SYT(\lambda) \,|\, \mathrm{maj}_{1,k}(T) = h,\, \mathrm{comaj}_{k,n}(T) = h'\},$$

where $SYT(\lambda)$ is the set of all standard Young tableaux of shape λ.

Stembridge's proof of Theorem 4.11 is rather complicated. Haglund [**19**] gave another proof of Theorem 4.11 that uses his conjectured combinatorial definition of $\tilde{H}_\mu(\overline{x};q,t)$. Haglund's conjecture has recently been proved by Haglund, Haiman and Loehr [**20, 21**]. We give two proofs to this decomposition rule.

First Proof of Theorem 4.11. Combine Theorems 4.1 and 4.8 with Corollary 4.9. □

A second proof of Theorem 4.11 is given in [**4**]. This proof is more straightforward and "combinatorial". It uses the mechanism of [**21**] but does not rely on Haglund's combinatorial interpretation of $\tilde{H}_{(1^k,n-k)}(\overline{x};q,t)$.

5. Sketch of the First Proof of Theorem 3.3

5.1. A k-th Analogue of the Polynomial Ring.

DEFINITION 5.1. *For every $1 \le k \le n$ let $\mathcal{I}_k$ be the ideal in $\mathbf{Q}[x_1,\ldots,x_n,y_1,\ldots,y_n]$ generated by*

(i) *the monomials $x_{i_1}\cdots x_{i_k}$ $(1 \le i_1 < \cdots < i_k \le n)$,*
(ii) *the monomials $y_{i_1}\cdots y_{i_{n-k+1}}$ $(1 \le i_1 < \cdots < i_{n-k+1} \le n)$, and*
(iii) *the monomials $x_i y_i$ $(1 \le i \le n)$.*

Denote

$$\mathcal{P}_n^{(k)} := \mathbf{Q}[x_1,\ldots,x_n,y_1,\ldots,y_n]/\mathcal{I}_k.$$

*For a monomial $m = \prod_{i=1}^n x_i^{e_i} \prod_{j=1}^n y_j^{f_j} \in \mathbf{Q}[x_1,\ldots,x_n,y_1,\ldots,y_n]$ define the x-*support *and the y-*support

$$\mathrm{Supp}_x(m) := \{i \,|\, e_i > 0\}, \qquad \mathrm{Supp}_y(m) := \{j \,|\, f_j > 0\}.$$

Let $M_n^{(k)}$ be the set of all monomials in $\mathbf{Q}[x_1,\ldots,x_n,y_1,\ldots,y_n]$ with

(i) $|\mathrm{Supp}_x(m)| \le k-1$ *(ii)* $|\mathrm{Supp}_y(m)| \le n-k$ *(iii)* $\mathrm{Supp}_x(m) \cap \mathrm{Supp}_y(m) = \emptyset$.

OBSERVATION 5.2. $\{p + \mathcal{I}_k \,|\, p \in M_n^{(k)}\}$ *is a basis for $\mathcal{P}_n^{(k)}$.*

Every monomial $m \in M_n^{(k)}$ has the form $m = x_{i_1}^{e_{i_1}} \cdots x_{i_{k-1}}^{e_{i_{k-1}}} \cdot y_{j_1}^{f_{j_1}} \cdots y_{j_{n-k}}^{f_{j_{n-k}}}$ (with disjoint supports of x-s and y-s). Let $u := \max_j f_j$ and define

$$\psi^{(k)}(m) := x_{i_1}^{e_{i_1}} \cdots x_{i_{k-1}}^{e_{i_{k-1}}} \cdot x_{j_1}^{-f_{j_1}} \cdots x_{j_{n-k}}^{-f_{j_{n-k}}} \cdot (x_1 \cdots x_n)^u.$$

PROPOSITION 5.3. *The map* $\psi^{(k)} : M_n^{(k)} \to M_n^{(n)}$ *is a bijection.*

DEFINITION 5.4. *For* $1 \le m \le n-1$ *let*

$$e_m^{(k)} := \begin{cases} e_m(\bar{x}) = e_m(x_1, \ldots, x_n), & \text{if } 1 \le m \le k-1; \\ e_{n-m}(\bar{y}) = e_{n-m}(y_1, \ldots, y_n), & \text{if } k \le m \le n-1. \end{cases}$$

For a partition $\mu = (\mu_1, \ldots, \mu_\ell)$ *with* $\mu_1 < n$ *let* $e_\mu^{(k)} := \prod_{i=1}^{\ell} e_{\mu_i}^{(k)}$.

Consider the natural S_n-action on $\mathcal{P}_n^{(k)}$. Let ${\mathcal{P}_n^{(k)}}^{S_n}$ be the algebra of S_n-invariants in $\mathcal{P}_n^{(k)}$. Then the set $\{e_\mu^{(k)} \,|\, \mu_1 < n\}$ forms a (vector space) basis for ${\mathcal{P}_n^{(k)}}^{S_n}$. It is easy to see that $\psi^{(k)} : \mathcal{P}_n^{(k)} \mapsto \mathcal{P}_n^{(n)}$ is an isomorphism, which sends invariants to invariants. Unfortunately, $\psi^{(k)}$ is not multiplicative and does not send the ideal generated by invariants (with no constant term) to its analogue; thus does not send a basis of the coinvariants to its analogue. However, the map $\psi^{(k)}$ may be used in finding a basis for $\Delta^*_{(k,1^{n-k})}$.

5.2. Straightening. Each monomial $m \in M_n^{(k)}$ can be written in the form

$$m = \prod_{i=1}^{k-1} x_{\pi(i)}^{p_i} \cdot \prod_{i=k+1}^{n} y_{\pi(i)}^{p_i},$$

where $p_1 \ge \ldots \ge p_{k-1} \ge 0$ and $0 \le p_{k+1} \le \ldots \le p_n$. Here $\pi = \pi(m)$, the *index permutation* of m, is the unique permutation that orders first the indices $i \in \mathrm{Supp}_x(m)$, then the indices $i \notin \mathrm{Supp}_x(m) \cup \mathrm{Supp}_y(m)$, and then the indices $i \in \mathrm{Supp}_y(m)$. The x-indices are ordered by weakly decreasing exponents, the y-indices are ordered by weakly increasing exponents, and indices with equal exponents are ordered in increasing (index) order.

For a monomial $m \in M_n^{(k)}$ with index permutation $\pi \in S_n$, $m = \prod_{i=1}^{k-1} x_{\pi(i)}^{p_i} \cdot \prod_{i=k+1}^{n} y_{\pi(i)}^{p_i}$, let the associated pair of exponent partitions

$$\lambda(m) = (\lambda_x(m), \lambda_y(m)) := ((p_1, p_2, \ldots, p_{k-1}), (p_n, p_{n-1}, \ldots, p_{k+1}))$$

be its *exponent bipartition*. Note that $\lambda(m)$ is a bipartition of the total bi-degree of m.

Define the *complementary bipartition* $\mu(m) = (\mu_x(m), \mu_y(m))$ of m to be the pair of partitions conjugate to the partitions $(p_i - d_i(\pi))_{i=1}^{k-1}$ and $(p_i - d_i(\pi))_{i=n}^{k+1}$ respectively; namely,

$$(\mu_x)_j := |\{1 \le i \le k-1 \,|\, p_i - d_i(\pi) \ge j\}| \qquad (\forall j)$$

and

$$(\mu_y)_j := |\{k+1 \le i \le n \,|\, p_i - d_i(\pi) \ge j\}| \qquad (\forall j).$$

If $k = n$ then, for every monomial $m \in M_n^{(n)}$, $\mu_y(m)$ is the empty partition. In this case we denote

$$\mu(m) := \mu_x(m).$$

With each $m \in M_n^{(k)}$ we associate the *canonical complementary partition*

$$\nu(m) := \mu_x(\psi^{(k)}(m)).$$

EXAMPLE 5.5. *Let* $m = x_1^2 y_2^4 x_3^2 y_5 x_6^3$ *with* $n = 7$ *and* $k = 5$. *Then*

$$m = x_6^3 x_1^2 x_3^2 y_5^1 y_2^4, \quad \lambda(m) = ((3,2,2,0),(4,1)), \quad \pi = 6134752 \in S_7,$$

$$\lambda(a_\pi^{(5)}) = ((1,0,0,0),(2,1)), \quad \mu(m) = ((2,2,2,0)',(2,0)') = ((3,3),(1,1)),$$

$$\psi^{(5)}(m) = x_6^7 x_1^6 x_3^6 x_4^4 x_7^4 x_5^3, \quad a_\pi = x_6^3 x_1^2 x_3^2 x_4^2 x_7^2 x_5^1, \quad \nu(m) = \mu(\psi^{(5)}(m)) = (4,4,4,2,2,2)' = (6,6,3,3).$$

DEFINITION 5.6.
1. *For two partitions* λ *and* μ, *denote* $\lambda \trianglelefteq \mu$ *if* λ *is weakly smaller than* μ *in dominance order. For two bipartitions* $\lambda^1 = (\mu^1, \nu^1)$ *and* $\lambda^2 = (\mu^2, \nu^2)$, *denote* $\lambda^1 \trianglelefteq \lambda^2$ *if* $\mu^1 \trianglelefteq \mu^2$ *and* $\nu^1 \trianglelefteq \nu^2$.

2. *For two monomials* $m_1, m_2 \in M_n^{(k)}$ *of the same total bi-degree* (p,q)*, write* $m_1 \preceq_k m_2$ *if:*
   (1) $\lambda(m_1) \trianglelefteq \lambda(m_2)$*; and*
   (2) *if* $\lambda(m_1) = \lambda(m_2)$ *then* $\mathrm{inv}(\pi(m_1)) > \mathrm{inv}(\pi(m_2))$.

A Straightening Algorithm:
For a monomial $m \in \mathcal{P}_n^{(k)}$*, let* $\pi = \pi(m)$ *be its index permutation,* $a_\pi^{(k)}$ *the corresponding descent basis element, and* $\nu = \mu(\psi^{(k)}(m))$ *the corresponding canonical complementary partition. Write*

$$m = a_\pi^{(k)} \cdot e_\nu^{(k)} - \Sigma,$$

where Σ *is a sum of monomials* $m' \prec_k m$*. Repeat the process for each* m'.

It is proved in [**4**] that this algorithm gives a basis. In particular,

Lemma 5.7. **(Straightening Lemma)** *Each monomial* $m \in \mathcal{P}_n^{(k)}$ *has an expression*

$$m = a^{(k)}_{\pi(m)} e^{(k)}_{\nu(m)} + \sum_{m' \prec_k m} n_{m',m} a^{(k)}_{\pi(m')} e^{(k)}_{\nu(m')},$$

where $n_{m',m}$ *are integers.*

Theorem 3.3 follows. □

6. Sketch of the Proof of Theorem 3.5

In this section we give a brief sketch of the proof of Theorem 3.5, which implies Theorem 3.3 as a special case. The idea is to generalize the kicking process for obtaining a basis. The kicking process was used in an early paper of Grasia and Haiman [**14**] to prove the $n!$-conjecture for hooks. We combine this process with a filtration.

6.1. Generalized Kicking-Filtration Process. For every triple $(A, c, \bar{A})$, where $[n] = A \cup \{c\} \cup \bar{A}$ and $|A| = k, |\bar{A}| = n-k$, define an $(A, c, \bar{A})$-permutation $\pi_{(A,c,\bar{A})} \in S_n$, in which the letters of A appear in decreasing order, then c, and then the remaining letters in increasing order. For example, let $n = 9, k = 4, c = 5, A = \{1, 6, 7\}$ then $\pi_{(\{1,6,7\},5)} = 7, 6, 1, 5, 2, 3, 4, 8, 9$.

Let $\leq_L$ be the reverse lexicographic order on the permutations in S_n (as words). For a given n and k, denote by π_t the t-th $(A, c, \bar{A})$-permutation in this order and $m_t := m_{\pi_t}$. Let $N := n\binom{n-1}{k-1}$ be the number of $(A, c, \bar{A})$-permutations.

For example, for $n = 4$ and $k = 3$, the complete list of permutations is

$$\pi_{(\{34\},2,\{1\})} = 4321,\ \pi_{(\{34\},1,\{2\})} = 4312,\ \pi_{(\{24\},3,\{1\})} = 4231,$$

$$\pi_{(\{24\},1,\{3\})} = 4213,\ \ldots,\ \pi_{(\{12\},4,\{3\})} = 2143,\ \pi_{(\{12\},3,\{4\})} = 2134.$$

and the order is

$$4321 <_L 4312 <_L 4231 <_L 4213 <_L 4132 <_L 4123 <_L 3241 <_L 3214 <_L 3142 <_L 3124 <_L 2143 <_L 2134.$$

Thus the permutations are indexed by $\pi_1 = 4321,\ \pi_2 = 4312,\ \pi_3 = 4231,\ \ldots, \pi_{11} = 2143,\ \pi_N = \pi_{12} = 2134$ and the corresponding monomials are $m_1 = x_4x_3y_1,\ m_2 = x_4x_3,\ m_3 = x_4y_1,\ m_{11} = y_3,\ m_N = m_{12} = 1$.

Let

$$I_0 := I^+_{(k,1^{n-k})}$$

and define

$$I_t := I_0 + \sum_{i=1}^{t} m_i \mathbf{Q}[\bar{x}, \bar{y}] \qquad (1 \leq i \leq N).$$

Clearly, $I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_N = \mathbf{Q}[\bar{x}, \bar{y}]$ and $\Delta^*_{(k,1^{n-k})} = \mathbf{Q}[\bar{x}, \bar{y}]/I_0 \cong \bigoplus_{t=1}^N (I_t/I_{t-1})$ as vector spaces. In particular, a sequence of bases for the quotients I_t/I_{t-1}, $1 \leq t \leq N$, will give a basis for $\Delta^*_{(k,1^{n-k})}$. It remains to prove that $m_t\, B_A\, C_{\bar{A}}$, where $B_A,\ C_{\bar{A}}$ are bases of coinvariant algebras in $\bar{x}_A$ and $\bar{y}_{\bar{A}}$ respectively, is a basis for I_t/I_{t-1}. This is an immediate consequence of the following lemma.

LEMMA 6.1. (1) *For each $1 \le t \le N$ there exists an explicit linear map*

$$f_t : m_t\mathbf{Q}[\bar{x}_A]/\langle\Lambda[\bar{x}_A]^+\rangle \ \cdot \ \mathbf{Q}[\bar{y}_{\bar{A}}]/\langle\Lambda[\bar{y}_{\bar{A}}]^+\rangle \longrightarrow I_t/I_{t-1},$$

defined by

$$f_t(m_t \cdot p \cdot q) = m_t \cdot p \cdot q \qquad (\forall p \in \mathbf{Q}[\bar{x}_A]/\langle\Lambda[\bar{x}_A]^+\rangle,\ q \in \mathbf{Q}[\bar{y}_{\bar{A}}]/\langle\Lambda[\bar{y}_{\bar{A}}]^+\rangle).$$

(2) *f_t is onto.*

Proof of Lemma 6.1. We shall start by defining a natural projection

$$\tilde{f}_t : m_t\mathbf{Q}[\bar{x}, \bar{y}] \longrightarrow I_t/I_{t-1}.$$

Clearly, $\tilde{f}_t$ is a surjective map (since, by definition, $m_t\mathbf{Q}[\bar{x}, \bar{y}] = I_t$). We claim that

$$m_t \cdot (\sum_{i \notin A}\langle x_i\rangle + \sum_{j \notin \bar{A}}\langle y_j\rangle + \langle\Lambda[\bar{x}]^+\rangle + \langle\Lambda[\bar{y}]^+\rangle) \ \subseteq \ I_{t-1} = \ker\,(\tilde{f}_t),$$

so that $\tilde{f}_t$ is well defined on the quotient

$$m_t\mathbf{Q}[\bar{x}, \bar{y}]/m_t(\sum_{i \notin A}\langle x_i\rangle + \sum_{j \notin \bar{A}}\langle y_j\rangle + \langle\Lambda[\bar{x}]^+\rangle + \langle\Lambda[\bar{y}]^+\rangle) \cong$$

$$m_t \cdot \mathbf{Q}[\bar{x}_A]/(\Lambda[\bar{x}_A]^+\mathbf{Q}[\bar{x}_A]) \cdot \mathbf{Q}[\bar{y}_{\bar{A}}]/(\Lambda[\bar{y}_{\bar{A}}]^+\mathbf{Q}[\bar{y}_{\bar{A}}])$$

and is exactly f_t of the lemma.

To prove this claim, first, let $i \notin A$. It is shown in [**4**] that $m_t x_i \in I_{t-1}$; thus $m_t x_i\mathbf{Q}[\bar{x}, \bar{y}] \subseteq I_{t-1}$. This is done by a combinatorial analysis of four complementary cases. Similarly, by considering four analogous cases, one can show that if $j \notin \bar{A}$ then $m_t y_j \in I_{t-1}$.

In order to prove Theorem 3.5, it remains to show that f_t is one-to-one. Indeed, for every $1 \le t \le N$

$$\dim\,(I_/I_{t-1}) \le \dim\, m_t\mathbf{Q}[\bar{x}_A]/\langle\Lambda[\bar{x}_A]^+\rangle\mathbf{Q}[\bar{y}_{\bar{A}}]/\langle\Lambda[\bar{y}_{\bar{A}}]^+\rangle \le$$

$$\dim\, \mathbf{Q}[\bar{x}_A]/\langle\Lambda[\bar{x}_A]^+\rangle\mathbf{Q}[\bar{y}_{\bar{A}}]/\langle\Lambda[\bar{y}_{\bar{A}}]^+\rangle = (k-1)! \cdot (n-k)!$$

If there exists $1 \le t \le N$, such that f_t is not one-to-one then there exists t for which a sharp inequality holds. Then

$$\dim\,\Delta^*_{(k,1^{n-k})} = \dim\,\mathbf{Q}[\bar{x}, \bar{y}]/I_0 = \dim\,\bigoplus_{t=1}^{N}(I_t/I_{t-1}) < N \cdot (k-1)! \cdot (n-k)! = n\binom{n-1}{k-1}(k-1)!(n-k)! = n!.$$

Contradicting the $n!$ theorem. This completes the proof of Theorem 3.5. □

7. Final Remarks

7.1. Haglund Statistics. Let ξ be a filling of the Ferrers diagram of a partition μ with the numbers $1, \ldots, n$. For any cell $u = (i, j) \in F_\mu$, let $\xi(u)$ be the entry in cell u. We say that $u = (i, j) \in F_\mu$ is a *descent* of ξ, written $u \in Des(\xi)$, if $i > 1$ and $\xi((i, j)) \ge \xi((i-1, j))$. Then $maj(\xi) = \sum_{u \in Des(\xi)}(leg(u) + 1)$. Two cells $u, v \in F_\mu$ *attack* each other if either

(a) they are in the same row: $u = (i, j)$ and $v = (i, k)$), or
(b) they are in consecutive rows, with the cell in the upper row strictly to the right of the one in the lower row: $u = (i+1, k)$ and $v = (i, j)$, where $j < k$).

The *reading order* is the total ordering on the cells of F_μ given by reading the cells row by row from top to bottom, and left to right within each row. For example, the reading order of $(4, 3, 2)$ is depicted on the left in Figure 2. An *inversion* of ξ is a pair of entries $\xi(u) > \xi(v)$ where u and v attack each other and u precedes v in the reading order. We then define $Inv(\xi) = \{\{u, v\} : \xi(u) > \xi(v) \text{ is an inversion}\}$ and $inv(\xi) = |Inv(\xi)| - \sum_{u \in Des(\xi)} arm(u)$.

For example, if ξ is the filling of shape $(4, 3, 2)$ depicted in Figure 2, then $Des(\xi) = \{(2, 1), (2, 2), (3, 2)\}$. There are four inversion pairs of type (a), namely $\{(2, 1), (2, 2)\}$, $\{(2, 1), (2, 3)\}$, $\{(2, 2), (2, 3)\}$, and $\{(1, 3), (1, 4)\}$, and one inversion pair of type (b), namely $\{(2, 2), (1, 1)\}$. Then one can check that $|Inv(\xi)| = 5$, $maj(\xi) = 5$ and $inv(\xi) = 2$. Finally, we can identify ξ with a permutation by reading the entries in the reading order. In the example of Figure 2, $\xi = 2\ 7\ 9\ 6\ 1\ 3\ 4\ 8\ 5$. Then we let $D(\xi) = Des(\xi^{-1})$. In our example, $\xi^{-1} = 5\ 1\ 6\ 7\ 9\ 4\ 2\ 8\ 3$ so that $D(\xi) = \{1, 5, 6, 8\}$.

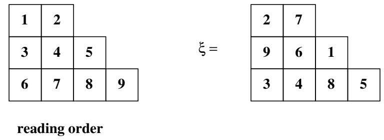

FIGURE 2. The reading order and a filling of $(4, 3, 2)$.

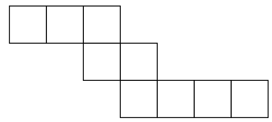

FIGURE 3. The skew shape corresponding to the composition $(3, 2, 4)$.

Recently, Haglund, Haiman and Loehr [**20, 21**] proved Haglund's conjectured combinatorial interpretation [**19**] of $\tilde{H}_\mu(\overline{x}; q, t)$ in terms of quasi-symmetric functions. That is, given a non-negative integer n and a subset $D \subseteq \{1, \ldots, n-1\}$, Gessel's quasi-symmetric function of degree n in variables $x_1, x_2, \ldots$ is defined by the formula

$$Q_{n,D}(\overline{x}) := \sum_{\substack{a_1 \le a_2 \le \cdots \le a_n \\ a_i = a_{i+1} \Rightarrow i \notin D}} x_{a_1} x_{a_2} \cdots x_{a_n}. \tag{1}$$

Then Haglund, Haiman and Loehr [**21**] proved

$$\tilde{H}_\mu(\overline{x}; q, t) = \sum_{\xi : \mu \simeq \{1, \ldots, n\}} q^{inv(\xi)} t^{maj(\xi)} Q_{n, D(\xi)}(\overline{x}). \tag{2}$$

Here the sum runs over all fillings ξ of the Ferrers diagram of μ with the numbers $1, \ldots, n$.

7.2. The Hilbert series of Δ_μ is equal to the coefficient of $x_1 x_2 \cdots x_n$ in $\tilde{H}_\mu(\overline{x}; q, t)$. Since the coefficient of $x_1 x_2 \cdots x_n$ in any quasi-symmetric function $Q_{n,D}(\overline{x})$ is 1, it follows that the Hilbert series of Δ_μ is given by

$$\sum_{k,r} \dim \Delta_\mu^{(h,k)} q^h t^k = \tilde{H}_\mu(\overline{x}; q, t)|_{x_1 x_2 \cdots x_n} = \sum_{\xi : \mu \simeq \{1, \ldots, n\}} q^{inv(\xi)} t^{maj(\xi)},$$

where the sum runs over all fillings ξ of the Ferrers diagram of μ with the numbers $1, \ldots, n$. No known basis realizes this remarkable identity for general Δ_μ. The k-th Haglund basis described in Subsection 3.2 above provides such a basis when μ is of hook shape.

Note also that Corollary 4.9 has an interesting interpretation relative to (2), as follows. Given a composition $\alpha = (\alpha_1, \ldots, \alpha_k)$ of n, let $Z_\alpha(\overline{x})$ denote the ribbon Schur function corresponding to α. For example, $Z_{(3,2,4)}(\overline{x})$ is the skew Schur function corresponding to the skew shape depicted in Figure 3. Gessel [**18**] proved that if $P(\overline{x})$ is a symmetric function of degree n then, for any set $D = \{i_1 < i_2 < \cdots < i_k\} \subseteq \{1, \ldots, n-1\}$, $\langle P(\overline{x}), Z_{\alpha(D)}(\overline{x})\rangle$ equals the coefficient of $Q_{n,D}(\overline{x})$ in the quasisymmetric function expansion of $P(\overline{x})$, where $\alpha(D)$ is the composition $(i_1, i_2 - i_1, \ldots, i_k - i_{k-1}, n - i_k)$ of n. This suggests that the coefficient of $Q_{n,D}(\overline{x})$ in the quasisymmetric function expansion of $\tilde{H}_\mu(\overline{x}; q, t)$ should have an algebraic meaning in terms of the Garsia-Haiman module Δ_μ. To be more precise, the set $\{Z_\lambda(\overline{x}) : \lambda \vdash n\}$ is a basis for the space Λ_n of homogeneous symmetric functions of degree n. Thus one could ask whether we can decompose $\Delta_\mu = \bigoplus_{\lambda \vdash n} R_\lambda^{(\mu)}$, where $R_\lambda^{(\mu)}$ is an S_n-module under the diagonal action that affords the representation whose character under the Frobenius map is $Z_\lambda(\overline{x})$. Corollary 4.9 provides such a decomposition in the case where μ is a hook.

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