Dual graded graphs for Kac-Moody algebras

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July, 2007 at FPSAC, Tianjin

Outline

Dual graded graphs: pairs of graphs invented by Fomin which encode insertion (Robinson-Schensted) algorithms. "Weighted" versions of Stanley's differential posets.

Kac-Moody algebras: generalization of Lie algebras depending on a *Cartan matrix* and possessing combinatorial data such as *weights*, *Weyl group*, ...

Our Aim: For each Kac-Moody algebra $\mathfrak{g}$ with Weyl group W we produce dual graded graphs (Γ_s, Γ_w) with:

vertex set: the Weyl group W, and
edges: "weighted" versions of the strong and weak orders of W.

This construction depends on the choice of a dominant integral weight and a positive central element of $\mathfrak{g}$.

Graded graphs

Definition: A weighted directed graph $\Gamma = (V, E)$ is *graded* if there is a *height function* $h : V \to \mathbb{Z}$ so that if $(v, w) \in E$ then $h(w) = h(v) + 1$. Let $m(v, w) \in \mathbb{Z}_{\geq 0}$ denote the weight of the edge (v, w).

If Γ is a graded graph, we define *up and down* linear operators on $\prod_{v \in V} \mathbb{Z}.v$ by

$$U_\Gamma(v) = \sum_{(v,w) \in E} m(v, w)\, w$$

and

$$D_\Gamma(v) = \sum_{(w,v) \in E} m(w, v)\, w.$$

We will always assume Γ is *locally finite* so that these operators make sense when extended by linearity and continuity.

Young's Lattice

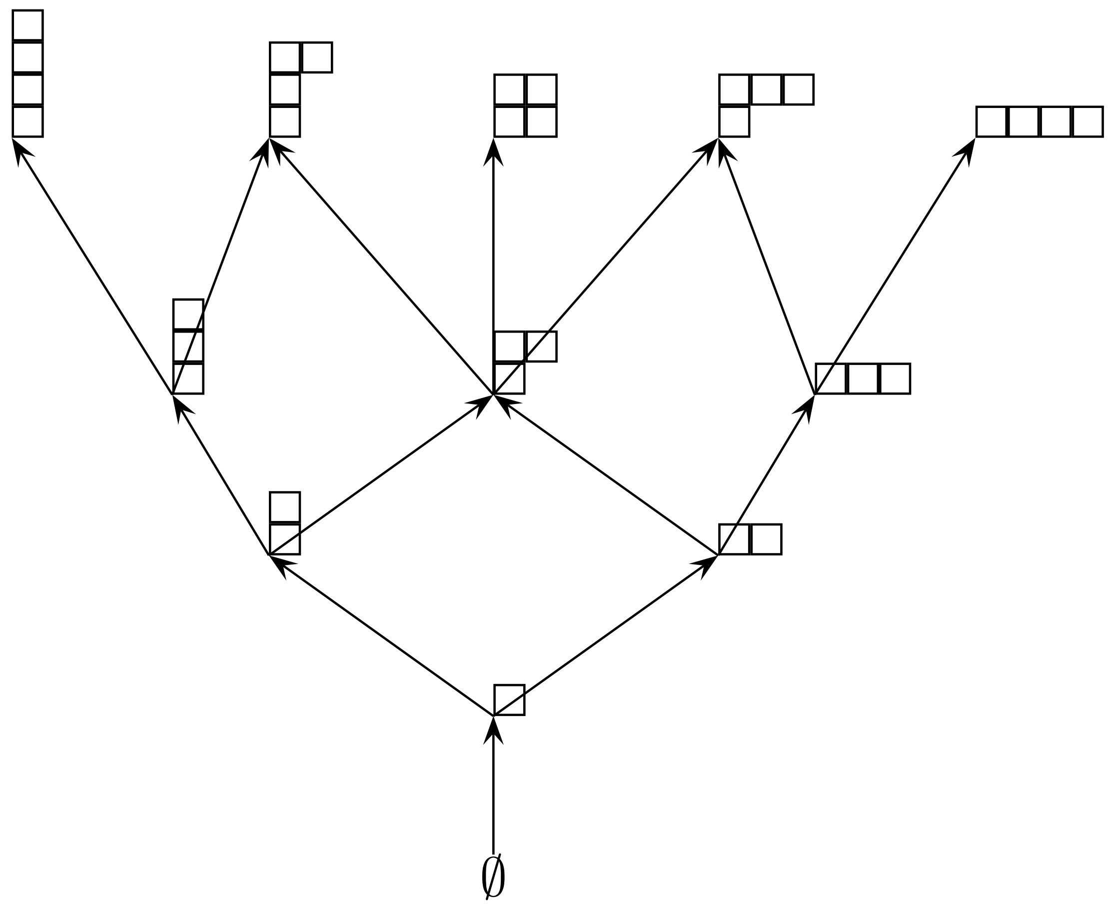


Figure 1: Young's lattice $\mathbb{Y}$ as a graded graph.

Dual graded graphs:

Definition: A pair (Γ, Γ') of graded graphs is *dual* if they have the same vertex set and

$$D_{\Gamma'} U_{\Gamma} - U_{\Gamma} D_{\Gamma'} = r \, \mathrm{Id}$$

for some integer $r \in \mathbb{Z}_{\geq 0}$, called the *differential coefficient*.

Example: Young's Lattice $\mathbb{Y}$. The pair $(\mathbb{Y}, \mathbb{Y})$ is dual with differential coefficient 1. For example,

$$\begin{aligned} DU(\yng(2)) &= D(\yng(3)) + D(\yng(2,1)) = (\yng(2)) + (\yng(2) + \yng(1,1)) \\ &= 2\,\yng(2) + \yng(1,1) \end{aligned}$$

$$UD(\yng(2)) = U(\yng(1)) = \yng(2) + \yng(1,1).$$

Tableaux and paths in dual graded graphs.

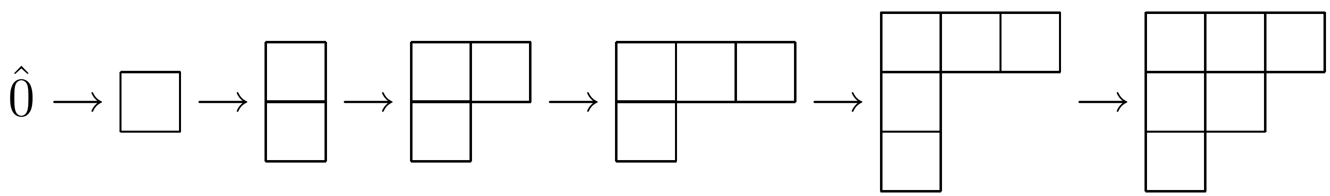
Assumption: Our graded graph Γ has a unique source (or minimum element) $\hat{0}$ with $h(\hat{0}) = n$.

Definition: A *tableau* of shape v is a path

$$\hat{0} = v_0 \to_{m_1} v_1 \to_{m_2} v_2 \to_{m_3} \cdots \to_{m_n} v_n = v$$

in Γ where each edge $v_i \to v_{i+1}$ has been marked with an integer m_{i+1} between 1 and $m(v_i, v_{i+1})$. We may think of there being $m(v, w)$ edges joining v to w, so the marking represents the choice of one such edge.

Example: In Young's lattice $\mathbb{Y}$



corresponds to

1	3	4
2	6	
5		

Robinson-Schensted identity

Let f_Γ^v denote the number of tableau of shape v.

Theorem (Fomin): Suppose (Γ, Γ') is a pair of dual graded graphs with differential coefficient r. Then

$$\sum_{v\colon h(v)=n} f_\Gamma^v \, f_{\Gamma'}^v = r^n n!. \tag{1}$$

Furthermore, a set of local bijections in (Γ, Γ') will give an algorithmic proof of (1).

Example: In Young's lattice $\mathbb{Y}$ we have

$$\sum_{\lambda\colon |\lambda|=n} (f_{\mathbb{Y}}^\lambda)^2 = n!.$$

Some known dual graded graphs and insertions

Γ	Γ'	Insertion
Young's lattice $\mathbb{Y}$	$\mathbb{Y}$	Robinson-Schensted
Fibonacci poset $\mathbb{FY}$	$\mathbb{FY}$	Fibonacci
Shifted Young's lattice $\mathbb{SY}$	Marked $\mathbb{SY}$	Shifted insertion
Marked strong order on cores	Weak order on cores	LLMS insertion

Shifted tableau:

1	2	4	5	8
	3	6	9	
		7	10	

Marked shifted tableau:

1	2^*	4	7^*	8
	3	5^*	9^*	
		6	10	

Marked strong tableau:

1^*	2^*
3^*	4
3^*	
4^*	

Weak tableau:

1	3
2	4
3	
4	

Kac-Moody algebras

A *Kac-Moody algebra* $\mathfrak{g}(A)$ depends on a Cartan matrix

$$A = (a_{ij})_{i \in I}$$

of integers where I is some indexing set.

The *Weyl group* W of $\mathfrak{g}(A)$ is a Coxeter group with generators $\{s_i \mid i \in I\}$ and relations

$$s_i^2 = 1 \quad (s_i s_j)^{m_{ij}} = 1$$

for some $m_{ij} \in \{2, 3, 4, 6, \infty\}$.

Other data:

1. roots $\alpha \in R$
2. simple roots $\{\alpha_i \mid i \in I\}$
3. simple coroots $\{\alpha_i^\vee \mid i \in I\}$
4. weights lattice P
5. fundamental weights $\omega_i \in P$

Strong and weak orders

Strong and weak orders are two partial orders on W.

The *length* $\ell(w)$ of $w \in W$ is the length of shortest expression of w in terms of the s_i.

Left weak order: transitive closure $\prec$ of the relations

$$v \prec s_i v \quad \text{whenever} \quad \ell(s_i v) = \ell(v) + 1$$

A *reflection* $s_\alpha \in W$ is an element conjugate to a generator s_i. They are labeled by *real roots* $\alpha \in R^{\mathrm{re}}$.

Strong (Bruhat) order: transitive closure $<$ of the cover relations

$$v \lessdot w \text{ if } w = v s_\alpha \text{ and } \ell(w) = \ell(v) + 1$$

The strong graph Γ_s

Pick a *dominant integral weight* $\Lambda \in P$.

Vertex set: Elements $w \in W$

Grading: $h = \ell : W \to \mathbb{Z}_{\geq 0}$

Edges: For each cover $v \lessdot w$ in the strong order set

$$m(v, w) = \langle \alpha^\vee, \Lambda \rangle$$

where $w = vs_\alpha$. This number $m(v, w)$ will always be a nonnegative integer.

Tableaux in Γ_s are called *strong tableaux*.

Every coroot $\alpha^\vee$ is a integral linear combination of simple coroots $\{\alpha_i^\vee \mid i \in I\}$. The function $\alpha^\vee \mapsto \langle \alpha^\vee, \Lambda \rangle$ is linear, so is determined by its value on simple coroots.

The weak graph Γ_w

Pick a *positive central element*

$$K = \sum_i a_i \alpha_i^\vee \in Z_+(\mathfrak{g}(A)).$$

Central: $\langle K, \alpha_i \rangle = 0$ for each $i \in I$

Positive: $a_i \geq 0$.

Vertex set: Elements $w \in W$

Grading: $h = \ell : W \to \mathbb{Z}_{\geq 0}$

Edges: Each cover $v \prec w = s_i v$ in the left weak order is weighted by

$$n(v, w) = \langle K, \omega_i \rangle = a_i.$$

Tableaux in Γ_w are called *weak tableaux*.

Main Theorem

The strong and weak graphs (Γ_s, Γ_w) form a pair of dual graded graphs with differential coefficient $r = \langle K, \Lambda \rangle$.

Corollary: Strong and weak tableaux satisfy

$$\sum_{\ell(w)=n} f^w_{\text{strong}} \, f^w_{\text{weak}} = n!.$$

The minimum element of (Γ_s, Γ_w) is the identity id.

Basic Properties

1. If $\mathfrak{g}$ is a finite dimensional simple Lie algebra then the construction produces nothing (since the center $Z(\mathfrak{g}) = 0$).
2. The richest example seems to be the case of the affine Lie algebras in which case there is a *canonical central element* K_{can}.
3. The construction is compatible with restriction to parabolics quotients W/W_J for $J \subset I$.
4. The construction is compatible with *folding* of Kac-Moody algebras: when $\mathfrak{g}(A)$ can be embedded into $\mathfrak{g}(B)$ as the fixed points of an automorphism.

Affine Schubert Calculus

Let K be a simple and simply-connected compact group and ΩK denote the based-loops into K.

The construction was inspired by the study of the dual Hopf algebras $H_*(\Omega K)$ and $H^*(\Omega K)$, together with their Schubert bases $\{\xi_w\}$ and $\{\xi^w\}$.

Roughly speaking, the up and down operators correspond to the *affine Chevalley rules* in homology and cohomology. These are combinatorial rules for multiplication by the unique Schubert class ξ_{s_0} (or ξ^{s_0}) in degree 2, written in the Schubert basis.

Thus "duality" of graded graphs corresponds to the pairing

$$H_*(\Omega K) \otimes H^*(\Omega K) \to \mathbb{Z}.$$

Hope: Our dual graded graphs can be related to Kac-Moody Schubert calculus. More precisely, *semistandard* generalizations of our tableaux should represent Schubert classes.

Remark: there is a general way to obtain dual graded graphs from dual Hopf algebras (independently discovered by Hivert-Nzeutchap and L.-Shimozono.)

LLMS insertion

LLMS = Lam-Lapointe-Morse-Shimozono

1. In type $\mathfrak{g} = \tilde{A}^{(1)}_{n-1}$, picking a local bijection one recovers the standard case of LLMS insertion.

2. Taking $n \to \infty$ we obtain the usual Robinson-Schensted insertion. (Alternatively take $\mathfrak{g} = A_\infty$.)

3. If we fold $\mathfrak{g}(A) = \tilde{C}^{(1)}_n$ into $\mathfrak{g}(B) = \tilde{A}^{(1)}_{2n-1}$ we obtain from LLMS insertion an explicit insertion algorithm for $\tilde{C}^{(1)}_n$.

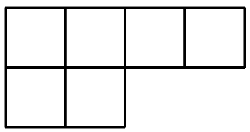
4. Taking $n \to \infty$ we obtain shifted insertion. (Alternatively take $\mathfrak{g} = C_\infty$.)

Open Problem: construct explicit insertion algorithms for all Kac-Moody dual graded graphs.

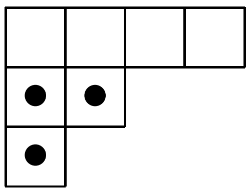
Cores

Definition: A *n-core* is a partition from which a n-ribbon cannot be removed.

Example: A 3-core:



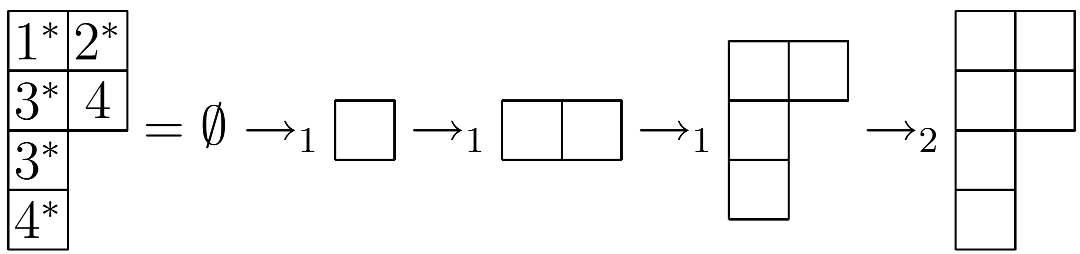
Not a 3-core:



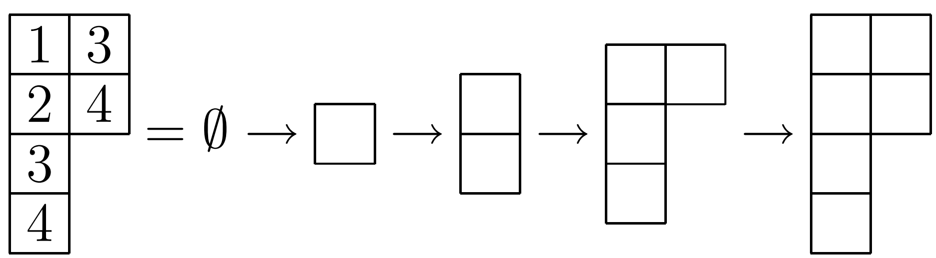
Weak and strong tableaux in the case $\mathfrak{g} = \tilde{A}^{(1)}_{n-1}$ (LLMS)

In the case $\mathfrak{g} = \tilde{A}^{(1)}_{n-1}$, weak and strong tableaux can be identified with nested sequences of n-cores. Weak tableaux have no markings (since $K_{\text{can}} = \sum_i \alpha_i^\vee$) but strong tableaux are marked.

A marked strong tableau:



A weak tableau:



w	$n=3$	$n=\infty$
1234	3 4 1* 2* 3* 4* 3 4 1 2 3 4	1 2 3 4 1 2 3 4
1243	3 4* 1* 2* 3* 4 3 4 1 2 3 4	4 1 2 3 4 1 2 3
1324	3* 4 1* 2* 3 4* 3 4 1 2 3 4	3 1 2 4 3 1 2 4
1342	3* 4* 1* 2* 3 4 3 4 1 2 3 4	3 1 2 4 4 1 2 3
1423	4* 3 1* 2* 3* 4 3 1 2 3	4 1 2 3 3 1 2 4
1432	4* 3* 1* 2* 3 4 3 1 2 3	4 3 1 2 4 3 1 2

2134	3 2* 1* 3* 4*	3 2 1 3 4	2 1 3 4	2 1 3 4
2143	4 3 2* 4* 1* 3*	4 3 2 4 1 3	2 4 1 3	2 4 1 3
2314	2* 4 1* 3 3* 4*	3 4 1 2 3 4	2 1 3 4	3 1 2 4
2341	2* 4* 1* 3 3* 4	3 4 1 2 3 4	2 1 3 4	4 1 2 3
2413	3 2* 1* 3* 4*	4 3 1 2 3	2 4 1 3	3 4 1 2
2431	4* 2* 1* 3 3*	4 3 1 2 3	4 2 1 3	4 3 1 2