06/2006

From Alternating Sign Matrices

To Orbital Varieties

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Plan of the talk

◇ Definition of the Temperley–Lieb model of loops

◇ Relation to Alternating Sign Matrices

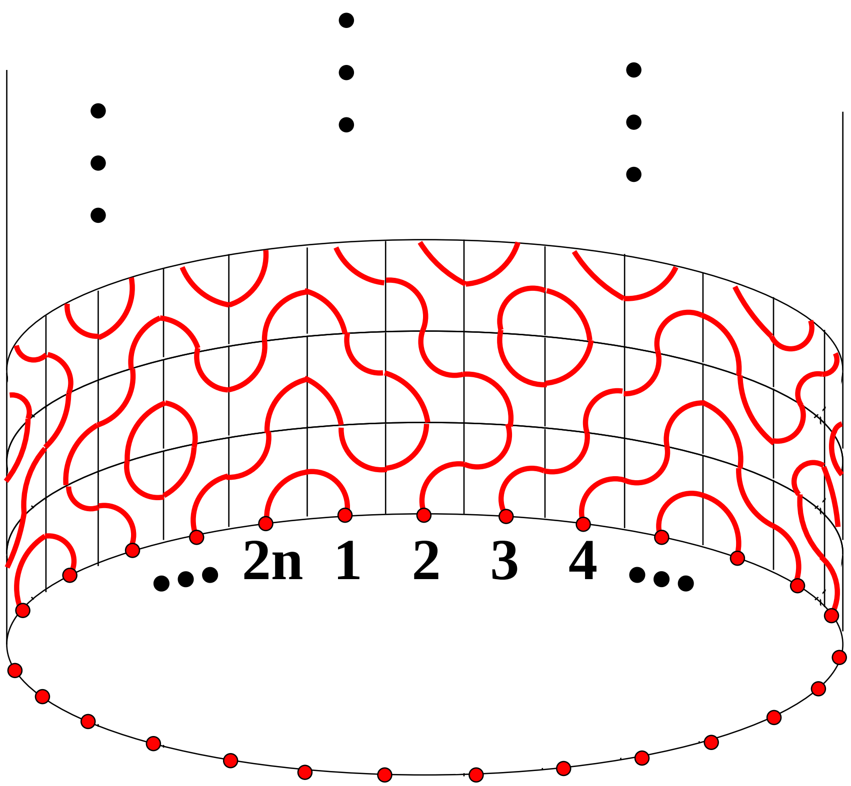
◇ Quantum Knizhnik–Zamolodchikov Equation

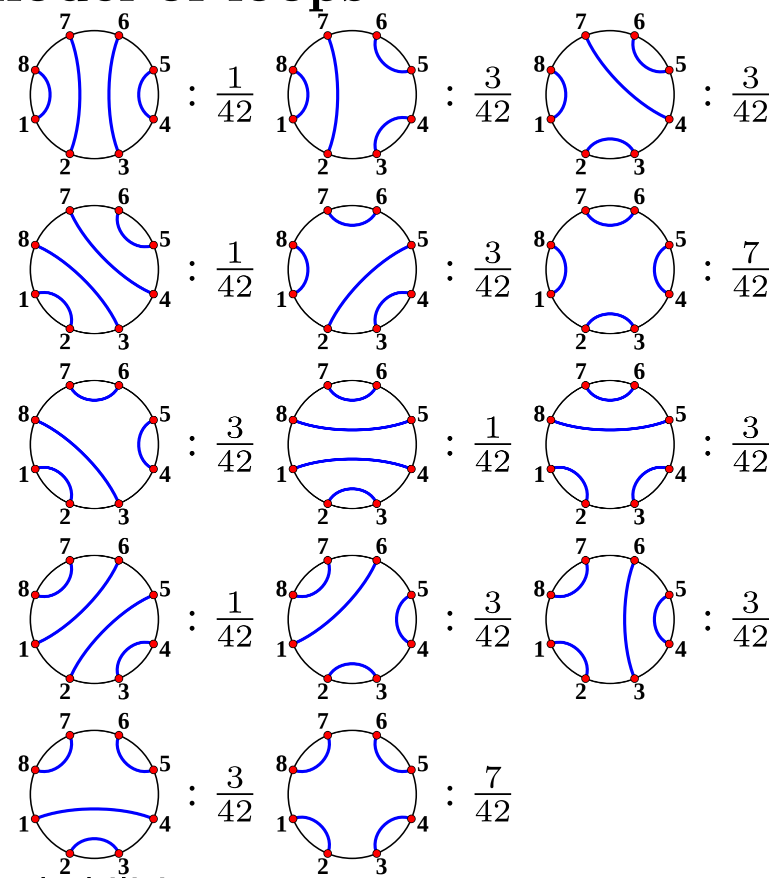
◇ Relation to $sl(N)$ Orbital Varieties

◇ Generalization to other orbital varieties / other boundary conditions

(see also: DF+ZJ math-ph/0410061, math-ph/0508059)

The Temperley–Lieb model of loops





The two types of plaquettes are chosen randomly with probabilities p, $1-p$.

Question: how do the external vertices connect to each other?

Temperley–Lieb model of loops cont'd

It is convenient to encode the probabilities as a vector Ψ indexed by **link patterns**, and to normalize it so that the smallest entry is 1.

Conjectures [de Gier, Nienhuis '01]

(1) The components can be chosen to be integers, the smallest being 1.

(2) The sum of components is the number of alternating sign matrices of size n:

$$A_n = \frac{1!4!7!\cdots(3n-2)!}{n!(n+1)!(n+2)!\cdots(2n-1)!}$$

$$\begin{pmatrix} 0 & + & 0 \\ + & - & + \\ 0 & + & 0 \end{pmatrix}$$

now a Theorem [PDF, PZJ oct '04]

(3) The largest component is A_{n-1}.

[Razumov, Stroganov '01] formulated a much more general conjecture that interprets combinatorially *each individual component*. [still unproven]

ASM enumeration: Izergin's determinant formula

Associate to each horizontal line of the grid a parameter x_i and to each vertical line a parameter y_i.

The weight $w(x, y)$ at a vertex depends on the parameters x, y of the lines and is equal to:

$$\begin{array}{ccc} + & \cdots & \mathbf{0} \\ & & \vdots \\ & & + \end{array} \quad or \quad \begin{array}{ccc} + & & \\ \vdots & & \\ \mathbf{0} & \cdots & + \end{array} \qquad a(x,y) = q^{1/2}x - q^{-1/2}y$$

$$\begin{array}{ccc} \mathbf{0} & \cdots & + \\ \vdots & & \\ + & & \end{array} \quad or \quad \begin{array}{ccc} & & + \\ & & \vdots \\ + & \cdots & \mathbf{0} \end{array} \qquad b(x,y) = q^{1/2}y - q^{-1/2}x$$

$$+ \ or \ - \qquad c(x,y) = (q - q^{-1})(x\,y)^{1/2}$$

$$A_n(x_1, \ldots, x_n; y_1, \ldots, y_n) \equiv \sum_{\text{6v DWBC configs}} \prod_{i,j=1}^{n} w(x_i, y_j)$$

Korepin wrote recursion relations that fix entirely A_n (in terms of A_{n-1}). Using them Izergin showed

$$A_n(x_1, \ldots, x_n; y_1, \ldots, y_n) = \frac{\prod_{i,j=1}^{n} a(x_i, y_j) b(x_i, y_j)}{\prod_{i<j}(x_i - x_j)(y_i - y_j)} \det_{i,j=1\ldots n} \left(\frac{c(x_i, y_j)}{a(x_i, y_j) b(x_i, y_j)} \right)$$

NB: $A_n(x_1, \ldots, x_n; y_1, \ldots, y_n)$ is a symmetric function of the x_i, and of the y_i.

Kuperberg ('98): set $q = \mathrm{e}^{2i\pi/3}$ and $x_i = y_i = 1 \Rightarrow$ recover Zeilberger's formula for A_n.

6 Vertex Model with DWBC at $q = e^{2i\pi/3}$: Okada formula

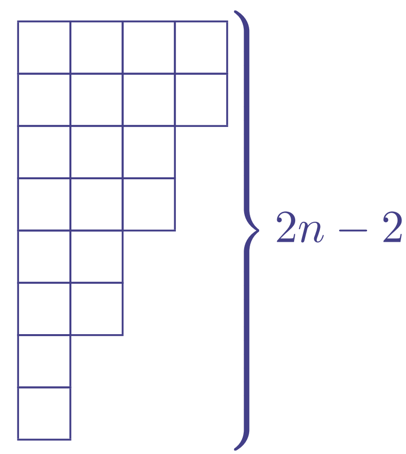
In the next 2 slides, set $q = e^{2i\pi/3}$.

Okada ('02): $A_n(x_1, \dots, x_n; y_1, \dots, y_n)$ is a symmetric function of the full set of parameters x_i, y_i.

$$z_i \equiv x_i \qquad z_{i+n} \equiv y_i \qquad i = 1 \dots n$$

It is a Schur function: (up to a prefactor)

$$A_n(z_1, \dots, z_{2n}) = s_Y(z_1, \dots, z_{2n}) \qquad Y =$$



It is entirely characterized by the following properties: (Stroganov, '04)

(i) It is a symmetric [homogeneous] polynomial of the z_i, of degree $n-1$ in each variable.

(ii) It satisfies the recursion relation

$$A_n(z_1, \dots, z_{2n})\Big|_{z_j = q\, z_i} = \prod_{\substack{k=1 \\ k \neq i,j}}^{2n} (q^2 z_i - z_k)\, A_{n-1}(z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_{j-1}, z_{j+1}, \dots, z_{2n}) \,.$$

Inhomogeneous T–L model of loops [PDF, PZJ '04]

Introduce local probabilities dependent on the column i via a parameter z_i respecting **integrability** of the model (i.e. satisfying Yang–Baxter equation). Form the new vector $\Psi(z_1, \ldots, z_{2n})$ of probabilities, normalized so that its components are **coprime** polynomials.

⋆ *Polynomiality.* The components of $\Psi(z_1, \ldots, z_{2n})$ are homogenous polynomials of total degree $n(n-1)$ and of partial degree at most $n-1$ in each z_i, with coefficients in $\mathbb{Z}[q]$, $q = \mathrm{e}^{2i\pi/3}$.

⋆ *Factorization and symmetry.* (...)

The sum of components is a symmetric polynomial of all z_i.

⋆ *Recursion relations.* The set of components $\Psi_\pi(z_1, \ldots, z_{2n})$ satisfies linear recursion relations when $z_j = q^2\, z_i$; in particular, the sum satisfies the Korepin/Stroganov recursion relation, and therefore

$$\sum_\pi \Psi_\pi(z_1, \ldots, z_{2n}) = A_n(z_1, \ldots, z_{2n})$$

qKZ and Affine Hecke representation [Pasquier]

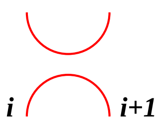
Consider the following set of equations: (level 1 qKZ)

$$\Psi(z_1,\ldots,z_{i+1},z_i,\ldots,z_{2n}) = \check{R}_i(z_{i+1}/z_i)\Psi(z_1,\ldots,z_{2n}), \quad i=1,2,\ldots,2n-1 \tag{1}$$

$$\Psi(z_2,z_3,\ldots,z_{2n},q^6\,z_1) = c\,\sigma^{-1}\Psi(z_1,\ldots,z_{2n}) \tag{2}$$

where Ψ is a vector-valued polynomial of degree $n(n-1)$, σ is rotation of link patterns and

$$\check{R}_i(z) = \frac{(q^{-1}-qz)+(1-z)e_i}{q^{-1}z-q}$$

$e_i =$ 

$=$ generator of Temperley–Lieb algebra TL($\beta=-q-q^{-1}$) acting on link patterns.

For $q=\mathrm{e}^{\pm 2i\pi/3}$, one recovers the previous eigenvector Ψ.

Rewrite Eqs. (1) by separating the action on link patterns and that on polynomials:

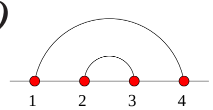
$$(q^{-1}z_{i+1}-qz_i)\partial_i\Psi = (e_i+q+q^{-1})\Psi \tag{1'}$$

where $\partial \equiv \frac{1}{z_{i+1}-z_i}(\tau_i-1)$ and τ_i switches z_i and z_{i+1}. The operators $(q^{-1}z_{i+1}-qz_i)\partial_i$ acting on polynomials form a representation of the Hecke algebra. Together with the cyclic shift of spectral parameters they generate a representation of affine Hecke…

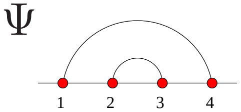
Rational limit and Hotta's construction

Consider $q = -\mathrm{e}^{-\hbar a/2}$, $z_i = \mathrm{e}^{-\hbar w_i}$, $\hbar \to 0$. In this limit the e_i form a representation of TL$(\beta = 2)$ which is a quotient of the symmetric group. The e_i generate the Joseph representation on orbital varieties, and Eq. (1') is related to Hotta's construction of this representation. Each Ψ_π is the *multidegree* of an orbital variety. NB: $\Psi_\pi(z_i = 0, a = 1) = \textit{degree}$, $\Psi(a = 0) = \textit{Joseph polynomial}$. Here the orbital varieties are the irreducible components of the scheme of upper triangular $N \times N$ matrices that square to zero, $N = 2n$. Torus action = conjugation by diagonal matrices and scaling.

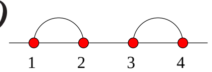
Example: $N = 4$. Two components:

O 

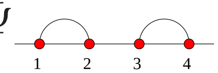
$$= \left\{ M = \begin{pmatrix} & 0 & m_{13} & m_{14} \\ & & m_{23} & m_{24} \\ & & & 0 \\ & & & \end{pmatrix} \right\}$$

Ψ 

$$= (a + z_1 - z_2)(a + z_3 - z_4)$$

O 

$$= \left\{ M = \begin{pmatrix} & m_{12} & m_{13} & m_{14} \\ & & 0 & m_{24} \\ & & & m_{34} \\ & & & \end{pmatrix} \ :\ m_{12}m_{24} + m_{13}m_{34} = 0 \right\}$$

Ψ 

$$= (a + z_2 - z_3)(2a + z_1 - z_4)$$

Other orbital varieties/boundary conditions

B-type orbital varieties: consider $(2r+1) \times (2r+1)$ matrices such that $M^T J + JM = 0$ where J is the antidiagonal matrix with 1's on the antidiagonal, and $M^2 = 0$.

The multidegrees of irreducible components of this scheme satisfy B-type qKZ equation at $q = -1$.

q-deform and set $q = \mathrm{e}^{2i\pi/3}$, $z_i = 1$.

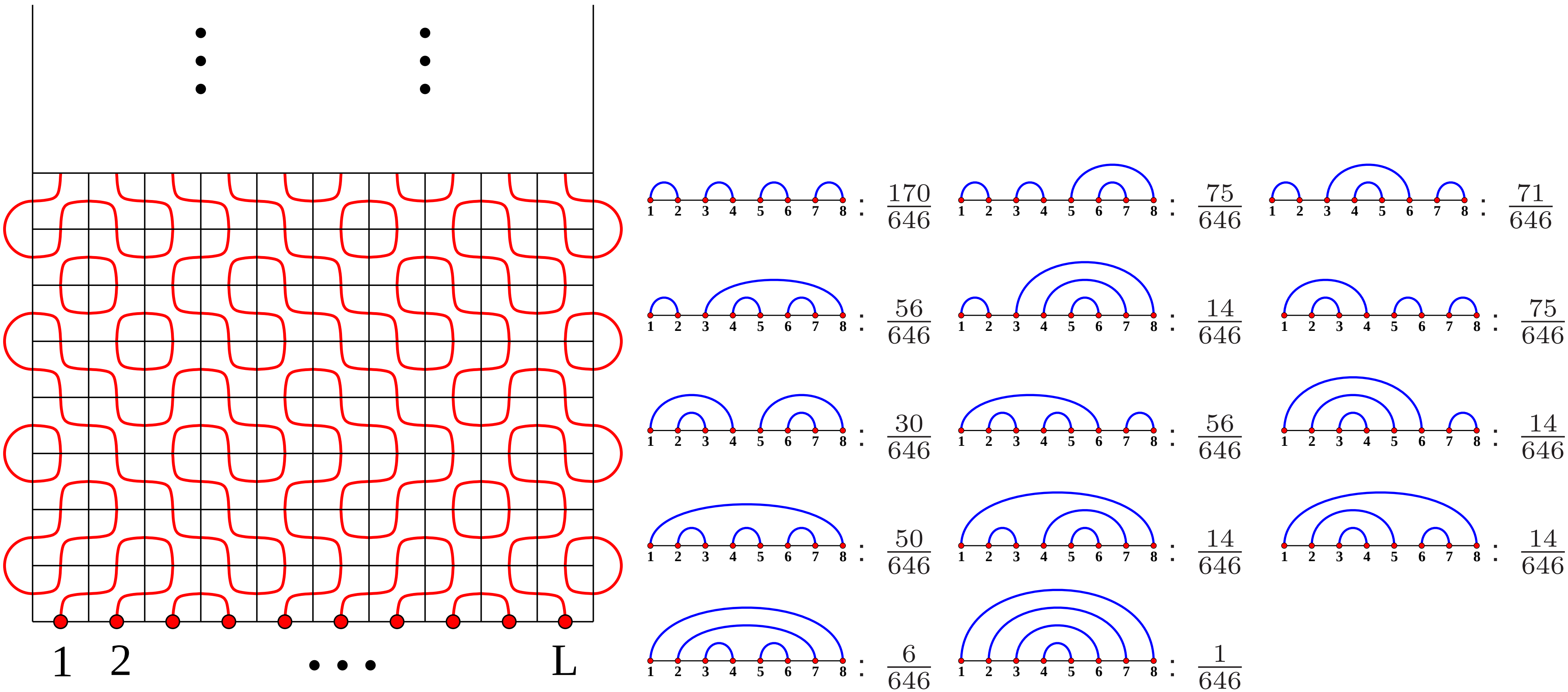
Results for r even:

Theorem [DF '05]: if one normalizes the solution of qKZ equation so that its smallest entry is 1, then the sum of components is $A_V(r)$, the number of Vertically Symmetric Alternating Sign Matrices of size $r+1$

Conjecture: the largest component is the number of Cyclically Symmetric Transpose Complement Plane Partitions in a hexagon of size $r \times r \times r$.

The $O(1)$ loop model: closed boundary conditions

The components are the (unnormalized) probabilities of the following model on a strip:



Other orbital varieties/boundary conditions

C-type orbital varieties: consider $(2r) \times (2r)$ matrices such that $M^T J + JM = 0$ where J is the antidiagonal matrix with 1's (resp. -1's) in the upper (resp. lower) triangle. and $M^2 = 0$.

Take its multidegrees, q-deform them, and set $q = \mathrm{e}^{2i\pi/3}$, $z_i = 1$. Conjectures: (r even)

◇ With the normalization that the smallest component is 1, the sum of components is the number of Cyclically Symmetric Self-Complementary Plane Partitions in a hexagon of size $r \times r \times r$.

◇ The largest entry is the sum of components at size $r - 1$.

D-type orbital varieties: consider $(2r) \times (2r)$ matrices such that $M^T J + JM = 0$ where J is the antidiagonal matrix with 1's on the antidiagonal, and $M^2 = 0$.

Take its multidegrees, q-deform them, and set $q = \mathrm{e}^{2i\pi/3}$, $z_i = 1$. Conjectures:

◇ With the normalization that the smallest component is 1, the sum of components is the number of Half-Turn Symmetric Alternating Sign Matrices of size r.

◇ The largest entry is the sum of components of the C-type solution at size $r - 1$.