Coincidences amongst skew Schur functions

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Compositions and partitions

A composition $\alpha_1 \ldots \alpha_k$ of n is a list of positive integers whose sum is n: $2213 \models 8$.

A composition is a partition if $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_k > 0$: $3221 \vdash 8$.

Any composition determines a partition: $\lambda(2213) = 3221$.

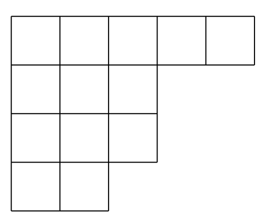
Skew diagrams and ribbons

The diagram $\lambda = \lambda_1 \geq \ldots \geq \lambda_k > 0$ is the array of boxes with λ_i boxes in row i.

For λ, μ the skew diagram λ/μ is the array of boxes contained in λ but not in μ.

A skew diagram λ/μ is a ribbon if

connected shape with no 2×2 square.



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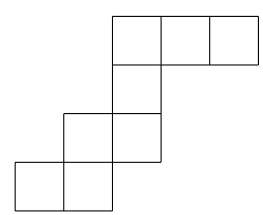

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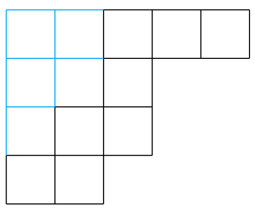

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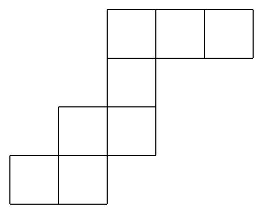

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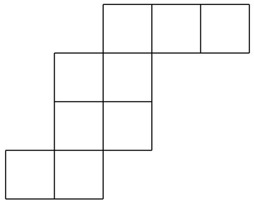
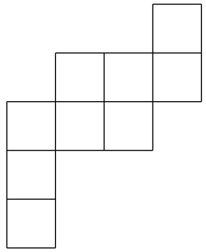
Symmetries of skew diagrams

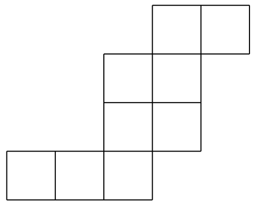
Given a skew diagram λ/μ:



Conjugation gives $(\lambda/\mu)^t = \lambda^t/\mu^t$:



Antipodal rotation gives $(\lambda/\mu)^*$:



Young tableaux

A semi-standard Young tableau (SSYT) T of shape λ/μ is a filling with $1, 2, 3, \ldots$ so rows weakly increase and columns increase.

Example

	1
2	2

Given a SSYT T we have

$$x^T := x_1^{\#1s} x_2^{\#2s} x_3^{\#3s} \ldots$$

Example

$$x_1 x_2^2$$

Skew Schur functions

We define skew Schur function of shape λ/μ by

$$s_{\lambda/\mu} = \sum_{T \text{ SSYT of shape } \lambda/\mu} x^T.$$

The classical Schur functions are s_λ when $\mu = 0$.

Let $\Lambda \subset \mathbb{Q}[[x_1, x_2, \ldots]]$ be the algebra of all symmetric functions

$$\Lambda := \Lambda_0 \oplus \Lambda_1 \oplus \cdots$$

where

$$\Lambda_n := \text{span}_{\mathbb{Q}}\{s_\lambda \mid \lambda \vdash n\}.$$

Example $s_{22/1} = x_1 x_2^2 + \ldots$

Equality of skew Schur functions

Question: When is

$$cs_{D_1}s_{D_2}\ldots s_{D_m} - c's_{D'_1}s_{D'_2}\ldots s_{D'_m} = 0?$$

Question: When are GL_n-representations the same?

Answer: Determine when

$$s_{\lambda/\mu} = s_{\nu/\rho}$$

for $\lambda/\mu, \nu/\rho$ connected. Denote by

$$\lambda/\mu \sim \nu/\rho.$$

Necessary conditions

For λ/μ the k-row overlap composition is $r^{(k)} := r_1^{(k)} r_2^{(k)} \ldots$ where $r_i^{(k)}$ is number of common columns for rows $i, \ldots, i+k-1$.

Example

		$\times$	$\times$	$\times$	$\times$	
$\times$	$\times$	$\times$				
$\times$	$\times$	$\times$				
$\times$						

$$\begin{aligned} r^{(1)} &= 4331 \\ r^{(2)} &= 131 \\ r^{(3)} &= 11 \\ r^{(4)} &= 0 \end{aligned}$$

Theorem If $D \sim E$ then $\lambda(r^{(k)}(D)) = \lambda(r^{(k)}(E))$ for all k.

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Example

$$\begin{array}{cccccc} & & \times & \times & \times & \times \\ \times & \times & \times & & & \\ \times & \times & \times & & & \\ \times & & & & & \end{array} \qquad \begin{aligned} r^{(1)} &= 4331 \\ r^{(2)} &= 131 \\ r^{(3)} &= 11 \\ r^{(4)} &= 0 \end{aligned}$$

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Back to ribbons

Example

$$\begin{array}{cccc} & & \times & \times \\ \times & \times & \times & \\ \times & \times & & \\ \times & & & \end{array} \quad \sim \quad \begin{array}{cccc} & \times & \times & \times \\ & \times & \times & \\ \times & \times & & \\ \times & & & \end{array}$$

and

$$\begin{array}{lcccc} r^{(1)} & : & 2321 & & 3221 \\ r^{(2)} & : & 121 & & 211 \\ r^{(3)} & : & 01 & & 10 \\ r^{(4)} & : & 0 & & 0 \end{array}$$

Corollary $\sim$ restricts to the subset of ribbons since they are the only skew diagrams with $r^{(2)} = 1 \ldots 1$.

Is this enough?

Example

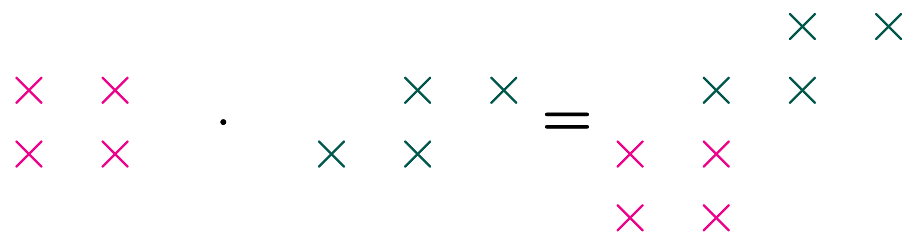
but

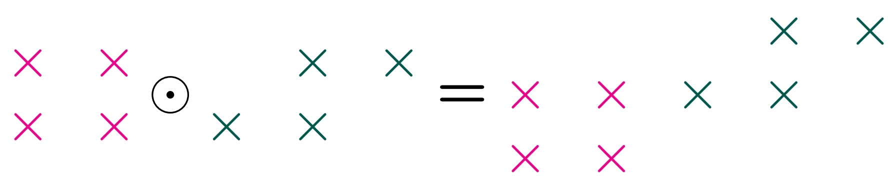
$r^{(1)}$	:	231	321
$r^{(2)}$	:	11	11
$r^{(3)}$	:	0	0

Question:

What is sufficient?

Operations on skew diagrams

Concatenation gives 

Near concatenation gives 

Ribbons and skew diagrams I

Observe if α is a ribbon then

$$\alpha = \times \star_1 \times \star_2 \ldots \star_k \times$$

where $\star_i = \cdot$ or $\odot$.

Example

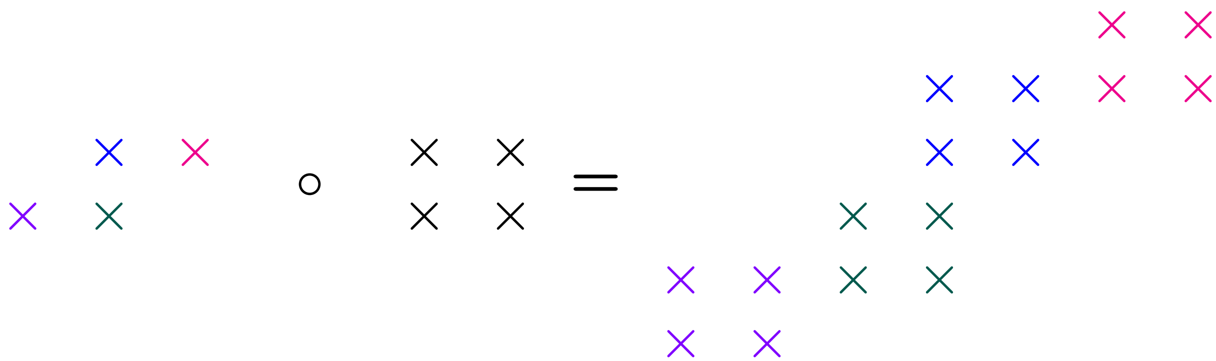
$$\begin{array}{ccc} & \times & \times \\ \times & \times & \end{array} = \times \odot \times \cdot \times \odot \times$$

If $\alpha = \times \star_1 \times \star_2 \ldots \star_k \times$ then

$$\alpha \circ D = D \star_1 D \star_2 \ldots \star_k D.$$

Ribbons and skew diagrams II

Example



Theorem If $\alpha \sim \alpha'$ then

$$\alpha' \circ D \;\sim\; \alpha \circ D \;\sim\; \alpha \circ D^*.$$

An important map

For a fixed skew diagram D we have

$$\begin{array}{ccc} \Lambda & \xrightarrow{(-)\circ s_D} & \Lambda \\ s_\alpha & \longmapsto & s_{\alpha\circ D} \end{array}$$

is well-defined.

Remark For $f \in \Lambda$ write f in ribbon Schur functions $f = p(s_\alpha)$ and set $f \circ s_D := p(s_{\alpha\circ D})$ so

$$s_\alpha \circ s_D = s_{\alpha\circ D}.$$

Ribbons and protrusions

If D is a skew diagram we say a ribbon ω protrudes from the top and bottom if

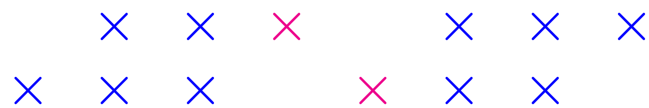
$$D \;=\; \begin{array}{cccc} & \times & \times & \times \\ \times & \times & \times & \end{array}$$

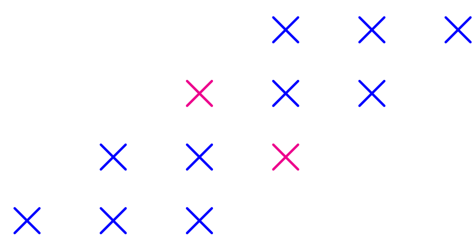
and the amalgamation of D along ω is

$$D \amalg_{\omega} D = \begin{array}{ccccccc} & & & & \times & \times & \times \\ & \times & \times & \times & \times & \times & \\ \times & \times & \times & & & & \end{array}$$

Inner and outer projections

If ω protrudes from D then

Outer projection gives 

Inner projection gives 

Note: At most one is a skew diagram $D \cdot_\omega D$.

$\circ$ wrt a ribbon

If

$$\alpha \circ D = D \star_1 D \star_2 \ldots \star_k D$$

then for ω protruding from top and bottom swap $\cdot$ for $\cdot_\omega$ and $\odot$ for $\amalg_\omega$ to get

$$\alpha \circ_\omega D.$$

Theorem If ωs are separated by at least one diagonal and $\alpha \sim \alpha'$ then

$$\alpha' \circ_\omega D \;\sim\; \alpha \circ_\omega D \;\sim\; \alpha \circ_{\omega^*} D^*.$$

Further avenues

Conjecture Operations $\circ_\omega$ and $*$ provide all necessary and sufficient conditions for $\sim$.

(McNamara and SvW upto n=18)

Conjecture All $\sim$ classes have cardinality power of 2.

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