Kempf Collapsing and Quiver Loci

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Motivation and Goal

Quiver polynomials generalize or specialize to:

- Schur polynomials [Porteous '71]
- Schubert and Grothendieck polynomials [Fulton '92] [Knutson,Miller '05]
- Quantum Schubert polynomials [Fulton '99]
- Fulton's "universal" Schubert polynomials [Fulton '99] [Buch,Kresch,Tamvakis,Yong '04]; [BKTY '05]
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A new explicit divided difference formulae for a large family of quiver polynomials which includes all these cases.

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Divided difference operators

$$F \in \mathbb{Z}[e^{\pm x_1}, e^{\pm x_2}, \ldots, e^{\pm x_m}] \qquad f \in \mathbb{Z}[x_1, x_2, \ldots, x_m]$$

$$\pi_i F = \frac{F - e^{x_{i+1}-x_i} s_i F}{1 - e^{x_{i+1}-x_i}} \qquad \partial_i f = \frac{f - s_i f}{x_i - x_{i+1}}$$

Demazure operator BGG operator

$$\pi_i^2 = \pi_i \qquad \partial_i^2 = 0$$

$$\pi_i \pi_j = \pi_j \pi_i \qquad \partial_i \partial_j = \partial_j \partial_i \qquad |i-j| \geq 2$$

$$\pi_i \pi_{i+1} \pi_i = \pi_{i+1} \pi_i \pi_{i+1} \qquad \partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1}$$

Define π_w and ∂_w using a reduced word for $w \in S_m$.

$$\mathfrak{G}_w = \pi^x_{w^{-1}w_0} \prod_{i+j \leq m} (1 - e^{-(x_i - y_j)}) \qquad \mathfrak{S}_w = \partial^x_{w^{-1}w_0} \prod_{i+j \leq m} (x_i - y_j)$$

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Quiver Representations

$Q = (Q_0, Q_1)$	Quiver = directed graph
Q_0	vertex set
Q_1	directed edge set

For $a \in Q_1$ tail $ta \xrightarrow{a} ha$ head

Representation V of Q:

vertex $i \in Q_0 \mapsto$ vector space $V_i = \mathbb{C}^{d(i)}$
arrow $a \in Q_1 \mapsto$ linear map $V_a \in M_{d(ta) \times d(ha)}(\mathbb{C})$

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$$\mathbb{C}^3 \longrightarrow \mathbb{C}^4$$

$\mathrm{Hom} = M_{3\times 4}(\mathbb{C}) \qquad G = GL(3) \times GL(4)$

$$\phi = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \in \mathrm{Hom}$$

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K-polynomial $K_V(Y)$

V vector space with positive $T \cong (\mathbb{C}^*)^m$-action

$Y \subset V$: T-stable algebraic subscheme

character group $T^* = \mathrm{Hom}_{\mathrm{group}}(T, \mathbb{C}^*) \cong \mathbb{Z}^m$

weight space decomp. of T-module $M = \bigoplus_{\lambda \in T^*} M_\lambda \subset \mathbb{C}[V]$

$$M_\lambda := \{m \in M \mid t \cdot m = \lambda(t)m \text{ for all } t \in T.\}$$

$$\mathrm{ch}_T(M) := \sum_{\lambda \in T^*} \dim(M_\lambda) e^\lambda$$ formal character

$$K_V(Y) := \mathrm{ch}_T(\mathbb{C}[Y])/\mathrm{ch}_T(\mathbb{C}[V])$$ K-polynomial

$$\in K_T^*(V) \cong R(T) = \mathbb{Z}[e^{\pm x_1}, e^{\pm x_2}, \ldots, e^{\pm x_m}]$$

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$V = M_{3\times 4}(\mathbb{C}) \qquad \mathbb{C}[V] = \mathbb{C}[z_{ij}]_{i,j=1,1}^{3,4}$

$t = \mathrm{diag}(x_1, x_2, x_3) \times \mathrm{diag}(y_1, y_2, y_3, y_4) \in T(3) \times T(4)$

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$$Z := \left[\begin{pmatrix} 0 & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & 0 & 0 \end{pmatrix} \right] \subset M_{3\times 4}(\mathbb{C})$$

$$\begin{aligned} K_V(Z) = {} & (1-e^{-(x_1-y_1)})(1-e^{-(x_1-y_2)})(1-e^{-(x_1-y_3)})(1-e^{-(x_1-y_4)}) \\ & (1-e^{-(x_2-y_3)})(1-e^{-(x_2-y_4)}) \\ & (1-e^{-(x_3-y_3)})(1-e^{-(x_3-y_4)}) \end{aligned}$$

Multidegree $[Y]_V$

$$[Y]_V \in H_T^*(V) \cong \mathrm{Sym}^\bullet(T^*) \cong \mathbb{Z}[x_1, x_2, \ldots, x_m]$$

$[Y]_V :=$ lowest degree term of $K_V(Y)$ Multidegree

$$e^\lambda = 1 + \lambda + \lambda^2/2! + \cdots$$

$[\Omega]_{\mathrm{Hom}}$: cohomological quiver polynomial

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				Z lin. subspace	$\subset$	V vector space
				$\circlearrowright$		$\circlearrowright$
T diagonal	$\subset$	L block diagonal	$\subset$	P lower block triangular	$\subset$	G product of GLs

A Kempf collapsing is a map

$$(G \times Z)/P =: G \times^P Z \xrightarrow{\kappa} V$$

$$(g, z)p = (gp, p^{-1} \cdot z) \qquad (g, z)P \mapsto gz$$

The image of κ is $G \cdot Z$.

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Geometric result

Theorem (Knutson,S.)

Let κ be a birational Kempf collapsing. Then

- $[G \cdot Z]_V = \partial_{G/P}[Z]_V$.
- *If $G \cdot Z$ has rational singularities then* $K_V(G \cdot Z) = \pi_{G/P} K_V(Z)$.

$$\partial_{G/P} := \partial_{w_{G/P}} \qquad \pi_{G/P} := \pi_{w_{G/P}}$$

$w_{G/P}$: minimal length coset rep of the longest element of $W(G)$ in $W(G)/W(L)$.

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Theorem

- *[Reineke '04] If Q is of type ADE, each quiver locus $\Omega \subseteq \mathrm{Hom}(Q, d)$ is the image of a birational Kempf collapsing, i.e., there exists a parabolic subgroup $P \subset G(Q, d)$ and a P-invariant linear subspace $Z \subset \mathrm{Hom}$ such that $G \times^P Z \twoheadrightarrow G \cdot Z = \Omega$ is birational.*
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Multidegree of Determinantal Variety

$$\Omega = \{A \in M_{3\times 4} \mid \mathrm{rank}(A) \leq 2\}$$

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Recipe for P and Z given Ω

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$$\phi \cong \bigoplus_{\alpha \in R^+} (I^\alpha)^{\oplus m(\alpha)}$$

Poset $\mathrm{Indec}_Q = \{I^\alpha \mid \alpha \in R^+\}$ of indecomposable Q-reps

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Recipe for P and Z given Ω (contd.)

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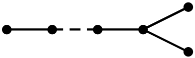
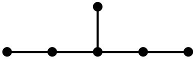
ADE Dynkin Diagrams

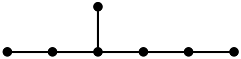
A_n



D_n



E_6



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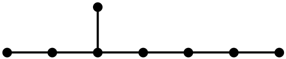

E_8